CHAPTER - II

SOME PROPERTIES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

ABSTRACT

Let S denote the class of normalised univalent functions f defined on the unit disk $E = \{z : |z| < 1\}$ having Taylor series expansion of the form

$$f(z) = z + \sum_{2}^{\infty} a_n z^n.$$

Recently Kulkarni-Thakare [5] have introduced the new subfamily of Univalent functions, $S(\alpha, \beta, \xi)$, in E, satisfying the condition,

$$\left| \frac{zf'/f - 1}{\left[2\xi (zf'/f - \alpha) - (zf'/f - 1) \right]} \right| < \beta,$$

where $\beta \in (0,1]$, $1/2 \leq \xi \leq 1$, $0 \leq \alpha < 1/2\xi$. In this chapter, we obtain several interesting different results of $S(\alpha, \beta, \xi)$ on the lines of Silverman H. [8], Sarangi - Uralegaddi, [7] and Gupta [2].

We also obtain very nice results for the new subfamily $D(\alpha,\beta,\xi)$, of univalent functions introduced by [5] .

[We have adopted the letter K to denote the class of convex functions, But since Gupta [2] has accepted $K(\alpha,\beta)$ as the class of close-to-convex functions, here in chapter II., we have used the same notation as that of Gupta [2]]

§ 1) INTRODUCTION:

In this chapter we obtain the several properties of holomorphic univalent functions that are the members of class $S(\alpha;\beta;\xi)$, the subfamily of the class of univalent functions S. This new subfamily $S(\alpha,\beta,\xi)$ of S, was introduced by Kulkarni-Thakare [5]. We say that f in S belongs to the class $S(\alpha,\beta,\xi)$ if f satisfies the condition

$$\left| \frac{zf'/f - 1}{[2\xi(zf'/f-\alpha) - (zf'/f-1)]} \right| < \beta ,$$

where $\beta \in (0,1]$, $1/2 \leq \xi \leq 1$, $0 \leq \alpha < 1/2\xi$. We observe that $S(0,1,1)$ is the class of starlike functions in E and $S(\alpha,1,1)$ is the class of starlike functions of order α, $0 \leq \alpha < 1$. The classes $S\left(0, \beta, \frac{1+\alpha}{2}\right)$ and $S(\alpha,1,\beta)$ are the classes studied by Lakshminarsimhan [6] and Juneja - Mogra [4] respectively. The class of functions considered by Gupta and Jain [3] is also a particular case of the family $S(\alpha,\beta,\xi)$.

We shall generalise our considerations for those members of $S(\alpha,\beta,\xi)$ that have negative coefficients.

We are motivated to carry out such study from the recent investigations carried out by Silverman H. [8], Gupta and Jain [3], Gupta [2], Sarangi - Uralegaddi [7], etc.

Let T be the subclass of S of holomorphic functions in E, that have the following power series representation

(1.1) $$f(z) = z - \sum_{2}^{\infty} |a_n| z^n.$$

We obtain very nice results, for those holomorphic functions which lie in both families $S(\alpha, \beta, \xi)$ and T. We denote $S^*(\alpha, \beta, \xi) = S(\alpha, \beta, \xi) \cap T$.

We state the lemma relating to the coefficients that completely characterises the members of $S^*(\alpha, \beta, \xi)$. The proof of which can be found in [5]

<u>Lemma 1.2</u> : A function $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$ is in $S^*(\alpha, \beta, \xi)$ if and only if

$$\sum_{2}^{\infty} |a_n| \left\{ (n-1) - \beta \left[n-1+2\xi\alpha - 2n\xi \right] \right\} \leqslant 2\beta\xi(1-\alpha).$$

This result is sharp.

We prove some distortion properties of univalent functions with negative coefficients.

§ 2) SOME THEOREMS :

Theorem 2.1 : If $f \in S^*(\alpha, \beta, \xi)$, then,

$$r - \left[\frac{2\beta\xi(1-\alpha)}{\{(n-1)-\beta(n-1+2\xi\alpha-2n\xi)\}}\right] r^2 \leqslant |f|$$

$$\leqslant r + \left[\frac{2\beta\xi(1-\alpha)}{\{(n-1)-\beta(n-1+2\xi\alpha-2n\xi)\}}\right] r^2.$$

For $|z| = r$

with equality for

$$f(z) = z - \frac{2\beta\xi(1-\alpha)z^2}{\{(n-1)-\beta(n-1+2\xi\alpha-2n\xi)\}}$$

Proof : We have $f(z) = z - \sum_2^\infty |a_n| z^n$, implies

$$|f(z)| \leqslant r + \sum_2^\infty |a_n| r^n, \quad \text{which gives}$$

$$|f(z)| \leqslant r + r^2 \sum_2^\infty |a_n|$$

In view of the lemma 1.2, the above inequality reduces to

$$|f(z)| \leqslant r + r^2 \left[\frac{2\beta\xi(1-\alpha)}{\{(n-1)-\beta(n-1+2\xi\alpha-2n\xi)\}}\right]$$

Similarly

$$|f(z)| \geq r - r^2 \left[\frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(n-1+2\xi\alpha-2n\xi)\}} \right]$$

Keeping our intentions in view, we go to state several special cases of Theorem 2.1.

Corollary:1 :- A function f having the form (1.1) belongs to $S^*(\alpha,1,1)$ i.e. starlike of order α, then

$$r - \left[\frac{1-\alpha}{n-\alpha} \right] r^2 \leqslant |f| \leqslant r + \left[\frac{1-\alpha}{n-\alpha} \right] r^2$$

Next is the similar characterization for those univalent holomorphic functions of Lakshminarsimhan [6] , having negative coefficients.

Corollary : 2 :- If a function having the coefficient expansion (1.1) is in $S^*\left(0, \beta, \frac{1+\alpha}{2}\right)$, then

$$r - \left[\frac{\beta(1+\alpha)}{\{(n-1)+\beta(n\alpha+1)\}} \right] r^2 \leqslant |f| \leqslant r + \left[\frac{\beta(1+\alpha)}{\{(n-1)+\beta(n\alpha+1)\}} \right] r^2$$

In the same vein we also have a corresponding result for the univalent holomorphic function studied by Juneja - Mogra [4]

Corollary : 3 :- If a function having the coefficient expansion (1.1) is in $S^*(\alpha, 1, \beta)$, then

$$r - \left[\frac{(1-\alpha)}{(n-\alpha)}\right] r^2 \leqslant |f| \leqslant r + \left[\frac{(1-\alpha)}{(n-\alpha)}\right] r^2.$$

Which is the same as obtained in corollary 1.

