CHAPTER - III

REGIONS OF UNIVALENCE FOR CONVEXITY AND CLOSE-TO-CONVEXITY.

ABSTRACT

In this third chapter of dissertation, we have attempted to obtain some results leading to the regions of univalence, particularly for convexity and close-to-convexity. In particular, we have investigated these regions for the univalent functions with second missing coefficient. We have also used the function like $P(z) = a_0 \prod_{k=1}^{n} (z-z_k)$, introduced in the first chapter, to determine these regions, under different conditions. Particular cases and sharp results, wherever possible are sited.

REGIONS OF UNIVALENCE FOR CONVEXITY AND CLOSE-TO-CONVEXITY.

§ INTRODUCTION :-

One of the classical and important problem of the Univalent function theory is to determine the regions of univalence of given holomorphic functions, subjected to the varieties of restrictions. This aspect of univalent function theory has motivated us to determine the radii of convexity and close-to-convexity of some holomorphic functions.

Alexander [1] determined the radius of starlikeness of $z\,p(z)$, where $p(z) \in P(n,1)$, Dieudonne [5] carried out such considerations for $z\,\left[p(z)\right]^{\alpha/n}$, where $p(z) \in P(n,1)$, α is a non zero real number, Basgöze [3] has also carried out several investigations under this heading. Barr [2] too has done a lot of research work in this field. Causey [4], Markes and Wright [9], Nunokawa [10], Pflatzgraff [11] Kulkarni - Thakare [8], Royster [12], Ziegler [13], have obtained very nice results by carrying out several researches on the estimation of regions of univalence.

§ SECTION - I

The first section of this chapter deals with the determination of radii of convexity of the univalent functions, which are holomorphic and having second missing coefficient. We in particular consider such holomorphic functions having Taylor series expansion of the form,

$$g(z) = z + az^3 + \sum_{4}^{\infty} a_n z^n,$$

In our course of investigations we need the following lemmas.

Lemma :1.1:- If $g(z) = z+az^3 + \sum_{4}^{\infty} a_n z^n \in S^*(\alpha)$, then

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqslant \frac{(1-\alpha)+a|z| - a(1-2\alpha)|z|^2 - (1-2\alpha)(1-\alpha)|z|^3}{(1-\alpha)+a|z|+a|z|^2 + (1-\alpha)|z|^3}$$

$0 \leqslant a \leqslant 1, \quad |z| = r < 1,$

The proof of this can be found in [6] .

Lemma :1.2:- If $z = re^{i\theta}$, $z_1 = Re^{i\phi}$, and α is real then

$$\frac{-r(R+r\cos\alpha)}{R^2-r^2} \leqslant \mathrm{Re}\left\{\frac{e^{i\alpha} z}{z-z_1}\right\} \leqslant \frac{r(R-r\cos\alpha)}{R^2-r^2}$$

Lemma :1.3: [2] For $< o < r < 1 \leq R$

$$\frac{R + r\cos\alpha}{R^2 - r^2} \leq \frac{1 + r\cos\alpha}{1 - r^2}$$

and

$$\frac{R - r\cos\alpha}{R^2 - r^2} \leq \frac{1 - r\cos\alpha}{1 - r^2}$$

§ 2) SOME THEOREMS :-

Theorem :2.1:- Let P be a polynomial of degree n belonging to P(n,1), Let f be a function such that, $f(0) = 0$ and

(2.2) $f'(z) = \left[p(z) \right]^{\beta/n} \cdot F(z)$, Where $\beta = \rho e^{i\xi}$ $(\rho > 0)$, ξ real and $F(z) = z^c \left[g(z)\right]^{\delta}$, where c is a non negative integer $0 \leq \delta$, and $g(z) = z + az^3 + \sum_{4}^{\infty} a_n z^n$ holomorphic and starlike of order α, with 2nd missing coefficient. Then for f and λ $(0 \leq \lambda < 1)$, we have the radius of convexity K_λ given by the polynomial, H (r), $(0 \leq \alpha < 1)$

$$H(r) = (1+c) + \frac{\delta\left[(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3\right]}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} -$$

$$- \frac{\rho r(1+r\cos\xi)}{(1-r^2)}$$

Proof :-

Logarithmic differentiation of (2.2) and accustomed computation yields,

$$\left[1+\frac{zf''}{f'}\right] = (1+c) + \delta\frac{zg'}{g} + \frac{\beta}{n} \qquad \frac{zp'}{p}$$

