SEMIGROUPS AND IDEALS

Introduction :

In this chapter, in first section preliminary definitions of semigroup and its different ideals are given with illustrations. Also some results about ideals are given .

In second section, we have defined semigroup with idemptoent ideals and its different properties are discussed .

In third section, Normal semigroups are defined and some of its properties are discussed.

In fourth section, we have defined B-pure Bi-ideal and some properties of B^* - pure semigroup have been discussed.

Section 1 : Semigroup, Ideals and Green's relations

Definition 2.1.1 :Semigroup : A non empty subset S together with binary operation . is called semigroup iff . is associative.

i.e. $a.(b.c) = (a.b).c \quad \forall a,b,c \in S.$

E.G. i) (N,x) where N is set of natural numbers.

ii) consider $(z_4, +_4, x_4)$ i.e. ring of residue classes modulo 4, Now if we define binary operation $\oplus$ on elements of Z as

$a \oplus b = a +_4 b - a \times_4 b$

Then $(Z_4, \oplus)$ is a semigroup.

e.g.

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	1	0	3
3	3	1	3	1

$(Z_4, \oplus)$ is a commutative semigroup.

Definition 2.1.2 :Subsemigroup : S be a semigroup. By a subsemigroup of S , we mean a nonempty subset A of S such that $A^2 \subseteq A$

i.e. $\forall x, y \in A$, $xy \in A$

E.g

i) (Set of all even natural numbers, x) is a subsemigroup of (N , x)

ii) $(Z_2, \oplus)$ is subsemigroup of $(Z_4, \oplus)$

Definition 2.1.3 : Left Ideal : A nonempty subset A of a semigroup S is called left ideal of S iff $SA \subseteq A$.

i.e. $\forall x \in S$, $y \in A$ we have $xy \in A$.

E.g.

i) Set of all even natural numbers is left ideal of semigroup (N , x)

ii) Let $S = \{1,2,3,4\}$

Define binary operation * on S as follows

$a * b = b \quad \forall a,b \in S$

So we get composition table as follows

	1	2	3	4
1	1	2	3	4
2	1	2	3	4
3	1	2	3	4
4	1	2	3	4

Obviously (S,*) is semigroup.

For if we consider any three elements

$a,b,c \in S$ then

$(a*b) * c = b * c = c$ and

$a * (b * c) = a * c = c$

So $(a * b) * c = a * (b * c)$

So * is associative

If we consider subset S' of S given by $S' = \{2,3\}$

Then S' is Left ideal of S

$\forall a \in S, b \in S'$ we have $a * b = b \in S'$

i.e. S' is left ideal.

Definition 2.1.4 Right Ideal : Non empty subset A of a semigroup S is called right ideal of S if $AS \subseteq A$

i.e. $\forall x \in A, y \in S$ we have $xy \in A$.

E.g

i) Set 2N is Right ideal of semigroup (N, x)

ii) Let $S = \{1,2,3,4\}$

Define binary operation * on S by $a * b = a \quad \forall a,b \in S$

i.e. we get composition table as follows

*	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4

Consider any three elements a,b,c, ε S

Then $(a * b) * C = a * C = a$ and

$(a * (b * C) = a * b = a$

So * is associative.

i.e. (S, *) is a semigroup.

Consider subset $S' = \{2,3\}$ of semigroup S.

Then $a \varepsilon S$ and $b \varepsilon S'$, $b * a = b \varepsilon S'$

So S' is right ideal.

Remarks :

1) For a commutative semigroup every left ideal is Right ideal also.
2) 2nd example given in 2.1.4 is an example of right ideal which is not left ideal.

Definition 2.1.5 Ideal : By Ideal, we mean subset of semigroup S which is both left and Right ideal of S.

Obviously every ideal is subsemigroup.

E.g. 1) Set 2N is an ideal of semigroup (N,x)

2) Z_2 is ideal of semigroup $(Z_4, \oplus)$

Definition 2.1.6 Simple Semigroup : A semigroup S is called simple if it contains no proper ideal.

E.g. Consider semigroup given in example 2 of 2.1.4. It can be easily seen that the set

$S = \{1,2,3,4\}$ is only left ideal as well as right ideal of S. So S itself is only ideal.

Hence it is an example of simple semigroup.

Definition 2.1.7 Bi-ideal : Subsemigroup A of S is called bi-ideal of S iff $ASA \subseteq A$.

i.e. $\forall$ x and $z \in A$, $y \in S$ we have $xyz \in A$.

E.g. Subsemigroup (2N, x) is bi-ideal of semigroup(N,x) .

Definition 2.1.8 Interior Ideal:

1) Subsemigroup of a semigroup S is called as an Interior ideal of S iff $SAS \subseteq A$.

2) By I(x), we denote the principal interior ideal of semigroup S, generated by $x \in S$

i.e. $I(x) = \{x, x^2\} \cup S x S$

E.g Subsemigroup (2N,x) is an interior ideal of semigroup (N,x)

Remark :

Any ideal of semigroup S is an interior ideal of S

For let A be any ideal of semigroup S

i.e. $SA \subseteq A$ and $AS \subseteq A$

Then trivially $SAS \subseteq AS \subseteq A$

i.e. $SAS \subseteq A$.

Definition 2.1.9 Regular Semigroup : Semigroup S is called regular if for each element a of S, $\exists$ an element $x \in S$ S. t . $a = axa$.

E.g. Semigroup (S,*) in example 2 of 2.1.3 is a regular semigroup.

