C H A P T E R - I

CHARACTERIZATION OF THE MAGNETOFLUID SCHEME

1.1 Introduction :

For the perfect fluid in which velocities are isotrophically distributed, the mean free paths between the oscillations and time are very short (Weinberg 1972). But generally this is not true. This necessitate the study of more relativistic fluids. So we study the relativistic magnetohydrodynamics comprising highly conducting self-gravitating charged perfect fluid (Shaha 1974).

Although the origin of magnetohydrodynamics is in the Minkowaski's electrodynamics of moving bodies, the discovery that celestial bodies are highly magnetized, is indirectly responsible for the development of magnetohydrodynamics. (Ferraro 1966). Bigelow in 1899 showed that sun is a great magnet, upto which earth is the only known celestial body which is magnetized. Hale in 1908 found that sun-spots are highly magnetic. These discoveries opened a new chapter in the theory of astrophysics.

Parker (1964) has applied the technique of classical magnetohydrodynamics (CMHD) to study the behaviour of magnetic field in solar winds, while Wilson (1968) applied this theory to study the sun-spots and Hewish (1969) to study spiral arms. The gravitational fields in astrophysical events are stronger than electromagnetic fields. This motivates the study of astrophysical events in context of relativistic magnetohydrodynamics (RMHD). The recent discoveries of pulsars, neutran stars, the cosmic microwave radiations and gravitational waves have accelerated the study of RMHD (Patil 1978).

Lichnerowicz (1955), Pandey and Gupta (1970) studied non-perfect schemes in relativistic hydrodynamics. Coburn (1963) examined the effect of self induction of entrophy and basic Cauchy problem in RMHD. In the generalized version of Godels universe, Raval and Vaidya (1966) considered imperfect fluids. The field equations of RMHD are derived by Lichnerowicz (1967). He has established the existence and uniqueness of RMHD solutions. Greenberg (1971 a,b) derived the Post-Newtonian approximations to the equation of RMHD. Lichnerowicz's (1967) theory helped Yodzis (1971) to study galactic cosmology, gravitational collapse and pulsar theory. Date (1972,73,a,b) considered the local behaviour of congruences in RMHD.

The thermodynamical differential equations considered by Lichnerowicz (1967) for the magnetofluid does not incorporate the possible effect of electromagnetic field on internal structure of the fluid. This fact is successfully tackled by Maugin (1972) with the use of action principle in the Eulerian description for general RMHD. Greco (1974) used Maugin (1972) scheme and determined the discontinuities, characteristic equations, associated rays and velocity of propogation. Maugin (1974) has derived the field equations of a non conducting charged perfect fluid interacting with electromagnetic fields and endowed with spins of magnetic origin. from variation principle.

Taub (1970) obtained equations of motion for self gravitating charged fluid with finite electrical conductivity. Self-gravitating magnetofluid schemes were investigated by Shaha (1974) while Asgekar (1976) studied self-gravitating matter distributions in the realm of RMHD. The necessary and sufficient conditions for the perfect magnetofluid to be Lie invariant along the preferred directions are found by Patil and Date (1978). With the use of Maugin (1972) scheme for a relativistic perfect magnetofluid, the relation between convective deformation of the matter tensor and deformation tensor is derived by Patil (1979). Fennelly (1980) introduced magnetic fields and fluids with a finite conductivity in Bianchi type-I universes and found the corresponding consequences. A.Chaljub et.al.(1980) have found some global solutions of Lichnerowicz equations in general relativity on an assymptotically Euclidean complete manifold.

This rapid progress in the field of RMHD has stimulated us to work on relativistic magnetofluid. The dynamical features of space time filled with the magnetofluid under particular symmetry character are to be investigated.