Making use of lemmas 1.1, 1.2, and 1.3 we are led to

$$\mathrm{Re}\left[1+\frac{zf''}{f'}\right]$$

$$\geqslant (1+c) + \frac{\delta\left[(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3\right]}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} -$$

$$- \frac{\rho r(1+r\cos\xi)}{(1-r^2)}$$

Thus $\mathrm{Re}\left[1+\frac{zf''}{f'}\right] > \lambda$, for $|z| < r$, if r is given by the above polynomial, $H(r)$.

For $\alpha = 0$, we can state the result for $S^*(0) = S^*$ functions, which is surprisingly a new result.

<u>Corollary</u> :- If $g(z) = z + az^3 + \sum_{4}^{\infty} a_n z^n$ belongs to S^*, the remaining conditions as usual in theorem 2.1 . We get the radius of convexity by the following polynomial, H(r).

$$H(r) = (1+c) + \delta \frac{\left[1 + ar - ar^2 - r^3\right]}{1 + ar + ar^2 - r^3} - \frac{\rho r\,(1+r \cos\xi_1)}{(1-r^2)}$$

We now proceed to arbitrate the region of λ - convexity of holomorphic functions whose derivatives involve polynomials of degree m + n so that m of the zeros lie in the annulus and also involve starlike function of order δ having second missing coefficient.

<u>Theorem : 2.2</u> :-

Let f be a function such that f(0) = 0 and (2.3) $f'(z) = z^{-1} \left[p(z)\right]^{\alpha} \left[F(z)\right]^{\beta}$

Where p is a polynomial of degree m+n, F is starlike of order δ $(0 \leq \delta < 1)$, $\alpha > 0$ and β is any non negative real number. Further let m of the zeros of p lie in the annulus $0 < d \leqslant |z| < D$ and the remaining n lie in $|z| \geqslant D$. Then for f, $K_\lambda \geqslant r$, where r is the root of the equation given by the polynomial $H(r) = 0$

where $$H(r) = \beta \frac{(1-\delta)+ar-a(1-2\delta)r^2-(1-2\delta)(1-\delta)r^3}{(1-\delta)+ar+ar^2+(1-\delta)r^3} - \frac{\alpha mr}{d-r} - \frac{\alpha n r}{D-r}$$

Proof :- By carrying out the customary calculations we obtain from (2.3)

$$\left\{1+\frac{zf''}{f'}\right\} = \alpha\frac{zp'}{p} + \beta z\frac{F'}{F}$$

Now $F(z)$ is starlike of order δ, with second missing coefficients so inview of lemma 1.1 we can use

$$\mathrm{Re}\left\{\frac{zF'}{F}\right\} \geqslant \frac{(1-\delta)+ar-a(1-2\delta)r^2-(1-2\delta)(1-\delta)r^3}{(1-\delta)+ar+ar^2+(1-\delta)r^3}$$

with $|z| = r$.

In view of this and taking into consideration the location of m and n zeros of $p(z)$, we obtain

$$\mathrm{Re}\left\{1+\frac{zf''}{f'}\right\} \geqslant \beta\,\frac{(1-\delta)+ar-a(1-2\delta)r^2-(1-2\delta)(1-\delta)r^3}{(1-\delta)+ar+ar^2+(1-\delta)r^3} - \frac{\alpha mr}{d-r} - \frac{\alpha nr}{D-r}$$

Thus f is convex of order λ with its radius of convexity $K_\lambda \geqslant r$, where r is the root of the equation $H(r) = 0$, $H(r)$ given as in statement of theorem.

A simple calculation shows that the root lies between 0 and 1.

For $\delta = 0$ we get the radius of convexity for starlike functions, which is again a surprising result and can be stated as follows.