For $1 = 1 * 1 * 1$, $2 = 2 * 2 * 2$, $3 = 3 * 3 * 3$ and

$4 = 4 * 4 * 4$

Remarks :

1) Every ideal of S is subsemigroup

obviously

For if A is an ideal of S

then $AS \subseteq A$ and $SA \subseteq A$

In particular $AA \subseteq A$ i.e. $A^2 \subseteq A$

2) **Every subsemigroup** of S need not be ideal.

We prove this by giving an example of a subsemigroup which is not ideal.

Let $S = \{ 1,2,3,4,5 \}$

Define binary operation * on elements of S by

$a * b = b \ \forall \ a,b \in S$

Then (S,*) is semigroup.

(S',*) is subsemigroup of S where $S' = \{ 1,2,3 \}$

But S' is not ideal of (S, *)

as $3 \in S'$, $4 \in S$ But $3 * 4 = 4 \notin S'$

3) Every ideal of S is bi-ideal.

Let A be any ideal of S.

i.e. $AS \subseteq A$ and $SA \subseteq A$.

Now $AS \subseteq A \Rightarrow ASA \subseteq AA \subseteq A$.

So A is bi-ideal.

4) Every bi-ideal of S is not an ideal of S.

Consider $(S,*)$ and $(S',*)$ as defined in (2).

$(S', *)$ is subsemigroup of $(S, *)$.

It is also bi-ideal.

As $x,z \in S'$ and $y \in S$

$x * y * z = z \in S'$

But as proved in (2) S' is not ideal

S' is only left ideal but not right ideal.

Proposition 1 :

For a subset A of regular semigroup S, the following conditions are equivalent.

1) A is an ideal of S

2) A is an interior ideal of S.

Proof :

(1) $\Rightarrow$ (2) Let A be an ideal of S.

$\Rightarrow$ A is subsemigroup of S

Also $SA \subseteq A$ and $AS \subseteq A$.

$\Rightarrow SAS \subseteq AS \subseteq A$.

So A is an interior ideal of S.

(2) $\Rightarrow$ (1) Let A be an interior ideal of S i.e. SAS $\subseteq$ A.

To prove that A is an ideal of S.

i.e. to prove that if a $\in$ A and s $\in$ S

The as $\in$ A and sa $\in$ A.

Now.. a ε A $\Rightarrow$ a ε S

As S is regular semigroup So $\exists$ x ε S s.t. a = a x a

Now as = a x a s = (ax) a s ε SAS $\subseteq$ A {as ax ε S}

sa = saxa = sa(xa) ε SAS { as x a ε S}

So as ε A and sa ε A $\forall$ a ε A and s ε S

i.e. SA $\subseteq$ A and AS $\subseteq$ A.

So A is an ideal of S.

Hence proved.

Definition 2.1.10 : Intra-regular Semigroup : A semigroup S is called intra-regular if for each element a $\in$ S, $\exists$ elements x and y in S, s.t. a = x a^2 y

E.g Semigroup (S, *) in example 2 of 2.1.3 is intra-regular semigroup.

For 1 = 1 * 1^2 * 1 3 = 3 * 3^2 * 3

2 = 2 * 2^2 * 2 4 = 4 * 4^2 * 4

Relation between regular and intra regular semigroups

Definition 2.1.11 : Semi-prime Subset :

Subset A of a semigroup S is called semiprime if a^2 ε A, a ε S $\Rightarrow$ a ε A.

E.g. Subset 2N of a semigroup (N , x) is semiprime.

Definition 2.1.12 :

Left regular Semigroup : A semigromup S is called left regular if for each element $a \in S$, $\exists$ an element $x \in S$ s.t. $a = x a^2$

E.g. Semigroup (S,*) in example 2 of 2.1.3 is left regular semigroup.

as $1 = 1*1^2$, $2 = 2 * 2^2$, $3 = 3 * 3^2$, $4=4 * 4^2$

Right regular semigroup is defined dually.

Proposition 2 :

For a subset A of an intra regular semigroup S following conditions are equivalent.

1) A is an ideal of S

2) A is an interior ideal of S

Proof : Let (1) hold.

i.e. A is an ideal of S

So $SA \subseteq A$ and $AS \subseteq A$

Now $SA \subseteq A \Rightarrow SAS \subseteq AS \subseteq A$

So $SAS \subseteq A$

i.e. A is an interior ideal.

Conversly, Let (2) hold

i.e. A is an interior ideal of S

Let $a \in A$ and $s \in S$

as $a \in A \Rightarrow a \in S$ and S is intra-regular

$\exists$ $x, y \in S$ s.t. $a = x a^2 y$

Consider

$as = x a^2 y S = (xa) a(y S) \in SAS \subseteq A$ {as xa, and $ya \in S$}

So $as \in A$

Similarly $sa \in A$

So $SA \subseteq A$ and $AS \subseteq A$

i.e. A is ideal of S.

Green's Equivalence relations :

Two elements of a semigroup S are said to be L equivalent if they generate same principal left ideal of S. R-equivalence is defined dually.

The join of equivalence relation L and R is denoted by D and their intersection by H. These equivalence relations were first introduced and studied by Green (1951). These equivalence relations are called Green's relations.

By $a^L b$ we mean $S' a = S'b$

where $S' = S$ if $1 \in S$ where 1 is identity elt

$= S \cup \{1\}$ if $1 \notin S$

By $a^R b$ we mean $aS' = bS'$

By La we mean set of all elements of S which are L equivalent, to a. We define a J b (a,b in S) to mean $S'aS' = S'bS'$ i.e. a and b are J equivalent iff they generate same two sided principal ideal.

