# CHAPTER - III

## P-SPACE FOR THE MAGNETOFLUID

### 3.1 Introduction :

The dynamical features of the magnetofluid space-time are exploited under the symmetry of p-space. The effect on the parameters associated with the time-like trajectories due to p-space is examined. Also the behaviour of magnetic lines in the p-space of the magnetofluid is investigated. Further the Ricci rotation ~~of~~ coefficients are introduced and their application is made in some dynamical relations.

The second section is devoted to the consequences of p-space. The space matter current tensor is described in this section. ~~Here the~~ A condition for ~~the~~ a transformation of p-space into c-space is obtained. It is deduced that for p-space $(T-3\psi)$ is constant and the divergence of the space matter current tensor vanishes identically. The following results are obtained in the third section :

1) (a) For p-space, the magnetic field $h^b$ is divergence free if and only if the value $\left(\frac{\mu h^2}{2} + 2\psi\right)$ conserves along the magnetic field lines.

   (b) The quantity $\left(\frac{\mu h^2}{2} + 2\psi\right)$ is left invariant along the magnetic field if and only if the four accelaration is orthogonal to magnetic field lines.

2) Streamlines are expansion free if and only if $(\psi = -\varsigma f + c)$, where $c$ is any arbitrary constant.

Further some consequences of the typical fluid flows in p-space of the magnetofluid are examined.

## 3.2 Space matter current tensor for the magnetofluid :

### a) Space matter current tensor through stress tensor -

The defining expression for the space matter current tensor $J^{*}_{abc}$ is (Shaha, 1974)

$$J^{*}_{abc} = 2\, P^{n}{}_{abc;n} \; . \tag{3.2.1}$$

From equation (1.4.12) we have

$$P^{n}{}_{.abc} = R^{n}{}_{.abc} \pm S^{n}{}_{.abc} \pm \psi\, g^{n}{}_{.abc}$$

This equation together with (1.4.13) and (1.4.14) produces

$$P^{n}{}_{.abc} = R^{n}{}_{.abc} + \tfrac{1}{2}\,[\, g^{n}{}_{.b}\, T_{ac} - g^{n}{}_{.c}\, T_{ab} + g_{ac} T^{n}{}_{.b} -$$

$$- g_{ab}\, T^{n}{}_{.c}] + \psi\, [g^{n}{}_{.c}\, g_{ab} - g^{n}{}_{.b} g_{ac}\,],$$

i.e.

$$2\, P^{n}{}_{.abc} = 2\, R^{n}{}_{.abc} + g^{n}{}_{.b}\, T_{ac} - g^{n}{}_{.c}\, T_{ab} + g_{ac} T^{n}{}_{.b} -$$

$$- g_{ab} T^{n}{}_{.c} + 2\psi [g^{n}{}_{.c} g_{ab} - g^{n}{}_{.b}\, g_{ac}\,] \; .$$

The divergence of this yields

$$2P^{n}{}_{.abc;n} = 2R^{n}{}_{.abc;n} + (g^{n}{}_{.b}\, T_{ac})_{;n} - (g^{n}{}_{.c}\, T_{ab})_{;n} +$$

$$+ (g_{ac} T^{n}_{.b})_{;n} - (g_{ab}T^{n}_{.c})_{;n} + [2\Psi(g^{n}_{.c}g_{ab} -$$

$$- g^{n}_{.b}\, g_{ac})]_{;n} ,$$

i.e. $2\, P^{n}_{.abc;n} = 2R^{n}_{.abc;n} + g^{n}_{.b}\, T_{ac;n} - g^{n}_{.c}\, T_{ab;n} +$

$$+ 2\Psi_{,n}[\, g^{n}_{.c}\, g_{ab} - g^{n}_{.b}\, g_{ac}\,] ,$$

i.e. $2P^{n}_{.abc;n} = 2\, R^{n}_{.abc;n} + T_{ac;b} - T_{ab;c} +$

$$+ 2\,\Psi_{,c}\, g_{ab} - 2\,\Psi_{,b} g_{ac} \quad . \qquad (3.2.2)$$

The well known Bianchi identities are

$$R^{n}_{.abc;n} + R^{n}_{.acn;b} + R^{n}_{.anb;c} = 0 ,$$

i.e. $R^{n}_{.abc;n} = -\,(R^{n}_{.acn;b} + R^{n}_{.anb;c}),$

i.e. $R^{n}_{.abc;n} = -\,(-R^{n}_{abn;c}) - R_{ac;b}$ .