<u>Corollary :-</u> Keeping $\delta = 0$ in the above Theorem 2.2 and the other condition remaining the same, we get the radius of convexity given by the polynomial,

$$\begin{aligned} & r^5(-2\beta) + r^4\left[(\beta(4+a) - \lambda a + \alpha a(m+n)\right] \\ & + r^3\left[(dD + 2\lambda dD + \alpha a(-Dm-Dn+m+n) + \lambda a(-D+d)\right] \\ & + r^2\left[(-2aD\beta + aD(3-\lambda) + d\lambda a(-D+1) - \alpha nD a)\right] \\ & + r\left[(-2D\beta - 2d\beta + Dda(\beta-\lambda)\right] \end{aligned}$$

So far we restricted ourselves to polynomials, now we shall prefer rational functions which are quotients of polynomials that are the members of P(m,1) and P(n,1) with $m \geqslant 1$, $n \geqslant 0$, more categorically, we have,

<u>Theorem :2.3:-</u> Let f be a function such that f(0) =0 and

$$(2.4)\quad f'(z) = \left(\frac{M(z)}{N(z)}\right)^{\beta} \left[F(z)\right]^{\alpha}$$

Where F(z) is starlike of order δ with second missing coefficient, M and N are polynomials of degree $m \geqslant 1$, $n > 0$ respectively and all the zeros of both M and N lie outside or on the unit circle, $\beta = \varrho e^{i\xi}$ ($\varrho > 0$, ξ real, $0 \leqslant \alpha$. Then for f and λ $(0 \leqslant \lambda < 1)$ we have $K_\lambda \geqslant r$, where r is the root of the equation, $H(r) = 0$, where $0 \leq \delta < 1$

$$\begin{aligned} H(r) = \; & r^6(2\alpha)+r^5\left[\delta-1+\alpha(1-3\alpha)+\varrho m\cos\xi(\delta-1)+\varrho n\cos\xi(1-\delta)-\lambda\right]+ \\ & + r^4\left[-a+\alpha(-2\delta-1)+\varrho m(-1+\delta-a\cos\xi)+\varrho n(-1+\delta+a\cos\xi)\right]+ \\ & + r^3\left[-a+1-\delta+\alpha(3\alpha-a-1)+\varrho ma(-1-\cos\xi)+\varrho na(-1+\cos\xi)- \right. \\ & \left. - \lambda a+\lambda\right] \\ & + r^2\left[-1+\delta+a+\alpha(3\delta-2)+\varrho m(-a-\cos\xi+\delta\cos\xi)+ \right. \\ & \left. +\varrho n(-a+\cos\xi+\delta\cos\xi)-\lambda+\lambda\delta\right] \\ & + r\left[a+\alpha a+\varrho m(-1+\delta)+\varrho n(-1+\delta)+a\lambda\right]+\left[1+\alpha+\lambda-\lambda\delta\right] \end{aligned}$$

Proof :- The routine calculation yields,

$$\frac{zf''}{f'} = \frac{\alpha\, zF'}{F} + 1 + \beta\left(\frac{zM'}{M} - \frac{zN'}{N}\right)$$

Let $z_1, z_2, \ldots\ldots z_m$ be the zeros of $M(z)$ and $z_{m+1}, \ldots\ldots z_{m+n}$ be the zeros of $N(z)$. We in view of lemmas 1.1, 1.2 and 1.3 we get

$$\left[\mathrm{Re}\ 1+\frac{zf''}{f'}\right] \geqslant 1+\frac{\alpha\left[(1-\delta)+ar-a(1-2\delta)r^2-(1-2\delta)(1-\delta)r^3\right]}{(1-\delta)+ar+ar^2+(1-\delta)r^3} -$$

$$- \frac{\rho m r\,(1+r\cos\xi)}{(1-r^2)} - \frac{\rho n r(1-r\cos\xi)}{(1-r^2)}$$

Evidently, $\mathrm{Re}\left\{1+\frac{zf''}{f'}\right\} > \lambda$ and the radius of convexity is given by the polynomial $H(r) = 0$.

$$\begin{aligned} H(r) = {} & r^6(2\alpha)+r^5\left[\delta-1+\alpha(1-3\alpha)+\rho m\cos\xi\,(\delta-1)+\rho n\cos\xi(1-\delta)-\lambda\right] + \\ & + r^4\left[-a+\alpha(-2\delta-1)+\rho m(-1+\delta-a\cos\xi)+\rho n(-1+\delta+a\cos\xi)\right] + \\ & + r^3\left[-a+1-\delta+\alpha(3\alpha-a-1)+\rho ma(-1-\cos\xi)+\rho na(-1+\cos\xi)-\lambda a+\lambda\right] + \\ & + r^2\left[-1+\delta+a+\alpha(3\delta-2)+\rho m(-a-\cos\xi+\delta\cos\xi)+\rho n(-a+\cos\xi+\delta\cos\xi)-\lambda+\lambda\delta\right] + \\ & + r\left[a+\alpha a+\rho m(-1+\delta)+\rho n(-1+\delta)+a\lambda\right]+\left[1+\alpha+\lambda-\lambda\delta\right] \end{aligned}$$

