CHAPTER - 0

MATHEMATICAL PRELIMINARIES

0.1 Introduction :

In this chapter we present some basic mathematical concepts and definitions used in this dissertation. Second section deals with curvature tensor and its properties. The third section deals with time-like, space-like and null congruences. The concepts of Tetrad formalism, Ricci rotation coefficients and Newman-Penrose formalism are given in fourth section. Last section includes the definitions of typical flows of the magnetofluid.

0.2 Riemann Christoffel curvature tensor :

The Riemann Christoffel curvature tensor of rank four is defined through

$$R^{a}_{.\,bcd} \equiv \frac{\partial}{\partial x^{c}} \Gamma^{a}_{bd} - \frac{\partial}{\partial x^{d}} \Gamma^{a}_{bc} + \Gamma^{e}_{bd} \Gamma^{a}_{ce} - - \Gamma^{e}_{bc} \Gamma^{a}_{de} . \qquad (0.2.1)$$

Here Γ^{a}_{bc} is the Christoffel symbol of second kind with expression

$$\Gamma^{a}_{bc} = \frac{1}{2} g^{ad} \left[g_{bd,c} + g_{dc,b} - g_{bc,d} \right] . \qquad (0.2.2)$$

Riemann Christoffel curvature tensor is the only tensor which can be constructed from the metric tensor and its first and second derivatives. It has 256 components of which 20 are independent. It satisfies the following properties.

$$R_{[abcd]} = 0, \quad R_{a[bcd]} = 0, \quad R_{abcd} = R_{[ab][cd]} = R_{[cd][ab]} \; . \tag{0.2.3}$$

The contraction of $R^{a}_{\cdot bcd}$ produces Ricci tensor

$$R_{ab} = g^{cd} R_{abcd} \; . \tag{0.2.4}$$

In terms of Christoffel symbols it can be expressed as

$$R_{ab} = \frac{\partial}{\partial x^{c}} \Gamma^{c}_{ab} - \frac{\partial}{\partial x^{b}} \Gamma^{c}_{ac} + \Gamma^{c}_{dc} \Gamma^{d}_{ab} - \Gamma^{c}_{db} \Gamma^{d}_{ac} \; . \tag{0.2.5}$$

It is the symmetric tensor of rank two, i.e.

$$R_{ab} = R_{(ab)} \; . \tag{0.26}$$

This Ricci curvature tensor has only 10 independent components.

0.3 Congruences and associated parameters :

Synge (1962) defined a congruence as a system of curves filling a portion of space, and such that in general a single curve passes through any assigned point. The null congruences exist in gravitational radiations, time-like congruences in cosmological models and space-like congruences in self-gravitating magnetofluids. This establishes the importance of the study of congruences in the theory of general relativity.

Parameters associated with time-like congruences :

Mathematically time-like curves are expressed in terms of parameters as

$$\bar{X}^a = X^a \left(m^\alpha , \tau \right), \qquad (0.3.1)$$

where m^α denotes Lagrangian coordinates and τ is the parameter along world line of the fluid element. The four-velocity vector u^a at a point along one of the streamlines is defined as

$$u^a = \frac{dx^a}{d\tau} \quad (m^\alpha \text{ fixed}), \qquad (0.3.2)$$

with normalising condition

$$u^a u_a = 1, \quad \left[u^a g_{ab} u^b = 1 \right], \qquad (0.3.3)$$

which gives

$$u^a{}_{;b}\, u_a = 0 . \qquad (0.3.4)$$

The expressions for expansion scalar θ, shear tensor σ_{ab} and rotation tensor W_{ab} are given by (Greenberg 1970,a)

$$\theta = u^a{}_{;a} \quad , \qquad (0.3.5)$$

$$\sigma_{ab} = u_{(a;b)} - \dot{u}_{(a} u_{b)} - \frac{1}{3}\,\theta\, P_{ab}, \qquad (0.3.6)$$