Thus we have

$$R^{n}_{.abc;n} = R_{ab;c} - R_{ac;b} \; . \qquad (3.2.3)$$

It follows from equations (3.2.2) and (3.2.3) that

$2\, P^{n}_{.abc;n} = 2[R_{ab;c} - R_{ac;b}\,] + T_{ac;b} - T_{ab;c} +$

$$+ 2\Psi_{,c}\, g_{ab} - 2\Psi_{,b}\, g_{ac} ,$$

i.e. $2\, P^{n}_{.abc;n} = 2[\, (T_{ab;c} - \tfrac{1}{2}\, g_{ab}T_{,c}) - (T_{ac;b} -$

$$- \tfrac{1}{2}\, g_{ac}\, T_{,b})\,] + T_{ac;b} - T_{ab;c} +$$

$$2\,\psi_{,c}\, g_{ab} - 2\psi_{,b}\, g_{ac} \, . \qquad [\text{ Vide } (1.3.1)\ ]$$

The simplified version of this is

$$2\, P^{n}{}_{.abc;n} = T_{ab;c} - T_{ac;b} - g_{ab}\,[\, T - 2\psi\,]_{,c} +$$

$$+\, g_{ac}\,[\, T - 2\psi\,]_{,b}\, . \qquad (3.2.4)$$

Accordingly the space matter current tensor in terms of stress energy tensor becomes

$$J^{*}{}_{abc} = T_{ab;c} - T_{ac;b} - g_{ab}[T - 2\psi]_{,c} + g_{ac}[T - 2\psi]_{,b}\, . \qquad (3.2.5)$$

b) <u>Space matter current tensor for perfect magnetofluid</u> :

In case of the perfect magnetofluid (3.2.5) yields

$$J^{*}{}_{abc} = [\, Au_a u_b - Bg_{ab} - \mu h_a h_b\,]_{;c} - [\, Au_a u_c - Bg_{ac} - \mu h_a h_c\,]_{;b} -$$

$$-\, g_{ab}[A - 4B + \mu h^2 - 2\psi\,]_{,c} + g_{ac}[\, A - 4B + \mu h^2 - 2\psi]_{,b},$$

i.e. $$J^{*}_{abc} = A_{,c} u_a u_b + Au_{a;c} u_b + Au_a u_{b;c} - \mu h_{a;c} h_b - \mu h_a h_{b;c} -$$

$$-\, A_{,b} u_a u_c - Au_{a;b} u_c - Au_a u_{c;b} + \mu h_{a;b} h_c +$$

$$+\, \mu h_a h_{c;b} + C_{,b}\, g_{ac} - C_{,c}\, g_{ab}\ , \qquad (3.2.6)$$

where $C = A - 3B + \mu h^2 - 2\psi$ . (3.27)

(3.2.6) can also be written as

$$J^{*}{}_{abc} = 2\, \{ u_a A_{,[c} u_{b]} + Au_{a;[c} u_{b]} + Au_a u_{[b;c]} -$$

$$- \mu h_{a;[c} h_{b]} - \mu h_a h_{[b;c]} + C_{,[b}\, g_{c]a}\, \}\, . \qquad (3.2.8)$$

We prove some important lemmas for the perfect magnetofluid.

Lemma (1) : P-space implies C-space when $\psi = \frac{1}{3} T$ .