Lastly we state a simple result, by putting $\delta = \alpha$ which is surprisingly, a new result for starlike functions with second missing coefficient.

Corollary :- We put $\delta = \alpha$, in (2.4) and the remaining conditions same as in theorem (2.3), we get the radius of convexity, which is given by the equation

$$r^6(2\alpha)+r^5(-3\alpha^2+\alpha-\lambda-1)+r^4(-\rho m a \cos\xi+\rho n a \cos\xi -3\alpha-a) +$$
$$+ r^3\left[3\alpha^2-\rho ma \cos\xi +\rho mn \cos\xi +a(-m-n-\alpha-2-\lambda)\right]+$$
$$+ r^2(2\rho n \cos\xi-\rho ma -\rho na+\alpha+a) + r(a\alpha+a\lambda+2a+1) +(1+\alpha)$$

§ SECTION - II

1. In the second section of this current chapter, we are going to undertake the study of searching the regions of univalence for close-to-convex functions. The several researchers have carried out investigations in obtaining the radii of discs of close-to-convex functions in the detailed discussion of which we are not going to enter.

For our analysis, we shall select $p \in P(n,1)$ to have the form, $p(z) = a_0 \prod_{1}^{n} (z-z_k)$ with $|z_k| \geqslant 1$ and a_0 is to be appropriately selected, so that the resulting functions are normalised. We obtain the radii of close-to-convexity to such integral forms, which consists of functions having the Taylor series expansions of the type

$$f(z) = a + az^3 + \sum_{4}^{\infty} a_n \ z^n$$

i.e. with second missing coefficient form.

In the course of investigation, we need the following lemmas.

Lemma :2.1.1 :- [2]

If $z = re^{i\theta}$, $z_1 = Re^{i\phi}$, where $0 < r < R$, then

$$\frac{-r}{R-r} \leqslant \operatorname{Re}\left\{\frac{z}{z-z_1}\right\} \leqslant \frac{r}{R+r}$$

Equality holds in the first inequality if and only if $z = \frac{r}{R} z_1$ and in the second inequality if and only if $z = \frac{-r}{R} z_1$

This can be found in Basgöze [3].

The following is the characterization of Kaplan [7] of close-to-convex functions.

Lemma :2.1.2:- Let f be a holomorphic function in $|z| < r$, $(r < 1)$ and $f'(z) \neq 0$, on $|z| = r$, If $f \in C$ then

$$-\pi < \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zf''}{f'}\right) d\theta < \pi + 2\pi$$

Where $0 \leq \theta_1 < \theta_2 < 2\pi$. Conversely each of the inequalities separately implies that $f \in C$.

MAIN THEOREMS

Theorem :2.3.1.:-

Let $f(z) = a + az^3 + \sum_{4}^{\infty} a_n z^n$, be $S^*(\alpha)$, $0 \leq \alpha < 1$ $p \in P(n,1)$, $\phi \in K$, K denoting the class of convex functions. $0 \leq \delta$, $0 \leq \varrho \leq 1$, then the function F defined by the expression

$$F(z) = \int_0^z \left(\frac{f(t)}{\phi(t)}\right)^{\delta} \left(p(t)\right)^{\varrho/n} dt \text{ is close-to-convex}$$

in the disc, the radius of which is given by $H(r) = 0$, where $H(r)$ is given by

$$\begin{aligned} H(r) = {} & r^4 (\alpha - 1 + 3\delta + 4\alpha^2\delta - 7\delta\alpha - 2\varrho + 2\varrho\alpha) + \\ & + r^3 (-a + 1 - \alpha - 3\delta + 7\delta\alpha - 4\alpha^2\delta + 3\delta a - 4\alpha\delta a - 2\varrho a) + \\ & + r^2 (4\alpha\delta a - 4\delta a - 2\varrho a) + r(\alpha + a - 1 + \delta a + \delta\alpha - \delta + 2\varrho\alpha - 2\varrho) + \\ & + (1 - \alpha + \delta - \delta\alpha) = 0. \end{aligned}$$