Remarks :

1) S is regular if $a \in aSa$

S is left regular if $a \in Sa^2$

S is Right regular if $a \in a^2 S$

S is intra rregular if $a \in Sa^2 S$

2) In terms of Green's equivalence relations

S is left regular iff $a \; L \; a^2$

S is Right regular iff $a \; R \; a^2$

S is intra regular iff $a \; J \; a^2$

For if S is left regular then for all elements $a \varepsilon S, \exists \; x \varepsilon S$ s.t. $x = xa^2$

Now $a = xa^2$ and $a^2 = a.a \; = a \; L \; a^2$

Conversly $a \; L \; a^2 \; = a \in L(a^2)$

i.e. $a \varepsilon \{a^2\} \cup Sa^2$

If $a = a^2$ then $a = a.a^2$

and if $a \varepsilon Sa^2$, $\exists \; x \varepsilon S$ s.t. $a = xa^2$

$\Rightarrow$ S is left regular.

Propositionm 3 :

A semigroup S is left [Right,Intra-] regular iff every Left (Right,two sided) ideal of S is semiprime.

Proof : Let S be intra regular and let A be any ideal of S.

Let $a \varepsilon A$, $a^2 \varepsilon S$

Then as $a \varepsilon Sa^2 S \subseteq SAS \subseteq A$

So every two sided ideal of S is semiprime.

Conversly, assume that every ideal of S is semiprime.

Let $a \varepsilon S$ Then $a^2 \varepsilon J(a^2)$

But $J(a^2)$ i.e. principal ideal generated by $a^2 \varepsilon S$ is semiprime.

So $a \varepsilon J(a^2)$

Hence $a \; Ja^2$

and so by remarks above S is intraregular. The proof of equivalence of Left (Right) regularity of S with semiprimality of all Left (Right) ideals of S is similar.

Proposition 4 :

The following statements concerning a semigroup S are equivalent.

1) S is union of simple semigroups

2) S is intra regular.

3) Every ideal of S is semiprime.

4) The principal ideals of S constitute a semilattice Y under intersection. In fact $J(a) \cap J(b) = J(ab)$ for every a and b in S.

Furthermore, S is union of semilattice Y of simple semigroups S_α ($\alpha \in Y$) each S being a J class of S_α

Proof :

Assume (1) holds

Let $a \in S$. Then a and a^2 both belong to same simple subsemigroup T of S

So $a \in Ta^2 T \subseteq Sa^2 S$

So S is intraregular.

i.e. (1) $\Rightarrow$ (2)

(2) $\Rightarrow$ (3) is clear from proposition (3) evidently (4)$\Rightarrow$(1)

The proof will be complete when we show (4) follows from (2) and (3) and this we do in several steps.

i) SaS is principal ideal $J(a)$ generated by a
For $a \in Sa^2 S \subseteq SaS$

ii) $J(ab) = J(ba)$ for every $a, b \in S$
For $(ab)^2 = a(ba)b \in SbaS = J(ba)$
And from (3) we infer that $ab \in J(ba)$
Hence $J(ab) \subseteq J(ba)$
and Equality follows by symmetry.

iii) $J(ab) = J(a) \cap J(b) \; \forall a, b \in S$
clearly $J(ab) \subseteq J(a) \cap J(b)$
conversly
Let $c \in J(a) \cap J(b)$
Say $c = uav = xby$ with $u, v, x, y \in S$
Then $c^2 = xbyuav \in J(byua) \subseteq J(abyu)$ by (ii)
By (3), this $= c \in J(abyu) \subseteq J(ab)$
Hence $J(a) \cap J(b) = J(ab)$ and equality follows

iv) By (iii) the set Y of principal ideals of S is a semilattice under intersection.
and mapping $a \to J(a)$ is homomorphism, of S upon Y .
The inverse image of the element $J(a)$ of Y is the set J_a of generators of $J(a)$,
i.e J class of to which a belongs.
In particular, J_a is subsemigroup of S and S is semilattice Y of mutually disjoint semigroups J_a
Proof of (4) will complete when we show that each J_a is simple

But principal factor $J(a)/I(a)=Ja \cup \{0\}$ is either 0-simple or null semigroup.

From this and the fact that J(a) is closed under multiplication, it is clear that J(a) must be simple.

Proposition 5 :

A semigroup S is a group iff it is left and Right simple.

Proof :

H.Weber defined a group as a semigroup G such that for any given elements a and b of G, $\exists$ unique elements x and y in G s.t. ax = b and ya = b.

Hunigton showed that it is not necessary to postulate uniqueness of x and y, that this followed as consequence.

If semigroup S is left and right simple

$\exists$ x and y in S s.t. $\forall$ a,b, $\in$ S

ax = b and ya = b

So S is group.

Conversly

As S is group, it has identity element

So aS = S for every a in S

and Sa = S for every a in S

So S is only left ideal as well as S is only right ideal.

So, S is left simple and Right simple.

Section 2 :

In this section we define semigroup with idempotent ideals and discuss some of its properties.

Definition 2.2.1 :

By a semigroup with Idempotent ideals we mean a semigroup in which every ideal is idempotent.

Proposition 6 :

The principal ideal $<a>$ of a semigroup S is idempotent iff $a \in SaSaS$

Proof :

Note that principal ideal $<a> = \{a\} \cup Sa \cup aS \cup SaS$

i.e. $<a> = S'aS'$

Assume that $a \in SaSaS$

i.e. $a \in (Sa)(SaS) \subseteq <a><a> = <a>^2$

So $<a> \subseteq <a>^2$

As converse inclusion is always true

we have $<a> = <a>^2$

Conversly, assume that $<a> = <a>^2$

Then $a \in <a> = <a>^2$

$= <a><a>$

$= <a>^2 <a>^2$

$= <a>^2 <a> <a>$

$= <a>^2 <a>^2 <a>$

$= <a>^5$

$= S'aS'.S'aS'.\ S'aS'.S'aS'.\ S'aS'$

$\subseteq$ Sa Sa S

So proof over.