Proof : We have the expression for the space matter current tensor (Asgekar, 1976)

$$J^{*}_{abc} = T_{ab;c} - T_{ac;b} + \frac{1}{3} g_{ac} T_{,b} - \frac{1}{3} g_{ab} T_{,c} . \qquad (3.2.9)$$

On employing the conditions for P-space (1.4.17) and C-space (1.4.11) and making use of the expressions (3.2.5) and (3.2.9) we have

$$\frac{1}{3} g_{ac} T_{,b} = g_{ac} (T-2\psi)_{,b}$$

OR $\frac{1}{3} g_{ab} T_{,c} = g_{ab} (T-2\psi)_{,c}$ ,

i.e. $\frac{1}{3} g_{ac} T_{,b} = g_{ac} T_{,b} - 2 g_{ac} \psi_{,b}$ ,

i.e. $g_{ac} [ \frac{1}{3} T-T ]_{,b} = - 2g_{ac} \psi_{,b}$ ,

i.e. $- \frac{2}{3} T_{,b} = - 2 \psi_{,b}$ .

In particular we have

$$\psi = \frac{1}{3} T . \qquad (3.2.10)$$

Hence the derivation is complete.

Lemma (2) : For P-space the quantity $(T-3\psi)$ is constant.

Proof : In case of P-space we have

$$J^{*}_{abc} = T_{ab;c} - T_{ac;b} + g_{ac}( T-2\psi)_{,b} - g_{ab} (T-2\psi)_{,c} = 0 .$$

This on contraction gives

$$T^{a}_{.b;a} - T^{a}_{.a;b} + g^{a}_{.a}\,[T-2\psi]_{,b} - g^{a}_{.b}[T-2\psi]_{,a} = 0 ,$$

i.e. $- T_{,b} + 4\,[T-2\psi]_{,b} - [T-2\psi]_{,b} = 0$ ,

i.e. $(2T - 6\psi)_{,b} = 0$ ,

i.e. $(T - 3\psi)_{,b} = 0$ ,

i.e. $(T - 3\psi) = \text{Constant}.$ (3.2.11)

This justifies the statement of the lemma.

Lemma (3) : For the relativistic fluid with particals of "zero rest mass" the divergence of $J^{*}_{abc}$ vanishes identically.

Proof : We have T = 0 , for the fluid with particles at zero rest mass (Radhakrishna 1973)

This constraint with (3.2.5) yields

$$J^{*}_{abc} = T_{ab;c} - T_{ac;b} \qquad (3.2.12)$$

This gives

$$J^{*a}_{.bc;a} = T^{a}_{b;ca} - T^{a}_{.c;ba} . \qquad (3.2.13)$$

It follows from contracted Bianchi identities and Ricci identities that

$$T^{a}_{.b;mn} - T^{a}_{.b;nm} = T^{a}_{.c}R^{c}_{.bmn} - T^{.c}_{b}\,R^{.a}_{c.mn} . \qquad (3.2.14)$$

From the equations (3.2.12) and (3.2.14) we get

$$J^{*a}_{.bc;a} = T^{a}_{.m}(R^{m}_{.bca} - R^{m}_{.cba}) + T^{m}_{.c}(R^{.a}_{m.ba}) -$$

$$- T^{.m}_{b}\,R^{.a}_{m.ca} .$$

As the stress energy tensor is symmetric

$$J^{*a}{}_{.bc;a} = T_{mb}\, R^{m}{}_{.c} - T^{m}{}_{.c} R_{mb} ,$$

i.e. $$J^{*a}{}_{.bc;a} = R_{mc}\, R^{m}{}_{.b} - R_{mb}\, R^{m}{}_{.c} ,$$

i.e. $$J^{*a}{}_{.bc;a} = 0 .$$

Hence the proof of the lemma is complete.

## 3.3 Consequences of P-space :

We prove some theorems for p-space in the magnetofluid.

Theorem (I) : For P-space of the magnetofluid the magnetic field is divergence free if and only if ( $\frac{\mu h^2}{2} + 2\psi$) conserves along the magnetic field lines.