Proof :- By routine computations, using logarithmic differentiation, we are led to

$$\left[1+\frac{zF''}{F'}\right]=\left[\delta\left\{\frac{zf'}{f}-\frac{z\phi'}{\phi}\right\}+\frac{\rho}{n}\frac{zp'}{p}+1\right]$$

Therefore because of Lemmas 2.1.1., 2.1.2, we have

$$\int_{\theta_1}^{\theta_2}\left[\mathrm{Re}\quad 1+\frac{zF''}{F'}\right]d\theta=\int_{\theta_1}^{\theta_2}d\theta+\delta\int_{\theta_1}^{\theta_2}\mathrm{Re}\frac{zf'}{f}\,d\theta-\delta\int_{\theta_1}^{\theta_2}\mathrm{Re}\frac{z\phi'}{\phi}\,d\theta$$

$$+\frac{\rho}{n}\int_{\theta_1}^{\theta_2}\mathrm{Re}\sum_{1}^{n}\left[\frac{z}{z-z_k}\right]d\theta$$

$$\geqslant(\theta_2-\theta_1)\left[1+\frac{\delta\left[(1-\alpha)+ar-a\;(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3\right]}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3}-\frac{\rho r}{(1-r)}-\frac{\delta}{2}\right]>-\pi$$

Thus upon using Kaplan's result, we conclude that, F(z) is close-to-convex in the disc, the radius of which is given by the polynomial, as stated in Theorem 2.3.1.

We consider the following functions for sharpness.

$$f_{a,\alpha}(z) = z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2-(1-\alpha-a)z+(1-\alpha)\right]^{1-\alpha}}\right]^{\frac{2(1-\alpha)}{3(1-\alpha)-a}}$$

$0 \leq a \leq 1$, whenever $\frac{a}{1-\alpha} \leq 1$, $p(z) = (1-t)^n$, $\phi(z) = \frac{z}{(1+z)}$

We shall see that theorem 2.3.1, subsumes as special cases several results which seem to be new.

For $\delta = 0$, and $\varrho = 1$, we shall get the theorem 2.5 as derived by Barr [pp, 19].

For $\delta = 0$ and $\varrho = n$, we shall get theorem 2.8 of Barr [pp, 22] from above theorem.

For $\alpha = 0$ we get a result for starlike functions with second missing coefficients, which seems to be new.

Corollary :- Suppose that $f \in S^*$, $p \in P(n,1)$, $\phi \in K$, K denoting the class of convex functions $0 \leq \delta \leq 1$, $0 \leq \varrho \leq 1$, then the function g defined by the expression

$$g(z) = \int_0^z \left(\frac{f(t)}{\phi(t)}\right)^{\delta} \left(p(t)\right)^{\varrho/n} dt$$ is close-to-convex in the disc, the radius of which is given by the equation $H(r) = 0$

$$r^4(-1+3\delta-2\rho)+r^3(-a+1-3\delta+3\delta a-2\rho a)$$

$$+r^2(-4\delta a-2\rho a)+r(a-1+\delta a-\delta-2\rho)+(1+\delta)=0$$

This result is not found in literature.

Theorem :2.3.2 :- Let f be holomorphic function belonging to the class $S^*(\alpha)$, starlike of order α, with second missing coefficient, $0\leq\alpha<1$, c is an integer greater than, ≥ 1 g is convex, $p \in P(n,1)$, δ, γ, λ are non negative real numbers then

$$F(z)=\int_0^z t^{c-1}\left(\frac{f(t)}{t}\right)^{\delta}\left(\frac{g(t)}{t}\right)^{\gamma}\left(p(t)\right)^{\lambda/n}dt$$

is close-to-convex, in a disc, the radius of which is given by the polynomial,

$$r^4\left[(1-\alpha)(2c-\gamma-4\alpha\delta-2\lambda+1)\right]+$$

$$+r^3\left[2(1-\alpha-a)(c-\delta-\gamma/2)-2\delta(1-2\alpha)(1-\alpha)-2a\lambda+(1-\alpha)-a\right]+$$

$$+r^2\left[4\alpha a\delta-4a\delta-2a\lambda\right]+r\left[2ac-a\gamma-2c+\gamma+2\alpha c-\alpha\gamma-2\lambda(1-\alpha)+a-(1-\alpha)\right]+$$

$$+\left[(1-\alpha)2\left[c-\delta-\gamma/2\right]+(1-\alpha)(2\delta+1)\right]$$