$$W_{ab} = u_{[a;b]} - \dot{u}_{[a} u_{b]} . \qquad (0.3.7)$$

The four accelaration is defined as

$$\dot{u}_a = u_{a;b}\, u^b \ , \tag{0.3.8}$$

and the three-space projection operator P_{ab} is described by

$$P_{ab} = g_{ab} - u_a u_b \ . \tag{0.3.9}$$

It follows from (0.3.6) to (0.3.9)

$$\sigma_{ab} = \sigma_{(ab)} \ , \quad w_{ab} = w_{[ab]} \ , \tag{0.3.10}$$

$$\sigma^a{}_a = 0 = w^a{}_a \ , \tag{0.3.11}$$

$$\sigma_{ab}\, u^b = w_{ab}\, u^b = \dot{u}_a u^a = 0 \ , \tag{0.3.12}$$

$$P_{ab} u^b = 0, \ P_a^{\cdot a} = 3, \ P_a^{\cdot c}\, P_c^{\cdot b} = P_a^{\cdot b} \tag{0.3.13}$$

Thus shear tensor σ_{ab} is symmetric, rotation tensor w_{ab} is antisymmetric, σ_{ab} and w_{ab} both are trace free. The magnitudes of the shear and rotation tensors are defined as

$$\begin{aligned} \sigma_{ab}\, \sigma^{ab} &= 2\, \sigma^2 \ , \\ w_{ab}\, w^{ab} &= 2\, w^2 \ . \end{aligned} \tag{0.3.14}$$

In terms of expansion, shear tensor, rotation tensor, projection operator and four accelaration the gradient of four velocity can be expressed as

$$u_{a;b} = \sigma_{ab} + w_{ab} + \frac{1}{3}\, \Theta\, P_{ab} + \dot{u}_a u_b \ . \tag{0.3.15}$$

The null congruences defined by Witten (1962) are very much significant as " all astrophysical information comes to us optically i.e. by photons - the history of photons is a null geodesic in space-time". The propogation equation for the expansion is obtained by Raychoudhari (1955). The theory of time-like and null congruences was initiated by Ehlers, Jorden, Kundth and Sachs (1961). The parameters associated to space-like congruences are developed by Greenberg (1970,b). Shaha (1974) has used the kinematical parameters in the study of magnetohydro-dynamics whereas the parameters associated with time-like and space-like congruences were applied by Ghunakikar (1974) in the study of self-gravitating charged fluids with null conductivity. The kinematical parameters are applied by Virkar (1978) in the study of self-gravitating elastic systems. The propogation equations for the kinematical parameters in the space-time filled with a viscous compressible, thermally conductivity, relativistic fluid with infinite electrical conductivity and constant magnetic permeability (magnetofluid) are computed by Asgekar (1979). By obtaining a new solution of Bianchi type VIII for homogeneous cosmological model consisting of perfect fluid, the ratio of matter shear to expansion is derived by Collins et.al. (1980). We examine the space-time of the magnetofluid under particular symmetry conditions and through the parameters associated with time-like trajectories.

0.4 Tetrad formalism :

The symmetric metric tensor has ten independent components while the tetrad field possesses sixteen independent components.

Hence the geometry obtained from tetrad formalism is more general than the Riemannian geometry (McCrea and Mikhail 1956).

The relativistic tetrad formalism was first used by Lichnerowicz (1955) for definite material scheme. The extended version of this is studied by Eisenhert (1964). McCrea and Mikhail (1956) used this tetrad procedure to derive the equations for "C-field theory" given by Hoyle (1948). Ellis (1967) used exploited a set of tetrad formalism for solving Einstein's field equations for pressure free matter. Stewart and Ellis (1968) applied this theory to study a fluid exhibiting local rotational symmetry. In the study of gravitational radiation Newman-Penrose (1962) used a tetrad of a complex null basis. Shaha (1974) has considered the tetrad field in relativistic definite magnetofluid schemes. Lord (1976) derived the special relativistic equations for the electron field given by Dirac in terms of tetrad.