Evolution of the stress energy tensor for the magnetofluid and the properties of this stress energy tensor are presented in second section. It is shown that this stress energy tensor is physically transparent. Third section deals with various space symmetries. Einstein space, C-space and P-space are defined and incompatibility of Einstein space is established in this section. The field equations are given in fourth section. Consequences of Maxwell equations are obtained and these are expressed in terms of Ricci rotation coefficients.

1.2 Evolution of a stress energy tensor for the relativistic magnetofluid :

(a) A stress energy tensor for the electromagnetic field :-

We have the expressions for "the polarization magnetization two-form (G_{ab})", and "electric field magnetic induction two-form (H_{ab})" (Grot and Eringen, 1966,a)

$$H_{ab} = (e_b u_a - e_a u_b) + \frac{1}{\sqrt{-1}} \epsilon_{abcd} \sqrt{-g} B^c u^d, \qquad (1.2.1)$$

$$G^{ab} = (L^a u^b - L^b u^a) + \frac{1}{\sqrt{-1}} \frac{e^{abcd}}{\sqrt{-g}} m_c u_d . \qquad (1.2.2)$$

with e^a as the four electric current,

ϵ_{abcd} as the Levi Civita's alternating symbol,

B^a is the magnetic induction vector

such that

$$B_a u^a = 0 , \qquad (1.2.3)$$

and

$$L^a = G^{ba} u_b . \qquad (1.2.4)$$

According to Lichnerowicz (1967) the approximations for perfect magnetohydrodynamics are infinite electrical conductivity (σ)

and constant magnetic permeability (μ). These assumptions necessiates the electric field vector $\bar{e}$ to vanish. Hence for perfect magnetohydrodynamics the expressions (1.2.1) and (1.2.2) get reduced to

$$H_{ab} = \frac{1}{\sqrt{-1}} \epsilon_{abcd} \sqrt{-g}\, B^c u^d , \qquad (1.2.5)$$

$$G^{ab} = \frac{1}{\sqrt{-1}} \frac{\epsilon^{abcd}}{\sqrt{-g}} m_c u_d , \qquad (1.2.6)$$

where magnetization vector m_a is defined by (Maugin 72)

$$m_a = B_a - h_a . \qquad (1.2.7)$$

Further if the magnetic induction depends linearly on the magnetic field then for isotropic perfect magnetohydrodynamics we have

$$B^a = \mu h^a . \qquad (1.2.8)$$

Thus it follows from (1.2.7) and (1.2.8) that

$$m^a = (1 - \mu^{-1}) B^a = (\mu - 1) h^a , \qquad (1.2.9)$$

which gives

$$u_a m^a = 0 . \qquad (1.2.10)$$

From the equations (1.2.5) and (1.2.6) the general form of the invariant ϕ of the electromagnetic field given by

$$\Phi = \frac{1}{2} F_{ab} F^{ba} , \qquad (1.2.11)$$

and the stress energy tensor for the electromagnetic field without involving $\langle \bar{A} , \bar{J} \rangle$ can be derived in the form

(Maugin 1972)

$$\phi = \frac{1}{2} \left(e_a e^a - B_a B^a \right), \tag{1.2.12}$$

$$T^{ab}_{(em)} = \phi\, g^{ab} - \left(e^a D^b + h^a B^b \right) + B^c h_c P^{ab} + $$
$$+ e_c D^c u^a u^b + u^a V^b + W^a u^b . \tag{1.2.13}$$

Here A_a is the electromagnetic potential, J^a is the four electric current and D^a is the electric displacement vector. The defining expressions for $\bar{V}$ and $\bar{W}$ are

$$V^b = \frac{1}{\sqrt{-1}} \frac{\epsilon^{bacd}}{\sqrt{-g}} e_a h_c u_d , \tag{1.2.14}$$

$$W^b = \frac{1}{\sqrt{-1}} \frac{\epsilon^{bacd}}{\sqrt{-g}} D_a B_c u_d . \tag{1.2.15}$$