In the next theorem we obtain the distortion properties of derivative of the normalised univalent functions having the power series expansion given by (1.1).

Theorem : 2.2: - If $f \in S^*(\alpha, \beta, \xi)$, then

$$1 - r\left[\frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(n-1+2\alpha\xi - 2n\xi)\}}\right] n \leqslant |f'|$$

$$\leqslant 1 + r\left[\frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(n-1+2\alpha\xi - 2n\xi)\}}\right] n$$

Equality holds for

$$f(z) = z - \frac{(2\beta\xi(1-\alpha))\, z^2}{\{(n-1) - \beta(n-1+2\alpha\xi - 2n\xi)\}}$$

$(z = \pm r)$.

Proof :- We have by (1.1)

$$f(z) = z - \sum_{2}^{\infty} |a_n|\, z^n,$$ which gives after differentiation and using the modulus inequality,

$$\left|f'(z)\right| \leq 1 + \sum_{2}^{\infty} n \left|a_n\right| \left|z\right|^{n-1}$$

In view of Lemma (1.2) we obtain,

$$\left|f'(z)\right| \leq 1 + r \left[\frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(n-1+2\alpha\xi - 2n\xi)\}}\right] n$$

Similarly,

$$\left|f'(z)\right| \geq 1 - r \left[\frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(n-1+2\alpha\xi - 2n\xi)\}}\right] n$$

We state several special cases of the above theorem.

Corollary :1 : - If a function f having the coefficient expansion (1.1) belongs to $S^*(\alpha,1,1)$, then, we have

$$1 - r\left[\frac{n(1-\alpha)}{(n-\alpha)}\right] \leq |f'| \leq 1 + r\left[\frac{n(1-\alpha)}{(n-\alpha)}\right]$$

Next is a similar result specialised for the univalent holomorphic functions considered by Lakshminarsimhan [6], but having negative coefficients.

Corollary : 2 : - If $f \in S^*(0, \beta, \frac{1+\alpha}{2})$ then,

$$1 - r\left[\frac{\beta(1+\alpha)}{\{(n-1) + \beta(n\alpha+1)\}}\right] n \leq |f'| \leq 1 + r\left[\frac{\beta(1+\alpha)}{\{(n-1) + \beta(n\alpha+1)\}}\right] n$$

Lastly, we state the result for the Class $S^*(\alpha, 1, \beta)$ studied by Juneja and Mogra [4], having the negative coefficient expansion (1.1).

Corollary : 3 :- If $f \in S^*(\alpha, 1, \beta)$, then, we get the same distortion as obtained in corollary 1.

In the next theorem we determine the radius of Convexity for the functions in $S^*(\alpha, \beta, \xi)$.

Theorem :2.3 :- If $f \in S^*(\alpha, \beta, \xi)$, then f is convex in the disc,

$$|z| < r = r(\alpha, \beta, \xi) = \inf_n \left[\frac{(n-1) - \beta(n-1+2\alpha\xi - 2n\xi)}{n^2\, 2\beta\xi(1-\alpha)} \right]^{1/(n-1)}$$

for $n = 2, 3, 4, \ldots\ldots$

Proof :- In view of the definition of convexity of function it is sufficient to show that,

$$\left| \frac{zf''}{f'} \right| \leq 1, \quad \text{for } |z| \leq r(\alpha, \beta, \xi).$$

We have by routine calculations,

$$\left| \frac{zf''}{f'} \right| = \left| \frac{\sum_2^\infty n(n-1)\, |a_n|\, z^{n-1}}{1 - \sum_2^\infty n\, |a_n|\, z^{n-1}} \right|$$

$$\leq \frac{\sum_2^\infty n(n-1)\, |a_n|\, |z|^{n-1}}{1 - \sum_2^\infty n\, |a_n|\, |z|^{n-1}}$$

Clearly, $\left|zf''/f'\right| \leqslant 1$, if

$$\sum_{2}^{\infty} n(n-1)\ |a_n|\ |z|^{n-1} \leqslant 1 - \sum_{2}^{\infty} n\ |a_n|\ |z|^{n-1}$$

alternately if $\sum_{2}^{\infty} n^2\ |a_n|\ |z|^{n-1} \leqslant 1$

By using the lemma (1.2)

$$\sum_{2}^{\infty} |a_n| \frac{\{(n-1) - \beta\ (n-1+2\alpha\xi\ -2n\xi)\}}{2\beta\xi(1-\alpha)} \leqslant 1$$

The expression will be true if,

$$n^2\ |z|^{n-1} \leqslant \frac{(n-1) - \beta(n-1+2\alpha\xi\ -2n\xi)}{2\beta\xi(1-\alpha)}$$

Solving this inequality for $|z|$, we obtain

$$|z| \leqslant \left[\frac{\{(n-1) - \beta\ (n-1+2\xi\alpha\ -2n\xi)\}}{n^2\ 2\beta\xi\ (1-\alpha)}\right]^{1/(n-1)}$$

The desired result follows by mere substitution of $|z| = r\ (\alpha, \beta, \xi)$ in the last expression. We state some particular cases.

Corollary : 1 :- If $f \in S^*(\alpha, 1, 1)$ then f is convex in the disc.

$$|z| < r = r\ (\alpha) = \inf_{n}\left[\frac{n-\alpha}{n^2(1-\alpha)}\right]^{1/(n-1)}$$

Corollary : 2: - If $f \in S^*(0, \beta, \frac{1+\alpha}{2})$, the subfamily of univalent holomorphic functions studied by Lakshminarsimhan [6], **but having negative coefficients.**

$$|z| < r = r\left(0, \beta, \frac{1+\alpha}{2}\right) = \operatorname*{Inf}_{n}\left[\frac{(n-1)+\beta(n\alpha+1)}{n^2\beta(1+\alpha)}\right]^{1/(n-1)}$$

Lastly we state a corollary due to Juneja and Mogra [4].