We consider the tetrad of vectors

$$M^{a}_{\ .\alpha} = (V^{a},\ W^{a},\ N^{a},\ u^{a}) . \qquad (0.4.1)$$

Here the Greek index denotes the tetrad suffix and Latin index denotes the coordinate suffix : V^{a}, W^{a}, N^{a} are the space like vectors. These tetrad vectors $M^{a}_{\ .\alpha}$ satisfy the following relations

$$M^{a}_{\ .\alpha}\, M^{\alpha}_{\ .b} = \delta^{a}_{\ .b} , \qquad (0.4.2)$$

$$M^{a}_{\ .\alpha}\ M^{\beta}_{\ .a} = \delta^{\beta}_{\ .\alpha} , \qquad (0.4.3)$$

$$M^{a}_{\cdot\alpha}\, M_{a\beta} = g_{\alpha\beta} , \tag{0.4.4}$$

$$M_{\alpha a}\, M^{\alpha}_{\cdot b} = g_{ab} . \tag{0.4.5}$$

The space-like vectors satisfy the relations

$$V^{a}V_{a} = N^{a}N_{a} = W^{a}W_{a} = -1 . \tag{0.4.6}$$

A second rank tensor X_{ab} in terms of tetrad vectors can be put in the form (Shaha 1974)

$$\begin{aligned} X_{ab} = {} & (X_{cd}\, u^{c}u^{d})\, u_{a}u_{b} + (X_{cd}\, V^{c}V^{d})\, V_{a}V_{b} + \\ & + (X_{cd}\, W^{c}W^{d})\, W_{a}W_{b} + (X_{cd}\, N^{c}N^{d})\, N_{a}N_{b} + \\ & + (X_{cd}\, u^{c}V^{d})\, u_{a}V_{b} + (X_{cd}\, V^{c}u^{d})\, V_{a}u_{b} + \\ & + (X_{cd}\, u^{c}W^{d})\, u_{a}W_{b} + (X_{cd}\, W^{c}u^{d})\, W_{a}u_{b} + \\ & + (X_{cd}\, u^{c}N^{d})\, u_{a}N_{b} + (X_{cd}\, N^{c}u^{d})\, N_{a}u_{b} + \\ & + (X_{cd}\, V^{c}W^{d})\, V_{a}W_{b} + (X_{cd}\, W^{c}V^{d})\, W_{a}V_{b} + \\ & + (X_{cd}\, V^{c}N^{d})\, V_{a}N_{b} + (X_{cd}\, N^{c}V^{d})\, N_{a}V_{b} + \\ & + (X_{cd}\, W^{c}N^{d})\, W_{a}N_{b} + (X_{cd}\, N^{c}W^{d})\, N_{a}W_{b} . \end{aligned} \tag{0.4.7}$$

It follows from (0.4.7) that

$$g_{ab} = u_{a}u_{b} - V_{a}V_{b} - W_{a}W_{b} - N_{a}N_{b} . \tag{0.4.8}$$

<u>Ricci rotation coefficients</u> :

The Ricci rotation coefficients are defined as

$$\gamma_{\alpha\beta\delta} = M_{\alpha a;b}\, M^{a}_{\cdot\beta}\, M^{b}_{\cdot\delta} . \tag{0.4.9}$$

They satisfy the conditions

$$\gamma_{\alpha\beta\delta} = -\gamma_{\beta\alpha\delta} \tag{0.4.10}$$

Kinematical parameters in terms of Ricci rotation coefficients :