Proposition 7 :

Let S be a semigroup and I be an ideal of S if $< a > = < a >^2$ for every element $a \in I$, then $I = I^2$, too.

Proof : Assume $< a > = < a >^2$ for each $a \in I$

Then $a \in \langle a \rangle^2 \subset I^2 \ \forall\ a \in I$

So $I \subseteq I^2$ .....(1)

Also for any ideal I, $I^2 \subseteq I$ .....(2)

From (1) and (2), $I^2 = I$

Proposition 8 :(by S.Lajos) :

A commutative semigroup is regular iff it is a semigroup with idempotent ideals.

Proof :

If S is semigroup with idempotent ideals then every element a of S can be represented in form a = xayaz

i.e. $a = a(xyz)a$ by proposition 6 and commutatively

$\in aSa$

$x,y,z \in S = xyz \in S$

So S is regular.

Assume conversly that S is regular and a be an arbitrary element of S

Then $\exists\ x \in S$, s.t. $a = axa$.

Now $a = axa = (ax)a \in \langle a\rangle\langle a\rangle = \langle a\rangle^2$

i.e. $\langle a\rangle \subseteq \langle a\rangle^2$

Now $\langle a\rangle^2 \subseteq \langle a\rangle$ is always true

So S is semigroup with idempotent ideals we have used proposition 7 also.

Remark :

Converse statement of proposition 7 does not hold.

i.e. In a semigroup S if I is any ideal of S and $I=I^2$ does not imply $\langle a\rangle = \langle a\rangle^2$ for $a \in I$

e.g. Let $I = S = \{0,1,a\}$ be commutative semigroup in which $a^2 = 0$

Every ideal of S is reproduced by S

i.e. SI=IS=I for every ideal I of S.

But principal ideal $\langle a\rangle$ is not idempotent.

$\langle a\rangle^2 = \langle 0\rangle \neq \langle a\rangle$

Proposition 9 :

In case of semigroup S, following assertions are equivalent.

1) S is semigroup with idempotent ideals

2) S is semigroup with idempotent principal ideals

3) $a \in SaSaS$ for every element a of semigroup S.

Proof :

(1) $\Rightarrow$ (2) Trivially

(2) $\Rightarrow$ (3) By proposition 6.

(3) $\Rightarrow$ (1)

Proof : If (3) holds, by proposition 6, every principal ideal is idempotent.

and by proposition 7, every ideal is idempotent.

Proposition 10 :

Every ideal of regular semigroup is idempotent.

Proof :

Let $a \varepsilon S$ as S is regular, $\exists$ $x \varepsilon S$ s.t. $a \varepsilon aSa$

i.e. $a = axa = (ax)a$

$\varepsilon < a >< a > = < a >^2$

So $< a > \subseteq < a >^2$ .....(1)

$< a >^2 \subseteq < a >$ .....(2) trivially.

So $< a > = < a >^2$

So, by proposition 7, every ideal I of regular semigroup is idempotent.

Remark :

1) In a similar way, it can be shown that every ideal of left regular, Right regular or intra-regular semigroup is idempotent.

For Intra-regular semigroup. $\forall a \varepsilon S \ \exists \ x$ and y in S s.t.

$a = xa^2y = (xa)\ (ay) \varepsilon < a >< a > = < a >^2$

So $< a > \subseteq < a >^2$

converse inclusion $< a >^2 \subseteq < a >$ always true.

So $< a > = < a >^2$

By proposition 7, it now follows that every ideal I of intra-regular semigroup is idempotent. Similarly, result can be proved for Left and Right regular semigroups.

2) Class of semigroups with idempotent ideals is properly wider than class of semigroups with prime ideals, even inside class of commutative semigroups.

Following example proves above statement

Example :

Consider commutative semigroup $\{0,a,b\}$ in which every element is idempotent and ab=0 obviously, every ideal of S is idempotent. But principal ideal $\langle a \rangle$ is not prime.
Because ab $\langle 0 \rangle$ but neither $a \varepsilon \langle 0 \rangle$ nor $b \varepsilon \langle 0 \rangle$

Definition 2.2.2 :

Ideal I of a semigroup S is said to be reproduced by S if SI = IS = I

Remark :

Class of semigroups reproducing their ideals is properly wider than class of semigroups with idempotent ideals.

See proposition 11 and example given in Remark following proposition 8.

Proposition 11 :

Every ideal I of semigroup S with idempotent ideals is reproduced by S.

Proof : $I = I^2 \subseteq SI \subseteq I$

whence $SI = I$

Similarly $IS = I$

So $SI = I = IS$

hence proved.