In case of thermodynamical perfect fluid with infinite electrical conductivity we have

$$e_a = L_a = D_a = 0 . \tag{1.2.16}$$

Making use of (1.2.12) and (1.2.16) in (1.2.13) we get

$$T^{ab}_{(em)} = -\frac{1}{2} B_c B^c g^{ab} - h^a B^b + B_c h^c p^{ab} .$$

From equation (1.2.8) it follows that

$$T^{ab}_{(em)} = \frac{1}{2} \mu^2 h^2 g^{ab} - \mu h^a h^b - \mu h^2 p^{ab} ,$$

i.e. $$T^{ab}_{(em)} = \frac{1}{2} \mu^2 h^2 g^{ab} - \mu h^a h^b - \mu h^2 \left(g^{ab} - u^a u^b \right) ,$$

i.e. $$T^{ab}_{(em)} = \mu h^2 u^a u^b - \mu \left(1 - \frac{\mu}{2} \right) h^2 g^{ab} - \mu h^a h^b .$$

Thus we get the final form for the stress energy tensor of electromagnetic field as

$$T^{ab}_{(em)} = \mu \left[u^a u^b - \left(1 - \frac{\mu}{2}\right) g^{ab} \right] h^2 - \mu h^a h^b . \qquad (1.2.17)$$

b) <u>The stress energy tensor for thermodynamical perfect fluid</u> :

We consider the stress energy tensor for the relativistic perfect fluid (Lichnerowicz 1967)

$$T^{ab} = (r + p) u^a u^b - p g^{ab} , \qquad (1.2.18)$$

where r is the energy density and p is the isotropic pressure of the fluid. The energy density r consists of a proper material density ϱ and a specific internal energy ϵ, where ϵ is the function of two thermodynamical variables of the fluid ϱ and p,

i.e. $$\epsilon = \epsilon (\varrho , p) . \qquad (1.2.19)$$

We put

$$r = \varrho (1 + \epsilon), \quad \varrho > 0 . \qquad (1.2.20)$$

Hence (1.2.18) becomes

$$T^{ab}_{(m)} = [\varrho (1+\epsilon)+p] u^a u^b - p g^{ab} ,$$

i.e. $$T^{ab}_{(m)} = \varrho (1+\epsilon+p/\varrho) u^a u^b - p g^{ab} .$$

If the specific enthalpy i is defined as

$$i = \epsilon + p/\varrho , \qquad (1.2.21)$$

then

$$T^{ab}_{(m)} = \varrho\,(1+i)\,u^a u^b - pg^{ab} \;,$$

which can be also put in the form

$$T^{ab}_{(m)} = \varrho\, f\, u^a u^b - pg^{ab} \;, \qquad (1.2.22)$$

where f is the fluid index with the expression

$$f = 1 + \epsilon + p/\varrho = 1 + i \;. \qquad (1.2.23)$$

The equation (1.2.22) characterizes the stress energy tensor for the thermodynamical perfect fluid.

c) Lichnerowicz's stress energy tensor for magnetofluid :

Considering the Minkowaski's energy tensor Lichnerowicz (1967) developed the stress energy tensor for the perfect fluid with infinite conductivity and constant magnetic permeability in the form

$$T^{ab}_{(L)} = (\varrho f + \mu h^2)\, u^a u^b - (p + \frac{\mu h^2}{2})\, g^{ab} - \mu h^a h^b \;,$$

where ϱ is the proper matter density,
f is the fluid index and
p is the isotropic pressure.

d) The stress energy tensor for Maugin's magnetofluid scheme :

A new stress energy tensor is constructed by Linear superpositions of stress energy tensors (1.2.17) and (1.2.22) (Maugin 72) .

$$T^{ab} = T^{ab}_{(em)} + T^{ab}_{(m)} \quad , \qquad (1.2.24)$$

where $T^{ab}_{(em)}$ is the stress energy tensor for electromagnetic field given by (1.2.17) and $T^{ab}_{(m)}$ is the stress energy tensor for the thermodynamical perfect fluid given by (1.2.22).