The values of the kinematical parameters in terms of Ricci rotation coefficients are (Shaha 1974)

$$\theta = u^{a}{}_{;a} = -(\gamma_{411} + \gamma_{422} + \gamma_{433}) , \tag{0.4.11}$$

$$\begin{aligned} \sigma_{ab} = {} & (\gamma_{411} + \tfrac{1}{3}\theta) V_a V_b + (\gamma_{422} + \tfrac{1}{3}\theta) \cdot \\ & W_a W_b + (\gamma_{433} + \tfrac{1}{3}\theta) N_a N_b + \\ & + \tfrac{1}{2}(\gamma_{412} + \gamma_{421})(V_a W_b + W_a V_b) + \\ & + \tfrac{1}{2}(\gamma_{423} + \gamma_{432})(W_a N_b + N_a W_b) + \\ & + \tfrac{1}{2}(\gamma_{431} + \gamma_{413})(N_a V_b + V_a N_b), \end{aligned} \tag{0.4.12}$$

$$\begin{aligned} w_{ab} = {} & \tfrac{1}{2}(\gamma_{412} - \gamma_{421})(V_a W_b - W_a V_b) + \\ & + \tfrac{1}{2}(\gamma_{423} - \gamma_{432})(W_a N_b - N_a W_b) + \\ & + \tfrac{1}{2}(\gamma_{431} - \gamma_{413})(N_a V_b - V_a N_b) , \end{aligned} \tag{0.4.13}$$

$$\dot{u}_a = \gamma_{414} V_a + \gamma_{424} W_a + \gamma_{434} N_a . \tag{0.4.14}$$

Newman-Penrose (N-P) formalism. (Spin Coefficients) :

To explain the concept of tetrad formalism orthonormal basis is selected ~~while for N-P formalism a complex null basis is~~

~~selected~~ while for N-P formalism a complex null basis is chosen. (Hawking and Israel 1979). The complex null basis $(l,n,m,\bar{m})$ is chosen to explain the concept of N-P formalism. Here l^a, n^a are two real null vectors and m^a, $\bar{m}^a$ are a pair of complex conjugate null vectors. These vectors satisfy the relations

$$l_a l^a = m_a m^a = n_a n^a = \bar{m}_a \bar{m}^a = 0 , \tag{0.4.15}$$

$$l_a n^a = 1 , \tag{0.4.16}$$

$$m_a \bar{m}^a = -1 , \tag{0.4.17}$$

$$l_a m^a = l_a \bar{m}^a = n_a m^a = n_a \bar{m}^a = 0 . \tag{0.4.18}$$

The N-P formalism is found to be amazingly useful in various fields. It is useful in the study of algebraically special fields (Hawking and Israel 1979). Flaherty (1976) expressed the spin coefficients elegantaly in the tabular form.

∇	$l_{a;b}m^a$	$\frac{1}{2}(l_{a;b}n^a + \bar{m}_{a;b}m^a)$	$\bar{m}_{a;b}n^a$
$;_a l^a$	κ	ϵ	π
$;_a \bar{m}^a$	ρ	α	λ
$;_a m^a$	σ	β	μ
$;_a n^a$	τ	γ	ν

Spin coefficients through Ricci rotation coefficients :

The relations between spin coefficients and Ricci rotation coefficients are developed by Newman and Penrose (1962).

$$\kappa = \gamma_{131} , \tag{0.4.19}$$

$$\pi = -\gamma_{241} , \tag{0.4.20}$$

$$\epsilon = \frac{1}{2} (\gamma_{121} - \gamma_{341}) , \tag{0.4.21}$$

$$\rho = \gamma_{134} , \tag{0.4.22}$$

$$\lambda = -\gamma_{244} , \tag{0.4.23}$$

$$\alpha = \frac{1}{2} (\gamma_{124} - \gamma_{344}) , \tag{0.4.24}$$

$$\sigma = \gamma_{133} , \tag{0.4.25}$$

$$\mu = -\gamma_{243} , \tag{0.4.26}$$

$$\beta = \frac{1}{2} (\gamma_{123} - \gamma_{343}) , \tag{0.4.27}$$

$$\nu = -\gamma_{242} , \tag{0.4.28}$$

$$\gamma = \frac{1}{2} (\gamma_{122} - \gamma_{342}) , \tag{0.4.29}$$

$$\tau = \gamma_{132} . \tag{0.4.30}$$

0.5. Special types of flows :

We study the following typical flows :