Corollary : 3 : - If $f \in S^*(\alpha, 1, \beta)$, **but having negative coefficient, then**

$$|z| < r = r(\alpha, 1, \beta) = \operatorname*{Inf}_{n}\left[\frac{(n-\alpha)}{n^2(1-\alpha)}\right]^{1/(n-1)},$$

Which is exactly same as in corollary 1. We also obtain the extreme points for the class $S^*(\alpha, \beta, \xi)$.

The extreme values on F, F being any compact family of univalent functions, of the real part of any continuous linear functional defined over the set of holomorphic functions occurs at one of the extreme points of the closed convex hull of F. Keeping this intention in mind, we try to obtain the extreme points of the family F; which will help us to solve many extremal problems for F. Here we particularly obtain the extreme points for the class $S^*(\alpha, \beta, \xi)$.

<u>Theorem : 2.4 :-</u> Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{2\beta\xi(1-\alpha)}{(n-1)+\beta(2n\xi+1-2\xi\alpha-n)} z^n$$

$n = 2, 3, \ldots\ldots$ Then $f \in S^*(\alpha, \beta, \xi)$ if **and only if it** can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad \text{where } \lambda_n \geqslant 0$$

$n = 1, 2, \ldots\ldots$ and $\sum_{1}^{\infty} \lambda_n = 1$,

<u>Proof</u> :- Suppose,

$$f(z) = \sum_{1}^{\infty} \lambda_n f_n(z) = z - \sum_{2}^{\infty} \lambda_n \frac{2\beta\xi(1-\alpha)}{(n-1)+\beta(2n\xi+1-2\xi\alpha-n)} z^n$$

Then,

$$\sum_{2}^{\infty} \left\{ \frac{(n-1)+\beta(2n\xi+1-2\xi\alpha-n)}{2\beta\xi(1-\alpha)} \lambda_n \frac{2\beta\xi(1-\alpha)}{(n-1)+\beta(2n\xi+1-2\xi\alpha-n)} \right\}$$

$$= \sum_{2}^{\infty} \lambda_n = 1 - \lambda_1 \leqslant 1.$$

Therefore $f \in S^*(\alpha, \beta, \xi)$

Conversely, suppose that $f \in S^*(\alpha, \beta, \xi)$, so by Lemma

$$|a_n| \leqslant \frac{2\beta\xi(1-\alpha)}{(n-1) + \beta(2n\xi+1-2\xi\alpha-n)} \qquad n = 2, 3, \ldots$$

Let
$$\lambda_n = \frac{(n-1) + \beta(2n\xi+1-2\xi\alpha-n)}{2\beta\xi(1-\alpha)} |a_n|$$

Then,

$$\sum_{2}^{\infty} \lambda_n \leqslant 1, \quad \lambda_n \geqslant 0 \quad \text{and} \quad \lambda_1 = 1 - \sum_{2}^{\infty} \lambda_n$$

Therefore,

$$f(z) = \sum_{1}^{\infty} \lambda_n \ f_n(z)$$

This completes the proof.

We state some particular cases of the above theorem.

Corollary : 1 :- If $f \in S^*(\alpha, 1, 1)$, then the extreme points of $S^*(\alpha, 1, 1,)$ are the functions

$$f(z) = z - \frac{(1-\alpha)}{(n-\alpha)} z^n \qquad (n = 2, 3, \ldots)$$

Where $f_1(z) = z$, and $f_n(z) = z - \dfrac{1-\alpha}{n-\alpha} z^n$.

which is same as proved by Silverman.H. [8] .

Corollary : 2 :- If $f \in S^*(0, \beta, \frac{1+\alpha}{2})$, then the extreme points are the functions

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{(1+\alpha)\beta}{(n-1) + \beta(n\alpha+1)} z^n$$

This is due to the class studied by Lakshminarsimhan [6].

Lastly, the following is the corollary for the class of functions introduced by Juneja - Mogra; [4].

Corollary : 3 : - If $f \in S^*(\alpha, 1, \beta)$, then the extreme points are the functions

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{(1-\alpha)}{(n-\alpha)} z^n$$

which is suprisingly the same as studied by Silverman H. [8].

§ 3) In continuation of the discussion of the properties of univalent function with negative coefficients we define the class $K(\alpha, \beta, \xi)$ with the usual restrictions on α, β, ξ as defined in $S(\alpha, \beta, \xi)$. This class $K(\alpha, \beta, \xi)$ is known as the close-to-convex functions of type β. Recently Gupta [2] has obtained some results in connection with $K(\alpha, \beta)$, the class of close-to-convex function of order α and type β.

Our attempt lies in generalising these results.

Definition :- Let S denote the class of functions of the form

$$f(z) = z - \sum_{2}^{\infty} |a_n| \, z^n$$

that are holomorphic in the unit disc. A function $f \in S$ is said to be in $K(\alpha, \beta, \xi)$, the class of close-to-convex functions of type β, if there exists a function $\psi(z) \in S$ such that $\psi(z) = z - \sum_{2}^{\infty} |b_n| \, z^n$, satisfying

$$\left| \frac{\dfrac{zf'}{\psi} - 1}{\left[2\xi\left(\dfrac{zf'}{\psi} - \alpha\right) - \left(\dfrac{zf'}{\psi} - 1\right) \right]} \right| < \beta .$$

where $\beta \in (0,1]$, $1/2 \leq \xi \leq 1$, $0 \leq \alpha \leq 1/2\xi$

For this class, we obtain the results concerning coefficient estimates, distortion theorems and covering theorems.

MAIN RESULTS :-

Theorem : 3.1 :- If $f \in K(\alpha, \beta, \xi)$ then

$$\sum_{2}^{\infty} \left\{ (1+\beta \;\; (2\xi - 1)\, n \, |a_n| - (1-\beta + 2\xi\alpha\beta) \, |b_n| \right\} \leqslant 2\beta\xi(1-\alpha)$$

Where b_n's are the coefficients of the function belonging to class S.