Proof :- By conventional calculations we can write down,

$$1+\frac{zF''}{F'}=(c-\delta-\gamma)+\delta\left(\frac{zf'}{f}\right)+\gamma\left(\frac{zg'}{g}\right)+\lambda\left(\frac{zp'}{p}\right)$$

and therefore in view of lemmas 2.1.1, 2.1.2, we have

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zF''}{F'}\right) d\theta$$

$$\geq (\theta_2 - \theta_1)\left[(c-\delta-\gamma/2)+\frac{\delta\left[(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3\right]}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} - \frac{\lambda r}{1-r}\right] > -\pi$$

On applying Kaplan's result, we conclude that F(z) is close-to-convex in $|z| < r$, where the radius of close-to-convexity is given by the polynomial as given in the statement of the theorem.

We consider the following functions for sharpness.

$$f_{a,\alpha}(z) = z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2-(1-\alpha-a)z+(1-\alpha)\right]^{1-\alpha}}\right]^{\frac{2(1-\alpha)}{3(1-\alpha)-a}}$$

Whenever $\frac{a}{(1-\alpha)} \leq 1$, $g(z) = \frac{z}{(1+z)}$, $p(z) = (1-t)^n$

It may be seen that, for $\gamma = 0$, $c = 1$, and $\lambda = \varrho$, we shall get the following new result.

<u>Corollary : 1 :-</u> Suppose $f \in S^*(\alpha)$, starlike of order α, with second missing coefficient, $p \in P(n,t)$, then F(z) given by

$$F(z)=\int_0^z \left(\frac{f(t)}{t}\right)^{\delta} \left(p(t)\right)^{\varrho/n} dt$$

is close to convex in $|z|< r$, where r is given by the polynomial

$$H(r) = r^4 \left[(1-\alpha)(1-4\alpha\delta-2\varrho)\right] + r^3\left[2-2\alpha-2a-4\delta+8\alpha\delta+2a\delta-4\alpha\delta^2\right]+$$

$$+ r^2\left[4\alpha a\delta -4a\delta -2a\varrho\right]+r\left[3a-3+3\alpha -2\varrho(1-\alpha)\right]+\left[(1-\alpha)\quad 3\right]$$

This result is not found in literature.

In the next development, we shall obtain a particular case of a result of Barr [2] but with a function having second missing coefficient for the following substitutions,

$c = \lambda = \delta = 1$ and $\alpha = \gamma = 0$. So we have

<u>Corollary :2 :-</u> Let

$$F(z) = \int_0^z \left(\frac{f(t)}{t}\right)\left(p(t)\right)^{1/n} dt$$

$f \in S^*$, starlike having second missing coefficient, $p \in P(n, 1)$ then F is close-to-convex for $|z|<r$, where r is given by the polynomial

$$H(r) = r^4 - r^3 (1+3a) - 6 ar^2 + r (3a - 5) + 3$$

This result is also a surprisingly new one.

Theorem :2.3.3:- Suppose that $F(z)$ is given by

$$F(z) = \int_0^z \left(p(t)\right)^{\beta/n} \left(\frac{g(t)}{t}\right)^{\delta} \left(f(t)\right)^{\xi} dt$$