1) Geodesic flow : The geodesic flow is characterised by

$$\dot{u}_a = 0 . \tag{0.5.1}$$

2) Essentially expanding flow : The essentially expanding flow is characterised by

$$\dot{u}_a = 0 , \quad \sigma_{ab} = 0 \quad \text{and} \quad \omega_{ab} = 0 . \tag{0.5.2}$$

3) Killing flow : The flow is said to be Killing if and only if $\underset{\bar{u}}{£}\, g_{ab} = 0$, where £ is the Lie derivative. This implies

$$\dot{u}_a = 0 , \quad \sigma_{ab} = 0 \quad \text{and} \quad \theta = 0 . \tag{0.5.3}$$

4) Born rigid flow : A continuous medium is said to be kinematically rigid if and only if the time-like vector is shear free and expansion free (Trautman, 1964),

i.e. $$\sigma_{ab} = 0 , \quad \theta = 0 . \tag{0.5.4}$$

This flow is defined to be Born rigid.

5) Harmonic flow : This flow is characterized by

$$\dot{u}_a = 0 , \quad \omega_{ab} = 0 , \quad \theta = 0 , \quad \sigma_{ab} \neq 0 . \tag{0.5.5}$$

6) Boost flow : This flow is characterized by

$$\sigma_{ab} = 0 , \quad \omega_{ab} = 0 , \theta = 0, \quad \dot{u}_a \neq 0 . \tag{0.5.6}$$

7) Steady state magnetofluid : The steady state magnetofluid is defined by (Yodzis 1971)

$$\sigma_{ab} = 0, \quad \theta = 0, \quad h_{a;b}\, u^b = 0, \; W_{ab;c} u^c = 0 \, , \tag{0.5.7}$$

where h^a is the magnetic field vector.

Effect of vanishing kinematical parameters on the Ricci rotation coefficients :

1) We have $\dot{u}_a = \gamma_{414} V_a + \gamma_{424} W_a + \gamma_{434} N_a$. (vide 0.4.14)

$$\therefore \quad \dot{u}_a = 0 \Rightarrow \gamma_{414} V_a + \gamma_{424} W_a + \gamma_{434} N_a = 0 \, ,$$

$$\Rightarrow \quad \gamma_{414} = \gamma_{424} = \gamma_{434} = 0 \, ,$$

$$\Rightarrow \quad \gamma_{A44} = 0 \, . \tag{0.5.8}$$

2) We have from (0.4.12)

$$\sigma_{ab} = (\gamma_{411} + \tfrac{1}{3}\theta) V_a V_b + (\gamma_{422} + \tfrac{1}{3}\theta) W_a W_b +$$

$$+ (\gamma_{433} + \tfrac{1}{3}\theta) N_a N_b + \tfrac{1}{2}(\gamma_{412} + r_{421})(V_a W_b + W_a V_b) +$$

$$+ \tfrac{1}{2}(\gamma_{423} + \gamma_{432})(W_a N_b + N_a W_b) + \tfrac{1}{2}(\gamma_{431} + \gamma_{413})(N_a V_b + V_a N_b)$$

$$\therefore \quad \sigma_{ab} = 0 \, .$$

$$\Rightarrow \left(\underset{411}{\gamma} + \tfrac{1}{3}\Theta\right) V_a V_b + \left(\underset{422}{\gamma} + \tfrac{1}{3}\Theta\right) N_a N_b + \left(\underset{433}{\gamma} + \tfrac{1}{3}\Theta\right) N_a N_b +$$

$$+ \tfrac{1}{2}\left(\underset{412}{\gamma} + \underset{421}{\gamma}\right)(V_a W_b + W_a V_b) + \tfrac{1}{2}\left(\underset{423}{\gamma} + \underset{432}{\gamma}\right)(W_a N_b + N_a W_b) +$$

$$+ \tfrac{1}{2}\left(\underset{431}{\gamma} + \underset{413}{\gamma}\right)(N_a V_b + V_a N_b) = 0 ,$$

$$\Rightarrow \underset{411}{\gamma} - \tfrac{1}{3}\left(\underset{411}{\gamma} + \underset{422}{\gamma} + \underset{433}{\gamma}\right) = 0 ,$$

$$\Rightarrow 2\underset{411}{\gamma} - \underset{422}{\gamma} - \underset{433}{\gamma} = 0 .$$