Proof : The expression for the space matter current tensor is (vide 3.2.6)

$$J^{*}{}_{abc} = A_{,c} u_a u_b + A u_{a;c} u_b + A u_a u_{b;c} - \mu h_{a;c} h_b - \mu h_a h_{b;c} - A_{,b} u_a u_c - A u_{a;b} u_c - A u_a u_{c;b} + \mu h_{a;b} h_c + \mu h_a h_{c;b} + C_{,b} g_{ac} - C_{,c} g_{ab} . \quad (3.3.1)$$

From the P-space condition, we have

$$J^{*}{}_{abc} = 0 ,$$

i.e. $$A_{,c} u_a u_b + A u_{a;c} u_b u_b + A u_a u_{b;c} - \mu h_{a;c} h_b - \mu h_a h_{b;c} - A_{,b} u_a u_c - A u_{a;b} u_c - A u_a u_{c;b} + \mu h_{a;b} h_c + \mu h_a h_{c;b} + C_{,b} g_{ac} - C_{,c}\, g_{ab} = 0 . \quad (3.3.2)$$

The result $J^{*}_{abc}u^a$ with equation (3.2.2) produces

$$A_{,c}u_b + Au_{b;c} - \mu u^a h_{a;c}h_b - A_{,b}u_c - Au_{c;b} +$$

$$+ \mu u^a h_{a;b}h_c + C_{,b}u_c - C_{,c}u_b = 0 . \qquad (3.3.3)$$

Similarly $J^{*}_{abc}\, u^a u^b = 0$ , implies

$$A_{,c} + C_{,b}\, u^b u_c - A_{,b}u^b u_c - Au_{c;b}u^b +$$

$$+ \mu h_{a;b}\, u^a u^b h_c - C_{,c} = 0 , \qquad (3.3.4)$$

and the result

$$J^{*}_{abc}\, u^a u^b h^c = 0$$

gives

$$(A-C)_{,c}h^c - (\varsigma f+\mu h^2)\,(-h_{c;b}u^c u^b) - \mu h^2 h_{a;b}u^a u^b = 0 ,$$

i.e. $[\, A-A+3B - \mu h^2+2\psi\, ]_{,c}\, h^c + [\varsigma f]h_{a;b}u^a u^b = 0$ .

By the usage of Maxwell equations (1.3.6) we get

$$[\, 3B - \mu h^2+2\psi\, ]_{,c}h^c + [r+p]\, h^c{}_{;c} = 0 ,$$

i.e. $[3(p + \mu\,(1-\frac{\mu}{2})\, h^2\,) - \mu h^2 + 2\psi\, ]_{,c}h^c + [r+p]h^c{}_{;c} = 0,$

i.e. $[3(p+ \mu/2(1-\mu)\, h^2) + \frac{3}{2}\,\mu h^2 - \mu h^2 + 2\psi\, ]_{,c}h^c +$

$$+ [\, r+p\, ]\, h^c{}_{;c} = 0 ,$$

i.e. $3[\, p+ \frac{\mu}{2}\,(1-\mu)\, h^2\, ]_{,c}\, h^c + [\, \frac{\mu h^2}{2} + 2\psi\, ]_{,c}h^c +$

$$+ [\, r+p\, ]\, h^c{}_{;c} = 0$$

Then it follows from the equation (2.2.10)

$$- 3\,[r+p]\, h^{c}{}_{;c} + \left[ \frac{\mu h^2}{2} + 2\psi \right]_{,c} h^{c} + [\, r+p\, ] h^{c}{}_{;c} = 0,$$

i.e. $$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,c} h^{c} = 2[r + p] h^{c}{}_{;c} \,. \qquad (3.3.5)$$

This shows that the quantity $\left( \frac{\mu h^2}{2} + 2\psi \right)$ is invariant along the magnetic field lines.

Corollarly : In p-space for the magnetofluid the quantity $\left( \frac{\mu h^2}{2} + 2\psi \right)$ is conserved along the magnetic field lines if and only if the four-accelaration is orthogonal to the magnetic field.

Proof : From the result (3.3.5) we have $\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^{b} = 2\,[\, r + p\,]\, h^{b}{}_{;b}$ .