Where $p \in P(n,1)$, $g \in K$, $0 \leqslant \alpha < 1$, $\delta \geqslant 0$, $\beta \geqslant 0$, and $f \in S^*(\alpha)$, with second missing coefficient. Then F is close-to-convex in $|z| < r$, where the radius of close-to-convexity is given by the polynomial,

$$H(r) = A r^5 + B r^4 + C r^3 + D r^2 + E r + F$$

the root lying between 0 and 1, where

$$A = (1-\alpha)\{-\beta h - 2\alpha\xi h - h - \pi\}$$

$$B = [-\delta h (1-\alpha) - \beta h(1-\alpha) - \beta ha - 2\alpha\xi ha - ha - \pi a]$$

$$C = \delta h(1-\alpha) - \delta ha - 2\beta ha - 2\xi h (1-\alpha)^2 - 2\xi ha + (1-\alpha)(h+\pi) - a(h+\pi)$$

$$D = \pi a - \pi(1-\alpha) + ha - h(1-\alpha) - \beta ha - \beta h (1-\alpha) - 2h\xi a(1-\alpha)$$

$$E = (\delta ha - \delta h(1-\alpha) - \beta h (1-\alpha) + ha + \pi a)$$

$$F = (1-\alpha)[\delta h + \xi h + h + \pi]$$

Proof :- By usual computations, we obtain

$$\left[1 + \frac{zF''}{F'}\right] = \delta\left(\frac{zg'}{g}\right) + \frac{\beta}{n} \sum_{1}^{n} \frac{z}{z-z_k} + \xi\left(\frac{zf'}{f}\right) + (1-\xi)$$

By applying a lemmas (1.1), (2.1.1.) we obtain,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[1+\frac{zF''}{F'}\right] d\theta \geqslant \frac{\delta h}{(1+r)} - \frac{\beta r h}{(1-r)} + \\ + \frac{\left[\xi h\left[(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)\, r^3\right]\right]}{(1-\alpha)+ar+ar^2+(1-\alpha)\, r^3} + (1-\xi)h > -\pi$$

Which in view of Kaplan's result helps to conclude that F(z) is close-to-convex in $|z| < r$, where the radius of close-to-convexity is given by the polynomial stated above in the statement of theorem.

For sharpness, we consider the following functions,:

$$f_{a,\alpha}(z) = z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2-(1-\alpha-a)z+(1-\alpha)\right]^{1-\alpha}}\right]^{\frac{2(1-\alpha)}{3(1-\alpha)-a}}$$

Whenever $\dfrac{a}{1-\alpha} \leqslant 1$, $p(z) = (1-t)^n$, $g(z) = \dfrac{z}{(1+z)}$.

For $\alpha = 0$, we get a very interesting result for starlike functions, with second missing coefficient.

<u>Corollary</u> :- Suppose that F(z) is given by

$$F(z) = \int_0^z \left(p(t)\right)^{\beta/n} \left(\frac{g(t)}{t}\right)^{\delta} \left(f(t)\right)^{\xi} dt$$

Then $F(z)$ is close-to-convex in $|z| < r$, given by the polyhomial $H(r) = Ar^5 + Br^4 + Cr^3 + Dr^2 + Er + F$, where

$A = (-\beta h - h - \pi)$

$B = (-\delta h - \beta h - \beta ha - ha - \pi a)$

$C = \left(\delta h - \delta ha - 2\beta ha - 2\xi h - 2\xi ha + (h+\pi) - a(h+\pi)\right)$

$D = \pi a - \pi + ha - h - \beta ha - \beta h - 2h\xi a$

$E = \delta ha - \delta h - \beta h + ha + \pi a$

$F = \delta h + \xi h + h + \pi$.

This result is **entirely** new one.

We conclude our findings of regions of close-to-convexity by considering the rational function $M(z)/N(z)$, $M \in P(m, R_1)$ and $N \in P(n, R_2)$ with $R_1, R_2 > 1$, $m \geqslant 1$, $n \geqslant 0$, in the integral form.

<u>Theorem :-2.3.4 :-</u> Let $0 \leq \alpha < 1$, $\delta \geqslant 0$, $\beta \geqslant 0$ $P \in P(n,1)$ and $f \in S^*(\alpha)$, starlike functions of order α, having second missing coefficients. Then $F(z)$ given by

$$F(z) = \int_0^z \left(p(t)\right)^{\beta/n} \left(\frac{f(t)}{t}\right)^{\delta} \left(\frac{M(t)}{N(t)}\right)^{\rho'} dt$$

is close-to-convex in $|z| < r$ where r is the root of the polynomial given by H(r) where,

$$H(r) = h \left[\frac{(1-\alpha) + ar - a(1-2\alpha) r^2 - (1-2\alpha)(1-\alpha) r^3}{(1-\alpha) + ar + ar^2 + (1-\alpha) r^3} - \right.$$

$$\left. - \frac{\beta\gamma}{(1-r)} + (1-\delta) + \varrho \left(\frac{M}{R_1-1} + \frac{N}{R_2+1} \right) \right]$$