Similarly $\quad -\underset{411}{\gamma} + 2\underset{422}{\gamma} - \underset{433}{\gamma} = 0 ,$

$$-\underset{411}{\gamma} - \underset{422}{\gamma} + \underset{433}{\gamma} = 0 .$$

Which give

$$\underset{411}{\gamma} = \underset{422}{\gamma} = \underset{433}{\gamma} . \tag{0.5.9}$$

Also

$$\underset{412}{\gamma} = -\underset{421}{\gamma} , \quad \underset{423}{\gamma} = -\underset{432}{\gamma} , \quad \underset{431}{\gamma} = -\underset{413}{\gamma} ,$$

i.e.

$$\underset{4AB}{\gamma} = -\underset{4BA}{\gamma} \quad (A \neq B) . \tag{0.5.10}$$

3) Now, (0.4.13) gives

$$W_{ab} = \left(\underset{412}{\gamma} - \underset{421}{\gamma}\right)(V_a W_b - W_a V_b) +$$

$$+ \left(\underset{423}{\gamma} - \underset{432}{\gamma}\right)(W_a N_b - N_a W_b) +$$

$$+ \left(\underset{431}{\gamma} - \underset{413}{\gamma}\right)(N_a V_b - V_a N_b) .$$

$$\therefore \quad W_{ab} = 0$$

$$\Rightarrow (\gamma_{412} - \gamma_{421})(V_a W_b - W_a V_b) + (\gamma_{423} - \gamma_{432})(W_a N_b - N_a W_b) +$$

$$+ (\gamma_{431} - \gamma_{413})(N_a V_b - V_a N_b) = 0 ,$$

$$\Rightarrow \gamma_{412} = \gamma_{421} , \ \gamma_{423} = \gamma_{432} , \ \gamma_{431} = \gamma_{413} ,$$

$$\text{i.e.} \quad \gamma_{4AB} = \gamma_{4BA} . \quad (A \neq B) . \tag{0.5.11}$$

4) Also we know

$$\theta = -(\gamma_{411} + \gamma_{422} + \gamma_{433}) \quad (\text{ vide } 0.4.11)$$

$$\therefore \ \theta = 0 \Rightarrow -(\gamma_{411} + \gamma_{422} + \gamma_{433}) = 0 ,$$

$$\Rightarrow \gamma_{411} = \gamma_{422} = \gamma_{433} = 0 . \tag{0.5.12}$$

Thus all the defining conditions for the typical flows can be expressed through Ricci rotation coefficients with the help of following table :

Type of flow	Characterization
1) Geodesic flow	$\gamma_{A44} = 0$.
2) Essentially expanding flow	$\gamma_{A44} = 0$, $\gamma_{4AB} = 0$.
3) Killing flow	$\gamma_{A44} = 0$, $\gamma_{411} = \gamma_{422} = \gamma_{433} = 0$, $\gamma_{4AB} = -\gamma_{4BA}$, $(A \neq B)$
4) Born rigid flow	$\gamma_{411} = \gamma_{422} = \gamma_{433} = 0$, $\gamma_{4AB} = -\gamma_{4BA}$, $(A \neq B)$.
5) Harmonic flow	$\gamma_{A44} = 0$, $\gamma_{4AB} = \gamma_{4BA}$, $(A \neq B)$, $\gamma_{411} = \gamma_{422} = \gamma_{433} = 0$.
6) Boost flow	$\gamma_{4AB} = 0$.
7) Steady state magnetofluid	$\gamma_{411} = \gamma_{422} = \gamma_{433} = 0$, $\gamma_{4AB} = -\gamma_{4BA}$, $(A \neq B)$, $\dot{h}_a = 0$, $\dot{W}_{ab} = 0$.