Section 3 : NORMAL SEMIGROUPS

Definition 2.3.1 :

i) **Normal Semigroup :** A semigroup S is called normal if $xS = Sx \quad \forall x \in S$

ii) Let $\beta(S)$ = set of all non empty subsets of semigroup S and $B(S)$ = set of all bi-ideals of semigroup S.

iii) Subsemigroup A of a semigroup S is called normal if $xA = Ax \ \forall \ x \in S$

iv) Semigroup S is called competely regular, if for any a of S, $\exists \ x \in S$ s.t. $a = axa$ and $ax = xa$

v) As used in previous part let us use following notations

$L(x)$ = principal Left ideal generated by $x = \{x\} \cup Sx$

$R(x)$ = principal Right ideal generated by $x = \{x\} \cup xS$

$B(x)$=principal bi-ideal of S generated by x

$= \{x, x^2\} \cup xSx$

Proposition 12 :

Let A be any ideal of semigroup S. Then

1) $A.B(x) = A.L(x) = Ax \ \forall x \in S$

2) $B(x).A = R(x).A = xA \ \forall x \in S$

Proof :

Let x be any element of S

$A.L(x) = A(x \cup Sx) = Ax \cup (A)(Sx)$

$= Ax \cup (AS)x \subseteq Ax \subseteq A.L(x)$

So $Ax = A.L(x)$ .....(1)

and $A.B(x) = A(x \cup x^2 \cup xSx)$

$= Ax \cup Ax^2 \cup A(xSx)$

$= Ax \cup (Ax)x \cup (AxS)x$

$\subseteq Ax$ (As A is an ideal)

$\subseteq A.B(x)$

So $A.B(x) = Ax$ .....(2)

From (1) and (2) $Ax = A.L(x) = A.B(x) \ \forall x \in S$

Similarly $A.B(x) = R(x).A = xA \ \forall x \in S$

Proposition 13 :

For an ideal A of a semigroup S, following conditions are equivalent

1) A is normal

2) $XA = AX \ \forall X \in \beta(S)$

3) $XA = AX \ \forall X \in B(S)$

4) $B(x).A = A.B(x) \ \forall x \in S$

5) $B(x).A = A.L(x) \ \forall x \in S$

6) $B(x).A = Ax \quad \forall \; x \in S$

7) $R(x).A = A.B(x) \quad \forall \; x \in S$

8) $R(x).A = A.L(x) \quad \forall \; x \in S$

9) $R(x).A = Ax \quad \forall \; x \in S$

10) $xA = A.B(x) \quad \forall \; x \in S$

11) $xA = A.L(x) \quad \forall \; x \in S$

Proof :

Let A is normal, X be any nonempty subset of S and $xa(x \in X, a \in A)$ be any element of XA.

Thus $xa \in xA = Ax \subseteq AX$ so $XA \subseteq AX$

Similarly, converse inclusion holds

So $AX=XA \quad \forall \; X \in \beta(S)$

So (1) $\Rightarrow$ (2)

Now (2) $\Rightarrow$ (3) clearly

(3) $\Rightarrow$ (4) clearly

It follows from proposition 12 and A is normal that if A is any ideal of semigroup S. Then $A.B(x)=A.L(x)=Ax$ $=xA=B(x).A=R(x).A \; \forall \; x \in S$

i.e. (1) and (4) to (11) are equivalent.

Hence proved.

Proposition 14 :

Let A and B any normal ideals of semigroup S then products AB and BA are also normal ideals of S and AB=BA.

Proof : It follows from proposition 13 that AB=BA holds

From any element x of S, we have

$x(AB)=(xA)B = (Ax)B = A(xB)=A(Bx)=(AB)x$

Proposition 15 :

For an ideal A of a regular semigroup S, following conditions are equivalent.

1) A is normal

2) $eA = Ae$ $\forall$ idempotents e of S

3) $B(e).A=A.B(e)$ $\forall$ idempotents e of S

4) $B(e).A = A.L(e)$ $\forall$ idempotents e of S

5) $B(e).A=Ae$ $\forall$ idempotents e of S

6) $R(e).A = A.B(e)$ $\forall$ idempotents e of S

7) $R(e).A=A.L(e)$ $\forall$ idempotents e of S

8) $R(e).A=Ae$ $\forall$ idempotents e of S

9) $eA =A.B(e)$ $\forall$ idempotents e of S

10) $eA = A.L(e)$ $\forall$ idempotents e of S

Proof :

(1) = (2) Trivially

(2) to (10) are equivalent can be proved in similar way as in proof that (1) and (4) to (11) are equivalent in proposition 13.

Now assume that (2) holds

In order to prove that (1) holds

Let x be any element of S. As S is regular y is S s.t. $x = xyx$ and yx is idempotent.

$xA = (xyx)A = x((yx)A) = x(A(yx))$

$= (xAy)x \subseteq Ax$

Similarly it can be proved that converse inclusion holds

So $xA = Ax \ \forall \ x \in S$

So (2) $\Rightarrow$ (1)

Hence proved

Proposition 16 :

Let A be normal ideal of a semigroup S and $x \in S$ then xA is an ideal of S.

Proof : Let A be any normal ideal of semigroup S and $x \in S$. Then $(xA)S = x(AS) \subseteq xA$ and $S(xA) = S(Ax) = (SA)x \subseteq Ax$

So xA is an ideal of S.

S. Lajos has given following

Theorem 17 :

Product of bi-ideal and of a non empty subset of a semigroup S is also a bi-ideal of S.

Proof :

Let A be any non empty subset of S and B be any bi-ideal of S.

To prove that BA is bi-ideal of S.

Let $x_1 y_1$ and $x_2\ y_2$ be any two elements of BA s.t. $x_1, x_2 \in B$ and $y_1\ y_2 \in A$.

$x_1 y_1 x_2 y_2 \varepsilon$ BA

So BA is subsemigroup of S.

Consider (BA) S (BA)

$= B(AS)BA$

$\subseteq$ (BSB) A

$\subseteq$ BA   As B is bi-ideal BSB $\subseteq$ B

So BA is bi-ideal of S.

Proposition 18 :

Any minimal ideal of a semigroup S is zero element of B(S)

Let A be minimal ideal of S

Then clearly $A \varepsilon B(S)$. { As every ideal is bi-ideal}

Let X be any bi-ideal of S.