Proof :- By definition of $K(\alpha,\beta,\xi)$, if $f \in K(\alpha,\beta,\xi)$ then there exists a function $\psi(z) = z - \sum_{2}^{\infty} |b_n| z^n$ in S

satisfying

$$\left| \frac{\frac{zf'}{\psi} - 1}{\left[2\xi\left(\frac{zf'}{\psi} - \alpha\right) - \left(\frac{zf'}{\psi} - 1\right)\right]} \right| < \beta$$

$$\Longrightarrow \left| \frac{zf' - \psi}{\left[2\xi(zf' - \alpha\psi) - (zf' - \psi)\right]} \right| < \beta$$

$$\Longrightarrow \left| \frac{\sum_{2}^{\infty} \left\{n|a_n| - |b_n|\right\} z^n}{(z - \sum_{2}^{\infty} n|a_n| z^n)(2\xi - 1) - (1 - 2\xi\alpha)(z - \sum_{2}^{\infty} |b_n| z^n)} \right| < \beta$$

which gives,

$$\operatorname{Re}\left\{ \frac{\sum_{2}^{\infty} \left\{n|a_n| - |b_n|\right\} z^n}{2\xi z(1-\alpha) - \sum_{2}^{\infty} \left[n|a_n|(2\xi - 1) + (1 - 2\xi\alpha)|b_n| z^n\right]} \right\} < \beta$$

We select the values of z, on the real axis so that, $\frac{zf'(z)}{\psi(z)}$ is real, simplifying the denominator of above expression and letting $z \to 1$, through real values we get,

$$\sum_{2}^{\infty}\left\{n\,|a_n| - |b_n|\right\} \leqslant \beta\left\{2\xi(1-\alpha) - \sum_{2}^{\infty}\left[n\,|a_n|\,(2\xi-1) + (1-2\xi\alpha)\,|b_n|\right]\right\}$$

$$\Rightarrow \sum_{2}^{\infty}\left\{\left(1+\beta(2\xi-1)\right)n\,|a_n| - (1-\beta+2\alpha\xi\beta)\,|b_n|\right\} \leqslant 2\beta\xi(1-\alpha)$$

<u>Corollary</u> :- Upon replacing ξ by 1, in the above inequality, we get the result of Gupta [2] , which we merely state

<u>Theorem :3.2:</u>- If $f \in K(\alpha,\beta)$, then

$$\sum_{2}^{\infty}\left\{(1+\beta)\,n\,|a_n| - (1-\beta+2\alpha\beta)\,|b_n|\right\} \leqslant 2\beta(1-\alpha).$$

In [8] it is shown that $\sum_{2}^{\infty} n\,|b_n| \leqslant 1$, which gives further that $\sum_{2}^{\infty}|b_n| \leqslant 1/2$, Hence the inequality of theorem, 3.1, can be expressed as

$$\sum_{2}^{\infty} n\,|a_n| \leqslant \frac{4\beta\xi(1-\alpha) + (1-\beta+2\alpha\beta\xi)}{2\left(1+\beta(2\xi-1)\right)}$$

We define the class $B(\alpha,\beta,\xi)$ in the following way :

A function $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$ is said to be in $B(\alpha, \beta, \xi)$ if there exists a function $\psi(z) = z - \sum_{2}^{\infty} |b_n| z^n$ in S satisfying

(i) $\sum_{2}^{\infty} \left\{ \left(1+\beta(2\xi-1)\right) n |a_n| - (1-\beta+2\alpha\beta\xi)|b_n| \right\} \leqslant 2\beta\xi(1-\alpha).$

and

(ii) $n |a_n| - |b_n| \geqslant 0$ for every n.

Theorem : 3.3 : -

$$B(\alpha,\beta,\xi) \subseteq K(\alpha,\beta,\xi)$$

Consider

$$\left| \frac{\frac{zf'}{\psi} - 1}{2\xi\left(\frac{zf'}{\psi} - \alpha\right) - \left(\frac{zf'}{\psi} - 1\right)} \right|$$

$$= \left| \frac{\sum_{2}^{\infty} \left\{ n |a_n| - |b_n| \right\} z^n}{2(1-\alpha)\xi z - \sum_{2}^{\infty} n |a_n| (2\xi-1) + (1-2\alpha\xi) |b_n| z^n} \right|$$

$$\leqslant \frac{\sum_{2}^{\infty} \left[n |a_n| - |b_n| \right]}{2\xi(1-\alpha) - \sum_{2}^{\infty} n |a_n| + (1-2\xi\alpha) |b_n|}$$

This expression is bounded above by β if the inequality of theorem holds :

Hence $B(\alpha,\beta,\xi) \subseteq K(\alpha,\beta,\xi)$.

Next we state the theorem concerning with coefficient estimate of the class $K(\alpha,\beta,\xi)$.

<u>Theorem : 3.4:-</u> If $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$ be in $B(\alpha,\beta,\xi)$ then,

$$|a_n| \leq \frac{2\beta\xi(1-\alpha)n + (1-\beta+2\alpha\xi\beta)}{n^2\left(1+\beta(2\xi-1)\right)}$$

with the extremal function

$$f_n(z) = z - \frac{2\beta\xi(1-\alpha)n + (1-\beta+2\alpha\beta\xi)}{n^2\left(1+\beta(2\xi-1)\right)} z^n$$

<u>Proof</u> :- By theorem 3.1 the inequality gives,

$$\left(1+\beta(2\xi-1)\right)n^2 |a_n| \leq 2\beta\xi(1-\alpha)+(1-\beta+2\alpha\beta\xi)\,|b_n|$$

$$\leq 2\beta\xi(1-\alpha)+(1-\beta+2\alpha\beta\xi)\,n^{-1}$$

Because $\psi(z) = z - \sum_{2}^{\infty} |b_n| z^n \in S$, $|b_n| \leq 1/n$

$$\Rightarrow |a_n| \leq \frac{2\beta\xi(1-\alpha)n + (1-\beta+2\alpha\xi\beta)}{n^2\left(1+\beta(2\xi-1)\right)}$$

which is the required condition.

Theorem : 3.5 :- (Distortion Theorem)

If $f \in B(\alpha, \beta, \xi)$, then for $|z| \leqslant r < 1$.