On coupling this with the consequence of Maxwell equations (1.3.6) we obtain

$$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^{b} = -2\,[r + p]\, \dot{u}_b h^{b} \,. \qquad (3.3.6)$$

This shows that the value $\left( \frac{\mu h^2}{2} + 2\psi \right)$ is conserved along the magnetic field lines if and only if the four accelaration is orthogonal to the magnetic field.

Remark : The result (3.3.6) in terms of Ricci rotation coefficients can be translated as

$$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^{b} = -2[r+p]\, [\gamma_{414} V_b + \gamma_{424} W_b + \gamma_{434} N_b] h^{b}.$$

If $h^{a}$ is along one of the tetrad vectors then this implies that

$( \frac{\mu h^2}{2} + 2\psi ) = K$ constant $\Rightarrow \underset{414}{\gamma} = 0$ or

$\underset{424}{\gamma} = 0$ or $\underset{434}{\gamma} = 0$ .

Theorem(2) : In p-space of the magnetofluid the stream-lines are expansion free if and only if $(\xi f+\psi)$ conserves along the flow.

Proof : The result, $J^*_{abc}\, g^{ab}$ with (3.3.2) gives

$$A_{,c} - \mu h_{a;c}h^a - \mu h_{b;c}h^b - A_{,b}u^b u_c - A\, u^b_{;b}u_c -$$

$$- A\, u^b u_{c;b} + \mu h^b_{;b}h_c + \mu\, h^b h_{c;b} + C_{,b}g_{ac}g^{ab} -$$

$$- 4\, C_{,c} = 0 \; . \qquad (3.3.7)$$

Also, $J^*_{abc}\, g^{ab}\, u^c = 0$ , gives

$$A_{,c}u^c - \mu h_{a;c}h^a u^c - \mu h_{a;c}h^a u^c - A_{,b}u^b - Au^b_{;b} +$$

$$+ \mu h_{c;b}\, h^b u^c + C_{,b}u^b - 4\, C_{,c}u^c = 0 \; ,$$

i.e. $$\mu h^2_{,b}\, u^b - Au^b_{;b} + \mu h_{c;b}h^b\, u^c - 3C_{,c}u^c = 0 \; . \qquad (3.3.8)$$

This with Maxwell equations produces

$$[\ \mu h^2 - 3C + \tfrac{1}{2}\,\mu h^2\ ]_{,b}\, u^b - [\ r+p\ ]\theta = 0 \; ,$$

i.e. $$[\ r+p\ ]\,\Theta = 3\,[\,\frac{\mu h^2}{2} - C\,]_{,b}u^b \; ,$$

i.e. $$[\ r+p\ ]\,\Theta = 3\,[\ 2p-r + \tfrac{3}{2}\,\mu h^2 - \tfrac{3}{2}\,\mu^2 h^2 + 2\psi]_{,b}u^b \; ,$$

i.e. $$[\ r+p\ ]\,\Theta = -g\,\{[\ r - \frac{\mu h^2}{2} + \frac{\mu^2 h^2}{2}\ ] -$$

$$- [\ \tfrac{2}{3}\,r + \tfrac{2}{3}\,p + \tfrac{2}{3}\,\psi\ ]\,\}_{,b}u^b \; ,$$

i.e. $[r+p]\ \Theta = 9[r+p]\ \Theta + \frac{2}{3}\ [\ r+p+\psi\ ]_{,b}u^b$ ,

i.e. $-8\ [\ r+p\ ]\Theta = \frac{2}{3}\ [\ r+p+\psi\ ]_{,b}u^b$,

i.e. $-\ 8\ (\varrho f)\ \Theta = \frac{2}{3}\ (\varrho f+\psi)_{,b}u^b$ . (3.3.9)

Consequentaly,

$$\Theta = 0 \Leftrightarrow (\ \varrho f+\psi\ )_{,b}\ u^b = 0\ .$$

Hence the theorem is proved.

Remark : On employing the Ricci rotation coefficients in (3.3.9) we have

$$8\ [r+p]\ [\underset{411}{\gamma} + \underset{422}{\gamma} + \underset{433}{\gamma}] = \frac{2}{3}\ [\varrho f+\psi\ ]_{,b}\ u^b\ ,$$

i.e. $\underset{411}{\gamma} + \underset{422}{\gamma} + \underset{433}{\gamma} = 0 \Leftrightarrow (\varrho f+\psi)_{,b}u^b = 0$ .