Then XA $\subseteq$ SA $\subseteq$ A

Then it follows from .Theorem 17 and minimality of A that

XA = A

Similarly we can prove AX = A

$\forall X \varepsilon B(S)$

$\Rightarrow$ A is zero element of B(S).

Proposition 19 :

Any minimal normal ideal of a semigroup is group.

Proof :

Let A be minimal normal ideal of a semigroup S

Let x be any element of S. Then we have

Ax = xA $\subseteq$ SA $\subseteq$ A

Then it follows from proposition 16 and minimality of A that $Ax = xA = A$

i.e. $Ax = xA = A \quad \forall\ x \in A$

So A is group.

Propositon 20 :

Following conditions about semigroup S are equivalent.

1) S is normal

2) $xS = SX \quad \forall\ X \in \beta(S)$

3) $XS = SX \quad \forall\ X \in B(S)$

4) $B(x).S = S.B(x) \quad \forall x \in S$

5) $B(x).S = S.\ L(x) \quad \forall x \in S$

6) $B(x).S = Sx \quad \forall\ x \in S$

7) $R(x).S = S.B(x) \quad \forall\ x \in S$

8) $R(x).S = S.L(x) \quad \forall\ x \in S$

9) $R(x).S = Sx \quad \forall\ x \in S$

10) $xS = S.B(x) \quad \forall\ x \in S$

11) $xS = S.L(x) \quad \forall\ x \in S$

12) B(S) is normal

13) $B(x).B(S) = B(S).B(x) \quad \forall x \in S$

14) $B(x).B(S) = B(S).L(x) \quad \forall x \in S$

15) $B(x).B(S) = B(S).x \quad \forall x \in S$

16) $R(x).B(S) = B(S).B(x) \quad \forall x \in S$

17) $R(x).B(S) = B(S).L(x) \quad \forall x \in S$

18) $R(x).B(S) = B(S) \quad \forall\ x \in S$

19) $xB(S) = B(S).B(x) \quad \forall\ x \in S$

20) $x\ B(S) = B(S).L(x) \quad \forall\ x \in S$

21) $xB(S) = B(S)\ x \quad \forall x \in S$

Proof :

Since semigroup S is itself an ideal of S it follows from proposition 13 that (1) To (11) are equivalent.

Assume (1) holds

Let A and X be any bi-ideals of S and a be any element of A. Then we have

$a X \subseteq aS = Sa \subseteq SA \subseteq B(S) A$

and so $A.B(S) \subseteq B(S).A$

Similarly $B(S).A \subseteq A.B(S)$ can be shown.

So we obtain that $A.B(S) = B(S).A$

So B(S) is normal.

So (1) $\Rightarrow$ (12)

Clearly (12) $\Rightarrow$ (13)

Assume (13) holds

In order to prove that S is normal
Let x be any element of S

Then for some $A \subseteq B(S)$, we have

$xS \subseteq B(x).S = A.B(x) \subseteq S.B(x) \subseteq Sx$

Similarly we can prove that converse inclusion holds

So S is normal and (13) $\Rightarrow$ 1

Remaining proof easily follows.

Hence proved.

Corollarly 21 :

Every one sided ideal of normal semigroup is a two sided ideal

Proof : Immediately follows from proposition 20

As S is normal

$= XS = SX \quad \forall X \in B(S)$

Proposition 23 :

For a semigroup S, the following conditions are equivaalent.

1) S is completely regular.

2) $a \in a^2 Sa^2 \; \forall a \in S$

3) S is left and right regular.

Proof :

(3) $\Rightarrow$ (2) Let (3) hold i.e.

S is left and right regular.

So if $a \in S \; \exists \; x$ and $y \in S$ s.t.

$a = xa^2$ and $a = a^2y$

To prove that $a \in a^2 Sa^2$

$$\begin{aligned} a = a^2y &= a\,a\,y \\ &= a^2y\,a\,y \\ &= a^2y\,x\,a^2y \\ &= a^2y\,x(a^2y) \\ &= a^2y\,x\,a \\ &= a^2(y\,x^2)\,a^2 \\ &\in a^2\,S\,a^2 \end{aligned}$$

as $x, y \in S$

$y\,x^2 \in S$

So (2) holds.

(2) $\Rightarrow$ (1) Let $a \in a^2 S a^2$

To prove that S is completely regular.

$a = a^2 x\ a^2$ for some $x \in S$

$= a(axa)\ a$ i.e. a is regular.

Consider $a(axa)$ and $(axa)\ a$

Claim

$a(axa) = (axa)\ a$

For $a(axa) = (aa)\ xa = a^2 xa = a^2 xa^2 xa^2$

and $(axa)a = a\ x(aa) = a\ xa^2 = a^2 xa^2\ xa^2$

So claim proved

$\Rightarrow$ S is completely regular.

i.e. (2) $\Rightarrow$ (1)

(1) $\Rightarrow$ (3) Let S is complotly regular.

i.e. $a \in S$, $\exists\ x \in S$ s.t. $a = axa$ and $ax = xa$

To prove that S is left and Right regular

Now

$a = axa = (ax)a = (xa)\ a = xa^2$

$a = axa = a(xa) = a(ax) = a^2 x$

i.e. $a = xa^2$ and $a = a^2 x$

So S is left regular and also right regular.

So (1) $\Rightarrow$ (3)

and hence proved.

Proposition 24 (A) :

A semigroup S is completely regular iff every bi-ideal of S is semiprime.

Proof :

First we assume that S is completely regular.

Let A be any bi-ideal of S.