(3.6) :-

$$r - \left[\frac{1+4\beta\xi-\beta-2\alpha\beta\xi}{4(1+\beta(2\xi-1))}\right] r^2 \leqslant |f| \leqslant r + \left[\frac{1+4\beta\xi-\beta-2\alpha\beta\xi}{4(1+\beta(2\xi-1))}\right] r^2$$

(3.7):-

$$1 - \left[\frac{1+4\beta\xi-\beta-2\alpha\beta\xi}{2(1+\beta(2\xi-1))}\right] r \leqslant |f'| \leqslant 1 + r \left[\frac{1+4\beta\xi-\beta-2\alpha\beta\xi}{2(1+\beta(2\xi-1))}\right]$$

Proof :- The inequality, $\sum_{2}^{\infty} n \, |a_n| \leqslant \dfrac{1+4\beta\xi-\beta-2\alpha\beta\xi}{2(1+\beta(2\xi-1))}$

can be regarded as the necessary condition, for f to be in $B(\alpha, \beta, \xi)$, even if we drop the condition (ii) in the definition of $B(\alpha, \beta, \xi)$, this condition can also be taken as the sufficient condition.

We can write down,

$$2 \sum_{2}^{\infty} |a_n| \leqslant \sum_{2}^{\infty} n \, |a_n| \leqslant \frac{1+4\beta\xi-\beta-2\alpha\beta\xi}{2(1+\beta(2\xi-1))}$$

Hence $$|f(z)| \leqslant r + \sum_{2}^{\infty} |a_n| \, r^n$$

$$= r + r^2 \sum_{2}^{\infty} |a_n|$$

$$= r + r^2 \left[\frac{1 + 4\beta\xi - \beta - 2\alpha\beta\xi}{4\left(1 + \beta(2\xi - 1)\right)} \right]$$

and similarly

$$|f(z)| \geqslant r - r^2 \sum_2^\infty |a_n|$$

$$= r - r^2 \left[\frac{1 + 4\beta\xi - \beta - 2\alpha\beta\xi}{4\left(1 + \beta(2\xi - 1)\right)} \right]$$

Thus (3.6) follows :

Further $f(z) = z - \sum_2^\infty |a_n| \; z^n$

$$f'(z) = 1 - \sum_2^\infty n\, |a_n|\, |z|^{n-1}$$

$$|f'(z)| \leqslant 1 + \sum_2^\infty n\, |a_n|\, |z|^{n-1}$$

$$\leqslant 1 + r \sum_2^\infty n\, |a_n|$$

$$\leqslant 1 + r \left[\frac{1 + 4\beta\xi - \beta - 2\alpha\beta\xi}{2\left(1 + \beta(2\xi - 1)\right)} \right]$$

Similarly $|f'(z)| \geqslant 1 - r \left[\dfrac{1 + 4\beta\xi - \beta - 2\alpha\beta\xi}{2\left(1 + \beta(2\xi - 1)\right)} \right]$

Hence proved (3.7)

We note that the bounds obtained above are sharp and the equalities are attained for the function

(3.8) :-

$$f(z) = z - \frac{1 + 4\beta\xi - \beta - 2\alpha\beta\xi}{4\left(1 + \beta(2\xi - 1)\right)} z^2, \qquad z = \pm r$$

Theorem : 3.9 :- Let $f \in B(\alpha, \beta, \xi)$. Then the disc $|z| < 1$ is mapped onto a domain that contains the disc

$$|w| < \frac{3 - 3\beta + 4\beta\xi + 2\alpha\beta\xi}{4\left(1+\beta(2\xi - 1)\right)}$$

The result is sharp and the extremal function is given by (3.8)

Proof :- By putting r equal to 1 in (3.6) we get the desired results.

Theorem :3.10 :- The class $B(\alpha, \beta, \xi)$ is convex.

Proof :- We shall make use of the definition of convexity.

Let $f_1(z) = z - \sum_2^\infty |a_n| z^n$ and $f_2(z) = z - \sum_2^\infty |c_n| z^n$

be in $B(\alpha, \beta, \xi)$ with respect to functions,

$$\psi_1(z) = z - \sum_2^\infty |b_n| z^n \quad \text{and} \quad \psi_2(z) = z - \sum_2^\infty |d_n| z^n$$

in S . For $0 \leqslant \lambda \leqslant 1$, we try to show that

$$F(z) = \lambda f_1(z) + (1-\lambda) f_2(z) = z - \sum_2^{\infty} |e_n(\lambda)| z^n \in B(\alpha,\beta,\xi)$$

with respect to $\Psi(z) = \lambda \psi_1(z) + (1-\lambda)\Psi_2(z)$

$$= z - \sum_2^{\infty} |t_n(\lambda)| z^n \in S.$$

The function $F(z)$ will belong to $B(\alpha,\beta,\xi)$ if

$$\sum_2^{\infty} \left\{(1+\beta(2\xi-1))\, n|e_n(\lambda)| - (1-\beta+2\alpha\beta\xi)|t_n(\lambda)|\right\} \leqslant 2\beta\xi(1-\alpha)$$

$n\, e_n(\lambda) - t_n(\lambda) \geqslant 0$ for each n

and since $n|a_n| - |b_n| \geqslant 0$

and $n|c_n| - |d_n| \geqslant 0$ for each n

Therefore, $n\, e_n(\lambda) - t_n(\lambda) \geqslant 0$ for each n

$$\therefore \sum_2^{\infty} n\, (1+\beta(2\xi-1))|e_n(\lambda)| - (1-\beta+2\alpha\beta\xi)|t_n(\lambda)|$$

$$= \lambda \sum_2^{\infty} (1+\beta(2\xi-1))\, n\, |a_n| - (1-\beta+2\alpha\beta\xi)\, |b_n| +$$

$$+ (1-\lambda) \sum_2^{\infty} (1+\beta(2\xi-1))\, n\, |c_n| - (1-\beta+2\xi\alpha\beta)\, |d_n|$$

$$\leqslant 2\beta\xi(1-\alpha)$$

and hence proved.

§ 4 - Sarangi - Uralegaddi [7] has obtained the radius of univalence of certain holomorphic functions with negative coefficients under the different conditions. We continue our discussion of univalency on the same line for our class $S^*(\alpha,\beta,\xi)$ and $P^*(\alpha,\beta,\xi)$ to obtain more surprising results.