Proof :- By conventional calculations we are led to

$$1 + \frac{zF''}{F'} = \delta \frac{zf'}{f} + \frac{\beta}{n} \sum_{1}^{n} \frac{z}{z-z_k} + (1-\delta) + \varrho \left[\frac{M}{R_1-1} + \frac{N}{R_2+1} \right]$$

Application of the Lemmas 2.1.1, 2.1.2, yields.

$$\int_{\theta_1}^{\theta_2} \mathrm{Re} \left[1 + \frac{zF''}{F'} \right] d\theta$$

$$\geqslant (\theta_2 - \theta_1) \left[\frac{\delta(1-\alpha) + ar - a(1-2\alpha) r^2 - (1-2\alpha)(1-\alpha) r^3}{(1-\alpha) + ar + ar^2 + (1-\alpha) r^3} - \right.$$

$$\left. - \frac{\beta r}{(1-r)} + (1-\delta) + \varrho \left[\frac{M}{R_1 - 1} + \frac{N}{R_2 + 1} \right] \right] > -\pi$$

If we treat $(\theta_2 - \theta_1)$ as h, then because of Kaplan's result, previously stated, we directly conclude that,

$$h\left[\frac{\delta(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3}-\right.$$

$$\left.-\frac{\beta r}{(1-r)}+(1-\delta)+\varrho\left[\frac{M}{R_1-1}+\frac{N}{R_2+1}\right]\right]>\pi$$

F(z) is close-to-convex in $|z|<r$, as given in the statement of the theorem.

The result is sharp for the following functions

$$f_{a,\alpha}(z) = z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a)}\left[(1-\alpha)z^2-(1-\alpha-a)z+(1-\alpha)\right]^{1-\alpha}}\right]^{\frac{2(1-\alpha)}{3(1-\alpha)-a}}$$

Whenever, $\frac{a}{1-\alpha}\leqslant 1 \qquad p(z)=(1-t)^n$.

This above stated result is quite new and not found in literature.

: R E F E R E N C E S :

1. Alexander, J. W. :-

Functions which map the interior of the unit circle upon simple regions, Ann. Math. 17 (1915) pp-12 - 22.

2. Barr, A.F. :-

The radius of univalence of certain classes of analytic functions, Ph. D. dissertation, Univ. of Mississipi, (1971).

3. Basgöze, T. :-

On the univalence of certain classes of analytic functions, Jour, Lond. Math., Soc. (2), 1, (1969) PP. - 140 - 144.

4. Causey, W. M. :-

The close-to-convex functions and univalence of an integral, Math. Z. 99 (1967), pp. - 207 - 212.

5. Dieudonne, J.:-

Sur quelques properties des polynomies, Actualite's Sci. Indust. (1934), pp. 114.

6. Dutton, Charles Everett :-

Geometric properties of some classes of univalent Starlike functions, Ph. D. dissertation, Temple University, (1978).

7. Kaplan, W. :-

Close-to-convex schlicht functions, Michigan Math. J. 1 (1952), pp. 169- 185.

8. Kulkarni, S.R.,(Thakare, N.K.):-

Some problems connected with univalent functions, Ph. D. Thesis (unpublished) Shivaji University, Kolhapur, (1981).

9. Merkes, E.P. and Wright, D.J. :-

On the univalence of certain integrals Proc. Amer. Math. Soc. 27 (1), (1971), pp. 97 - 100.

10. Nunokawa, W. :-

On the univalence of certain integrals, Proc. Japan, Academy, 45 (10) (1969- a),pp. 841 - 845.

11. Pflatzgraff, J.A. :-

Univalence of the integral of $\left[f'(z)\right]^{\lambda}$, Bull. Lond. Math. Soc. 7, (1975) pp. 254 - 256 .

12. Royster, W. C. :-

On the univalence of certain integrals, Michigen Math. Jour., 12 (1965), pp. 385 - 387.

13. Ziegler, M. R. :-

Some integrals of univalent functions, Indian Jour. Math. 11 (3), (1969), pp. 145 - 151.

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