Let $a^2 \in A$ and $a \in S$

Then it follows from proposition 23 that

$a \in a^2 S a^2 \in ASA \subseteq A$

$\Rightarrow$ A is semiprime.

Conversly, Let every bi-ideal of S is semiprime.

Then since any one sided ideal of a semigroup S is bi-ideal, every left and right ideal of S is semiprime

$\Rightarrow$ S is left and right regular.

$\Rightarrow$ S is completely regular.

Hence proved.

Section 4 :

In this section definitions of B-pure Bi-ideal and B^* - pure semigroup are given. Some properties of B^* -pure semigroup are discussed.

Definition 2.4.1 :

Semigroup S is called normal if $aS = Sa \ \forall \ a \in S$

Definition 2.4.2 :

Bi-ideal A of a semigroup S is called B-pure if $A \cap xS = xA$ and $A \cap Sx = Ax \ \forall \ x \in S$

Definition 2.4.3 :

Semigroup S is called B^* - pure if every bi-ideal of it is B-pure.

Proposition 24 (B) :

Normal regular semigroup is B^* - pure, semigroup

Proof :

First we prove that every bi-ideal of normal regular semigroup is ideal.

Let S be normal regular semigroup and let B be any bi-ideal of S i.e. $BSB \subseteq B$

To prove that B is ideal

i.e. $BS \subseteq B$ and $SB \subseteq B$

Let $bs \in BS$ i.e. $b \in B$ and $s \in S$

As S is regular $\exists$ $q \in S$ s.t.

$b = bqb$

So $bs = bq\ b\ s$ $\{$ as S is normal $\}$

$= b\ q\ S'b$

$\in B\ S\ B$

$\subseteq B$

So $BS \subseteq B$

Similarly $SB \subseteq B$

i.e. B is ideal.

So in normal regular semigroup every bi-ideal is ideal.

Now let A be any bi-ideal of S

To prove that A is B-pure

i.e. $A \cap xS = xA$

Let $a \in A \cap xS$

i.e. $a \in A$ and, $a \in xS$

i.e. $\exists\, p$ and $s, s' \in S$ s.t.

$a = apa$ (regularity) and $a = xs = s'x$

So $a = apa$

$= xspapa$

$\in x\,SA\,SA$

$= x(SA)\,SA$ (every bi-ideal is ideal in Normal regular semigroup)

$\subseteq x\,ASA$

$\subseteq xA$ A is bi-ideal

So $A \cap xS \subseteq xA$ .....(1)

For converse inclusion consider following.

$xA \subseteq xS$ as A is bi-ideal of S

also for normal regular semigroup every bi-ideal is ideal

$\Rightarrow xA \subseteq A$

So $xA \subseteq A \cap xS$ .....(2)

From (1) and (2) $A \cap xS = xA$

Similarly $A \cap Sx = Ax$ can be proved.

So A is B-pure bi-ideal.

But as A was any arbitrary bi-ideal

We have S is B^* - pure.

Hence proved.

Let E(S) denote set of all idempotent elements of a semigroup S.

Proposition 25 :

Let S be a B^* - pure semigroup. Then S has following properties

1) $aS = a^2S$ and $Sa = Sa^2 \ \forall \ a \in S$

2) For every $a \in S$, a^2 is completely regular .

3) S is normal.

4) E(S) is contained in the centre of S.

Proof :

(1) Let a be any element of S

Now aS is bi-ideal of S

$\{$ As $aS.S.aS \subseteq aS \}$

As S is B^* - pure, Bi-ideal aS of S is B-pure.

Then $aS = aS \cap aS = a(aS) = a^2S$

Similarly $Sa = Sa^2$

(2) Let a be any element of S

Then by (1) we have

$a \in aS = a^2S$ and $a \in Sa = Sa^2$

i.e. $a^2 \in aS$ and $a^2 \in Sa$

i.e. $a^2 \in aS \cap Sa = a^2S \cap Sa^2 = (a^2)^2S \cap S(a^2)^2$

Then it follows that

a^2 is both left and right regular.

So S is completely regular.

(3) Let a be any element of S

Then Sa is bi-ideal of S.

As S is B^* - pure semigroup,

12359

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Sa is B-pure Bi-ideal of S.

By (1) we have

$aS = a^2S \subseteq (Sa)\ S = Sa \cap SS \subseteq Sa$

So $aS \subseteq Sa$

Similarly $Sa \subseteq aS \ \forall\, a \in S$

i.e. $Sa = aS \ \forall\, a \in S$

So S is normal.

(4) Let $a \subseteq S$ be any element and $e \in E(S)$

Then by (3) above

$Sa = aS$ and $Se = eS$

i.e. $x, y, p, q \in S$ s.t.

$ea = ax \quad ae = ye \quad ae = ep \quad ea = qe$

Consider

$ea = qe = qee = eae = eep = ep = ae$

So $ea = ae$

i.e. E(S) is contained in the centre of S.

Definition 2.4.4 :

A semigroup S is called <u>Archimedian</u> if for each element a and b of S, $\exists$ +ve integer n s.t. $a^n \in SbS$

Proposition 26 :

For a B^* - pure semigroup S, the following conditions are equivalent.

1) S is Archimediann

2) $SaS = SbS \ \forall\, a,b \in S$

3) $aS = bS \ \forall \ a,b \in S$

4) $aSa = bSb \ \forall a,b \in S$

5) S has exactly one idempotent element.

6) Every bi-ideal of S is archimedian.