<u>Theorem :4.1</u>:- Let $F(z) = z - \sum_{2}^{\infty} |a_n|\ z^n \ \in \ S^*(\alpha,\beta,\xi)$ and $f(z) = 1/2\ [z\ F(z)]'$. Then $f(z)$ is starlike of order λ and type δ in the disc $|z| < r = r(\alpha,\beta,\xi,\lambda,\delta)$ given by

$$|z| \leqslant \left[\frac{\{(n-1) - \beta(n-1+2\xi\alpha-2n\xi)\}\ (2\delta-\lambda-1)}{2\beta\xi(1-\alpha)\left(\frac{n+1}{2}\right)(n-\lambda)} \right]^{1/(n-1)}$$

<u>Proof</u> :- By definition of $f(z)$ we have

$$f(z) = 1/2\ [z\ F(z)]'$$

Now $$zF(z) = z^2 - \sum_{2}^{\infty} |a_n|\ |z|^{n+1}$$

$$(zF(z))' = 2z - \sum_{2}^{\infty} (n+1)\ |a_n|\ z^n$$

$$1/2\ (zF(z))' = z - \sum_{2}^{\infty} \left(\frac{n+1}{2}\right) |a_n|\ z^n$$

Now $f'(z) = 1 - \sum_{2}^{\infty} \frac{n(n+1)}{2} |a_n| z^{n-1}$

$$\Longrightarrow \quad z f'(z) = z - \sum_{2}^{\infty} \frac{n(n+1)}{2} |a_n| z^n$$

$$\frac{zf'(z)}{f(z)} = \frac{z - \sum_{2}^{\infty} \frac{n(n+1)}{2} |a_n| z^n}{z - \sum_{2}^{\infty} \left(\frac{n+1}{2}\right) |a_n| z^n}$$

$$\frac{zf'(z)}{f(z)} - \delta = \frac{\left(z - \sum_{2}^{\infty} \frac{n(n+1)}{2} |a_n| z^n\right) - \delta\left(z - \sum_{2}^{\infty} \frac{n+1}{2} |a_n| z^n\right)}{z - \sum_{2}^{\infty} \left(\frac{n+1}{2}\right) |a_n| z^n}$$

$$\left|\frac{zf'(z)}{f(z)} - \delta\right| = \left|\frac{(1-\delta) - \sum_{2}^{\infty} \left(\frac{n+1}{2}\right) |a_n| z^{n-1} (n-\delta)}{1 - \sum_{2}^{\infty} \frac{n+1}{2} |a_n| z^{n-1}}\right|$$

$$\leqslant \frac{(1-\delta) + \sum_{2}^{\infty} \left(\frac{n+1}{2}\right) |a_n| |z|^{n-1} (n-\delta)}{1 - \sum_{2}^{\infty} \left(\frac{n+1}{2}\right) |a_n| |z|^{n-1}}$$

Hence $\left| \dfrac{zf'(z)}{f(z)} - \delta \right| \leqslant \delta - \lambda$ if

$$(1-\delta) + \sum_{2}^{\infty} \frac{n+1}{2} |a_n| |z|^{n-1} (n-\delta) \leqslant (\delta-\lambda)\left\{1 - \sum_{2}^{\infty} \frac{n+1}{2} |a_n| |z|^{n-1}\right\}$$

$$\Longrightarrow \frac{\sum_{2}^{\infty} |a_n| |z|^{n-1} \left\{\left(\frac{n+1}{2}\right)(n-\lambda)\right\}}{(2\delta - \lambda - 1)} \leqslant 1$$

On account of the lemma 1.2, we get

$$\sum_{2}^{\infty} |a_n| \frac{\{(n-1) - \beta(n-1+2\xi\alpha - 2n\xi)\}}{2\beta\xi(1-\alpha)} \leqslant 1$$

$$\frac{\sum_{2}^{\infty} |a_n| |z|^{n-1} \left\{\frac{n+1}{2}(n-\lambda)\right\}}{(2\delta - \lambda - 1)} \leqslant \frac{\sum_{2}^{\infty} |a_n| \{(n-1) - \beta(n-1+2\xi\alpha - 2n\xi)\}}{2\beta\xi(1-\alpha)}$$

solving for $|z|$, we get, the desired result

$$|z| \leqslant \left[\frac{[(n-1) - \beta(n-1+2\xi\alpha - 2n\xi)]\,[(2\delta - \lambda - 1)]}{2\beta\xi(1-\alpha)\left(\frac{n+1}{2}\right)(n-\lambda)}\right]^{1/n-1}$$

We state some particular cases,

Corollary : 1:- If $F(z) \in S^{*}(\alpha,1,1)$ and $\delta=1$, $\lambda=\beta$ we get,

$$|z| \leqslant \left[\frac{2\,(n-\alpha)\,(1-\lambda)}{(n+1)\,(1-\alpha)(n-\lambda)}\right]^{1/(n-1)}$$

which is the same as obtained by Sarangi - Uralegaddi [7] .

Corollary : 2 :- If $F(z) \in S^{*}(0,\beta,\frac{1+\alpha}{2})$, the class studied by Lakshminarsimhan, T.V. [6], **but having negative coefficients.**

$$|z| \leqslant \left[\frac{2\left[(n-1)+\beta(1+n\alpha)\right]\,(2\delta-\lambda-1)}{\beta(1+\alpha)\,(1-\alpha)\,(n+1)\,(n-\lambda)}\right]^{1/(n-1)}$$

Corollary :3 : - If $F(z) \in S^{*}(\alpha,1,\beta)$, the class studied by Juneja - Mogra [4] **but having negative ooefficients.**

$$|z| \leqslant \left[\frac{2\,(n-\alpha)\,(2\delta-\lambda-1)}{(1-\alpha)\,(n+1)\,(n-\lambda)}\right]^{1/(n-1)}$$

We also state and prove another theorem,

Theorem :4.2:- If $F(z) = z - \sum_{2}^{\infty} |a_n|\, z^n \in S^{*}(\alpha,\beta,\xi)$ and $f(z) = 1/2\,(zF(z))'$, then $\mathrm{Re}\, f'(z) > \lambda$ for $0 \leqslant \lambda < 1$ and $|z| < r = r(\alpha,\beta,\xi,\lambda)$, given by

$$\operatorname*{Inf}_{n}\left[\frac{(1-\lambda)\ \left[(n-1)-\beta\ (n-1+2\alpha\xi-2n\xi)\right]}{n(n+1)\ \beta\xi\ (1-\alpha)}\right]^{1/(n-1)}$$

<u>Proof</u> :- We show that

$$|f'(z)-1| \leqslant (1-\lambda), \text{ for } |z| < r\ (\alpha, \beta, \xi, \lambda)$$