Hence the necessary and sufficient condition for the conservation of $(\varrho f+\psi)$ along the world line is

$$\underset{411}{\gamma} + \underset{422}{\gamma} + \underset{433}{\gamma} = 0\ .$$

## 3.4 The special flows of the magnetofluid in P-space :

The generalization of "Ferraro's theorem of isorotation" is derived by Yodzis (1971) in the form

$$w^2_{\ ,a}\ h^a = \frac{1}{4}\ R_{abcd}\ u^a w^{bc} h^d\ ,$$

where $w^2$ is the magnitude of the rotation tensor $w_{ab}$ . The compatibility condition for the killing flow of the steady state magnetofluid is derived by Khade (1973). For self-gravitating magnetofluid admitting an essentially expanding flow, Khade (1973) has proved that the magnitude of the magnetic field vector cannot be conserved along the flow while the pressure is conserved along the magnetic lines. Shaha (1974) showed that for definit material scheme in RMHD when matter energy density is uniform the Born rigid flow implies adiabatic flow. In case of boost flow of the magnetofluid with uniform heat flux vector the law of conservation of baryons characterizes the adiabatic flow (Asgekar 1976). For the geodesic flow of the charged fluid with the electric field orthogonal to magnetic one, Asgekar (1978) has obtained the following results (a) the magnetic lines are divergence free if and only if the electric field vector is normal to the vortex lines and (b) the electric field vector is divergence free if the magnetic field vector is orthogonal to the vortex lines when the residual charge vanishes. We examine the magnetofluid flows under the symmetry of p-space.

We have the equations

$$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^b = 2\left[ r+p \right] \dot{u}_b h^b \qquad (3.3.6)$$

and

$$-8 \left[ r+p \right] \theta = \frac{2}{3} \left[ \rho f+\psi \right]_{,b} u^b . \qquad (3.3.9)$$

For different fluid flows these two equations produce the following results ;

Geodesic flow :

From (3.3.6) and (0.5.1) we have

$$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^b = 0 \,. \qquad (3.4.1)$$

Essentially expanding flow :

Making use of (0.5.2) in (3.3.6) we get

$$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^b = 0 \,. \qquad (3.4.1)$$

Killing flow :

It follows from equations (0.5.3), (3.3.6) and (3.3.9) that

$$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^b = 0 \,, \qquad (3.4.1)$$

and

$$\left[ \varsigma f + \psi \right]_{,b} u^b = 0 \,. \qquad (3.4.2)$$

Born rigid flow :

By the usage of (0.5.4), (3.3.9) takes the form

$$\left[ \varsigma f + \psi \right]_{,b} u^b = 0 \,. \qquad (3.4.2)$$

Harmonic flow :

On account of (0.5.5), (3.3.6) and (3.3.9) becomes

$$\left[ \frac{\mu h^2}{2} + 2\psi \right]_{,b} h^b = 0 \,, \qquad (3.4.1)$$

and

$$\left[ \varsigma f + \psi \right]_{,b} u^b = 0 \,. \qquad (3.4.2)$$

Boost flow :

Equation (0.5.6) with (3.3.9) produces

$$[\,\varrho f + \psi\,]_{,b}\, u^b = 0 \,. \qquad (3.4.2)$$

Steady state magnetofluid :

Employing (0.5.7), (3.3.9) can be put in the form

$$[\,\varrho f + \psi\,]_{,b}\, u^b = 0 \,. \qquad (3.4.2)$$

Thus from (3.4.1) and (3.4.2) we observe that the value of the potential $\psi$ coincides with $\left( c - \frac{\mu h^2}{4} \right)$ along magnetic field lines when the flow of the magnetofluid is either geodesic, or essentially expanding or killing or harmonic. Moreover $\psi = c - \varrho f$ along the flow lines when the flow is either Born rigid, or killing or harmonic or boost or steady state magnetofluid.