Proof :

(1) $\Rightarrow$ (2)

Let a and b be any elements of S. Then since S is archimedian, $\exists$ positive integer n ,s.t. $a^n \in SbS$

By proposition (25(1))

$SaS = Sa^n S \subseteq S(SbS)S = (SS)b(SS) \subseteq SbS$

Similarly $SbS \subseteq SaS$

So $SaS = SbS$

(2) $\Rightarrow$ (3)

Let $SaS = SbS$

To prove that $aS = bS$

$aS = a^2S = aaS \subseteq SaS = SbS = bSS \subseteq bS$

So $aS \subseteq bS$

Similarly $bS \subseteq aS$

So $aS = bS$

(3) = (4) Let $aS = bS \ \forall \ a,b \in S$

In particular $a^2S = b^2S$

Now $aSa = aaS \subseteq a^2S = b^2S = bbS = bSb$

So $aSa \subseteq bSb$

Similarly $bSb \subseteq aSa$

so $aSa = bSb$

(4) $\Rightarrow$ (5) Let e and f be any two idempotents of S

Then as $eSe = fSf$ $\forall$ x and y in S

s.t. $e = f\,xf$ and $f = eye$

Now $e = fxf = ffxf = fe = eyee = eye = f$

since E(S) is nonempty by proposition 25(2), S has exactly one idempotent.

(5) $\Rightarrow$ (6)

Let A be any bi-ideal

a and b be any elements of A.

Then as a^2, b^2 are completely regular by proposition 25(2) , elements x and y s.t.

$a^2 = a^2x\,a^2$ $\quad b^2 = b^2y\,b^2$

Since a^2x and b^2y are idempotents we have

$a^2x = b^2y$

Then

$$\begin{aligned} a^3 = a.a^2 &= a\ (a^2x\ a^2) \\ &= a(b^2y)\ a^2 \\ &= ab\ (bya^2) \\ &\in Ab\ (ASA) \\ &\subseteq AbA \end{aligned}$$

= A is archimedian.

(6) $\Rightarrow$ (1)

Trivially true.

Definition 2.4.5 :

Semigroup S is called weakly commutative if $\forall$ a,b $\in$ S, $\exists$ +ve integer n s.t.

$(ab)^n \in bSa$.

Proposition 27 :

Let S be a semigroup s.t. $aS = a^2S$ and $Sa = Sa^2$ $a \in S$. Then following conditions are equivalent.

1) E(S) is contained in centre of S

2) S is normal

3) S is weakly commutative.

Proof :

(1) $\Rightarrow$ (2)

Let $a \in S$. The $a^2 \in S$ and a^2 is completely regular

So a^2 is regular

$\exists$ $x \in S$ s.t. $a^2 = a^2 x a^2$

Let a^2y be any element of $aS(=a^2S)$

Since xa^2 is idempotent $x a^2 \in E(S)$

$a^2y = (a^2x a^2) y$

$= a^2((xa^2)y)$

$= a^2(y(xa^2))$ { As completely regular

$= a^2 (y(xa^2) (xa^2))$ = left regular and xa^2 is idempotent }

$= (a^2y a^2) a^2$ { E(S) is contained in centre of S}

$\in Sa^2$

$= Sa$

So $aS \subseteq Sa$

Similarly $Sa \subseteq aS$

So $aS = Sa$

$\Rightarrow$ S is normal

(2) $\Rightarrow$ (3) Let a and b be any elements of S

Then as S is normal

$ab \in Sb$ and $ab \in aS$

$(ab) \in (Sb)(aS)$

$= (bS)(Sa)$

$= b(SS)a$

$\subseteq bSa$

Thus S is weakly commutative.

(3) $\Rightarrow$ (1)

Let a be any element of S and e be any idempotent of S

Since S is weakly commutative, we have

$(ae)^n \in eSa$ for some +ve integer n.

Then

$ae = aee \in aeS = (ae)^n S \subseteq (eSa)S \subseteq eS$

So $ae \in eS$

So $\exists$ x in S s.t. $ae = ex$

Similarly, $\exists$ y in S s.t.

$ea = ye$

So $ae = ex$

$= eex$

$= eae$

$= yee$

$= ye$

$= ea$

So E(S) is contained in centre of S.

Proposition 28 :

For a semigroup S, following conditions are equivalent.

1) S is B^* - pure.

2) S is normal and $Sa = Sa^2 \ \forall \ a \in S$

Proof :

(1) $\Rightarrow$ (2) Trivial

(2) $\Rightarrow$ (1)

Let (2) hold.

Let A be any bi-ideal of S and x be any element of S.

Let $a = x^2 s$ be any element of $A \cap xS (= A \cap x^2 S)$

Then it can be seen that x^2 is regular, so

$\exists\ y \in S$ s.t. $x^2 = x^2 y\ x^2$

Since

$ya \in Sa = Sa^2$, $\exists$ element $z \in S$ s.t.

$ya = za^2$

Then since S is normal we have

$a = x^2 s = (x^2 y\ x^2)\ s = (x^2 y)(x^2 s) = (x^2 y)\ a$

$= x^2 (ya) = x^2 (za^2) = x((xz)a)\ a$

$\in x(Sa)a = x\ (aS)\ a \subseteq x\ (ASA) \subseteq xA$

So $A \cap xS \subseteq xA$ .....(1)

Let xa $(a \in A)$ be any element of xA

Then $xa \in Sa = Sa^2 = aSa \subseteq ASA \subseteq A$

So $xA \subseteq A$

Since $xA \subseteq xS$

We have $xA \subseteq A \cap xS$ .....(2)

So From (1) and (2) $A \cap xS = xA$

Similarly it can be proved that $A \cap Sx = Ax$

So Bi-ideal A is B-pure.

As A was any arbitrary bi-ideal,

S is B^* - pure semigroup.