We have,

$$|f'(z)-1| \leqslant \left\{\left[1-\sum_{2}^{\infty}\frac{n(n+1)}{2}\ |a_n|\ |z|^{n-1}\right]-1\right\}$$

$$|f'(z)-1| \leqslant \sum_{2}^{\infty}\frac{n(n+1)}{2}\ |a_n|\ |z|^{n-1}$$

Hence $|f'(z)-1| \leqslant (1-\lambda)$ if

$$\sum_{2}^{\infty} n\ \frac{n+1}{2}\ |a_n|\ |z|^{n-1} \leqslant (1-\lambda)$$

By Lemma 1.2

$$\sum_{2}^{\infty}|a_n|\left[\frac{(n-1)-\beta(n-1+2\alpha\xi-2n\xi)}{2\ \beta\xi(1-\alpha)}\right] \leqslant 1$$

$$\sum_{2}^{\infty}\frac{n(n+1)\,|a_n|\,|z|^{n-1}}{2(1-\lambda)} \leqslant \frac{|a_n|\ \{(n-1)-\beta(n-1+2\alpha\xi-2n\xi)\}}{2\beta\xi(1-\alpha)}$$

$$|z| \leqslant \left[\frac{(1-\lambda)\left[(n-1) - \beta(n-1+2\alpha\xi - 2n\xi)\right]}{n(n+1)\beta\xi(1-\alpha)} \right]^{1/(n-1)}$$

gives the desired result.

We state some corollaries -

Corollary : 1 :- If $F \in S^*(\alpha,1,1)$, with the remaining conditions as stated in Theorem, 4.2 then,

$$\text{Re } f'(z) > \lambda \quad \text{in}$$

$$|z| \leqslant \left[\frac{2(1-\lambda)(n-\alpha)}{n(n+1)(1-\alpha)} \right]^{1/(n-1)}$$

This is a new result for starlike functions of order α

Corollary : 2 :- If $F \in S^*\left(0,\beta,\frac{1+\alpha}{2}\right)$ and other restrictions remaining same, then Re $f'(z) > \lambda$ in

$$|z| \leqslant \left[\frac{4(1-\lambda)\beta(n\alpha+1)}{n(n+1)} \right]^{1/(n-1)}$$

F having negative coefficient.

Corollary : 3 : - If $F \in S^*(\alpha,1,\beta)$, then,

Re $f'(z) > \lambda$, but having negative coefficient, then

$$|z| \leqslant \left[\frac{2(1-\lambda)(n-\alpha)}{n(n+1)(1-\alpha)} \right]^{1/(n-1)}$$

§ 5 : We also obtain the new interesting results on the above lines but for different class, $D(\alpha,\beta,\xi)$ defined by Kulkarni - Thakare [5]. $D(\alpha, \beta, \xi)$ is a subfamily of S of normalised univalent functions f that are holomorphic in the open unit disc E and satisfy the condition

$$\left| \frac{f' - 1}{[2\xi(f' - \alpha) - (f' - 1)]} \right| < \beta$$

Regarding the class $D(\alpha, \beta, \xi)$ we note down the following points.

The class $D(\alpha) = D(0, \alpha, 1)$ is precisely the class of functions in E, studied by Caplinger [1]. The class $D(\alpha, 1, \beta) = D(\alpha, \beta)$ is the class of holomorphic functions discussed by Juneja - Mogra [4].

Where $\beta \in (0,1]$, $1/2 \leqslant \xi \leqslant 1$, $0 \leqslant \alpha < 1/2\xi$

Let T be the subclass of S of holomorphic functions in E, having the power series representation. We define $P^*(\alpha, \beta, \xi) = T \cap D(\alpha, \beta, \xi)$ we have the following theorem based on the above subclasses of S, before this we state the following coefficient theorem the proof of which can be found in [5].

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Lemma : 5.1 : (coefficient) :- A holomorphic function

$f(z) = z - \sum_{2}^{\infty} |a_n| \; z^n$ is in $P^*(\alpha, \beta, \xi)$ if and only if

$$\sum_{2}^{\infty} n \; |a_n| \; \{1+\beta\,(2\xi-1)\} \leqslant 2\beta\xi(1-\alpha).$$

This result is sharp.

We shall apply this coefficient theorem in case of the

following theorems :

Theorem :5.2:- Let $F(z) = z - \sum_{2}^{\infty} |a_n| \; z^n$ be in $P^*(\alpha, \beta, \xi)$

$f(z) = 1/2\,\left(zF(z)\right)'$. Then F(z) is starlike of order λ and

type δ in $|z| < r$ given by

$$|z| \leqslant \left[\frac{n\,\{1+\beta(2\xi-1)\}\;(2\delta-1-\lambda)}{\beta\xi(1-\alpha)(n+1)\;(n-2\delta-\lambda)} \right]^{1/(n-1)}$$

Proof :- We show that $\operatorname{Re}\left[\frac{z\,f'(z)}{f(z)}\right] > \lambda$

for $|z| < r\;(\alpha, \beta, \xi, \delta, \lambda)$

$$F(z) = z - \sum_{2}^{\infty} |a_n| \; z^n$$

$$\left(zF(z)\right)' = 2z - \sum_{2}^{\infty} (n+1) \; |a_n| \; z^n$$

$$1/2\,\left(zF(z)\right)' = f(z) = z - \sum_{2}^{\infty} \frac{n+1}{2} \; |a_n| \; z^n$$

Now

$$\frac{zf'(z)}{f(z)} = \frac{z - \sum_{2}^{\infty} n\,\frac{(n+1)}{2}\,|a_n|\,z^n}{z - \sum_{2}^{\infty} \frac{(n+1)}{2}\,|a_n|\,z^n}$$

$$\left|\frac{zf'}{f} - \delta\right| \leqslant \frac{(1-\delta) + \sum_{2}^{\infty} \frac{(n+1)}{2}\,|a_n|\,|z|^{n-1}\,(n-\delta)}{1 - \sum_{2}^{\infty} \frac{(n+1)}{2}\,|a_n|\,|z|^{n-1}}$$

$$\leqslant (\delta - \lambda)$$

Also making use of the coefficient inequality the lemma 5.1

and solving for $|z|$, we get

$$|z| \leqslant \left[\frac{n\left\{1+\beta(2\xi-1)\right\}(2\delta-1-\lambda)}{\beta\xi(1-\alpha)(n+1)(n-2\delta-\lambda)}\right]^{1/(n-1)}$$

We state some particular cases in the following :

<u>Corollary :1</u> :- If $F(z) \in P^{*}(\alpha)$, where

$$P^{*}(\alpha) = D(\alpha) \cap T$$

and $f(z) = 1/2\cdot\left(zF(z)\right)'$, then $F(z)$ is starlike of order λ , type δ in

$$|z| \leqslant \left[\frac{n\,(1+\alpha)\,(2\delta-1-\lambda)}{\alpha\,(n+1)(n-2\delta-\lambda)}\right]^{1/(n-1)}$$

This new result would be obtained for the class $D(\alpha)$ defined by Caplinger [1]

Corollary :2:- If $F(z) \in D(\alpha, 1, \beta) \cap T$, and $f(z) = 1/2 \left(zF(z)\right)'$, then $F(z)$ is starlike of order λ, type δ, **but having negative coefficients.**

$$|z| \leqslant \left[\frac{n \quad (2) \quad (2\delta - 1 - \lambda)}{(1-\alpha)(n+1)\ (n-2\delta - \lambda)} \right]^{1/(n-1)}$$

This would be a new result for Juneja-Mogra [4] .

Corollary :3:- If $F(z) \in D(\alpha, \beta, 1) \cap T = P^{*}(\alpha, \beta)$ and $f(z) = 1/2 \left(zF(z)\right)'$, then $F(z)$ is starlike of order λ, type δ, **This class was studied by Gupta and Jain [3] .**

$$|z| \leqslant \left[\frac{n(1+\beta)\ (2\delta - 1 - \lambda)}{\beta(1-\alpha)(n+1)(n-2\delta - \lambda)} \right]^{1/(n-1)}$$

We state and prove a very new result for the combined class $D(\alpha, \beta, \xi)$ and T,

Theorem :3.3:- If $F(z) = z - \sum_{2}^{\infty} |a_n| z^n \in D(\alpha, \beta, \xi) \cap T$ and $f(z) = 1/2 \left(zF(z)\right)'$. then Re $f'(z) > \lambda$ for $|z| < r$

Proof :- We prove that $|f'(z) - 1| \leqslant (1-\lambda)$

By the definition of $f(z)$, we have

$$\left|f'(z) - 1\right| \leqslant \sum_{2}^{\infty} \frac{n(n+1)}{2} \quad |a_n| \; |z|^{n-1}$$

Hence $\left|f'(z) - 1\right| \leqslant (1-\lambda)$ if

$$\sum_{2} n \frac{(n+1)}{2} \quad |a_n| \quad |z|^{n-1} \leqslant 1 - \lambda \qquad 0 \leqslant \lambda < 1$$

But we have

$$\sum_{2}^{\infty} \frac{n\,|a_n| \; \{1 + B(2\xi - 1)\}}{2\beta\xi\,(1-\alpha)} \leqslant 1$$

Therefore,

$$\sum_{2}^{\infty} \frac{n(n+1) \;\; |a_n|\,|z|^{n-1}}{2(1-\lambda)} \leqslant \frac{\sum_{2}^{\infty} n \;\; |a_n| \; (1+\beta(2\xi - 1)}{2\beta\,\xi(1-\alpha)}$$

Implies,

$$|z| \leqslant \left[\frac{(1-\lambda)\;\left(1+\beta\;(2\xi - 1)\right)}{(n+1)\;\beta\xi(1-\alpha)}\right]^{1/(n-1)}$$

We put some particular cases :

<u>Corollary :1:-</u> If $F(z) \in D(0,\alpha,1) \cap T$ and $f(z) = 1/2(zF(z))'$

Then, $\operatorname{Re} f'(z) > \lambda$, for $|z| < r$, **but having negative coefficients**

$$|z| \leqslant \left[\frac{(1-\lambda)(1+\alpha)}{\alpha(n+1)}\right]^{1/(n-1)}$$

This is due to Caplinger [1] .

Corollary :2: - If $F(z) \in D(\alpha, 1, \beta) \cap T$, and $f(z) = 1/2\left(zF(z)\right)'$ then Re $f'(z) > \lambda$, for $|z| < r$, but having negative coefficients

$$|z| \leqslant \left[\frac{2\,(1-\lambda)}{(n+1)\,(1-\alpha)}\right]^{1/(n-1)}$$

This is a new result obtained for the class defined by Juneja - Mogra [4].

Corollary :3:- If $F(z) \in P^{*}(\alpha, \beta) = D(\alpha, \beta, 1) \cap T$, and $f(z) = 1/2\,\left(zF(z)\right)'$, then Re $f'(z) > \lambda$ for $|z| < r$, given by

$$|z| \leqslant \left[\frac{(1-\lambda)(1+\beta)}{(n+1)\,\beta(1-\alpha)}\right]^{1/(n-1)}$$

This is a new result obtained for the class defined by Gupta and Jain [3].

R E F E R E N C E S

1. Caplinger, T.R. :-

On Certain Classes of analytic functions, Ph.D. Diss. University of Mississipi, (1972).

2. Gupta, V.P. :-

Convex Class of Starlike functions, Yokohama Mathematical Journal, Vol 32, (1984) PP. 55 - 59.

3. ———— and Jain, P.K. :-

Certain classes of Univalent functions with negative coefficients, Bull. Austral. Math. Soc. 14(1976),PP-409-416.

4. Juneja,O.P. and Mogra, M.L. :-

Radii of convexity for certain classes of Univalent analytic functions, Pacific, Jour. Math. 78, (1978) PP - 359 - 368.

5. Kulkarni, S.R. :-(Thakare,N.K.) :-

Some problems connected with Univalent functions, Ph.D. Thesis(1982)(unpublished), Shivaji University,Kolhapur.

6. Lakshminarsimhan, T.V. :-

On subclasses of functions starlike in the disc. Jour. Indian Math. Soc. 41(1977), PP - 233 - 243.

7. Sarangi, S.M. and Uralegaddi, B.A. :-

The radius of Univalence of certain analytic functions with negative coefficients, The Mathematics Student, Vol. 47, No. 4 (1979), PP-305-309.

8. Silverman H. :-

Univalent functions with negative coefficients, Proc. Amer. Math. Soc. Vol. 51, No.1, Aug.(1975) PP-109-116.

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