By utilizing (1.2.17) and (1.2.27) in (1.2.24) we get

$$T^{ab} = \varrho f u^a u^b - p g^{ab} + \mu \left[u^a u^b - \left(1 - \frac{\mu}{2}\right) g^{ab} \right] h^2 - \\ - \mu h^a h^b ,$$

i.e. $$T^{ab} = (\varrho f + \mu h^2) u^a u^b - \left[p + \mu\left(1 - \frac{\mu}{2}\right) h^2 \right] g^{ab} - \mu h^a h^b . \qquad (1.2.25)$$

<u>Note</u> : The stress energy tensor (1.2.25) characterizing the magnetofluid is used throughout this dissertation.

<u>Remark</u> : This stress energy tensor reduces to the stress energy tensor given by Lichnerowicz (1967) if $\left(1 - \frac{\mu}{2} = \frac{1}{2}\right)$.

$\mu = 1$

<u>Various aspects of the stress energy tensor for the magnetofluid</u> :

The various contractions of tensor (1.2.25) can be summarized as follows

$$T^{ab} u_a = \left[r + \frac{\mu^2 h^2}{2} \right] u^b , \qquad (1.2.26)$$

$$T^{ab} u_a u_b = r + \frac{\mu^2 h^2}{2} , \qquad (1.2.27)$$

$$T^{ab} g_{ab} = T = r - 3p - 2\mu h^2 + 2\mu^2 h^2 , \qquad (1.2.28)$$

$$T^{ab}h_a = - [p - \frac{\mu^2 h^2}{2}] h^b , \quad (1.2.29)$$

$$T^{ab}h_a h_b = [p - \frac{\mu^2 h^2}{2}] h^2 , \quad (1.2.30)$$

$$T^{ab} u_a h_b = 0 , \quad (1.2.31)$$

$$T^{ab}T_{ab} = (r + \frac{\mu^2 h^2}{2})^2 + 3 (p - \frac{\mu^2 h^2}{2})^2 + + 2 \mu h^2 (2p - \mu^2 h^2 + \mu h^2), \quad (1.2.32)$$

$$T^{ab}P_{ab} = \frac{3}{2} \mu^2 h^2 - 3p - 2 \mu h^2 . \quad (1.2.33)$$

The equation (1.2.27) gives the eigen value ($r + \frac{(\mu h)^2}{2}$) with respect to eigen vector u_a. Moreover it is the energy density. The trace T of the stress energy tensor (1.2.25) is ($r-3p - 2\mu h^2 + 2 (\mu h)^2$) which is given by (1.2.28). The equation (1.2.30) is the eigen value corresponding to eigen vector h^a and the expressions (1.2.32) and (1.2.33) provides the invariants for the magnetofluid.

Energy conditions :

The famous Hawking and Ellis (1968) energy condition to be satisfied by all known forms of matters and equations of state is

$$T^{ab} u_a u_b \geqslant \frac{1}{2} T . \quad (1.2.34)$$

For the magnetofluid given by (1.2.25) we have

$$r + \frac{\mu^2 h^2}{2} \geqslant \frac{1}{2} [r-3p-2\mu h^2 + 2 (\mu h)^2],$$

i.e. $r + 3p + \mu\,(2-\mu)\,h^2 \geqslant 0$, (1.2.35)

which is true since the magnetic permeability μ is the constant and always less than unity. This establishes the fact that the stress energy tensor given by (1.2.25) is physically transparent.

1.3 Field equations :

The field equations for the magnetofluid consists of the following equations.

a) Einstein equations :

$$R_{ab} - \frac{1}{2} R\, g_{ab} = -\, K\, T_{ab} , \qquad (1.3.1)$$

where T_{ab} is the stress energy tensor given by (1.2.25).
If we take the gravitational constant $K = -1$ and velocity of light $c = 1$, then the field equations are

$$R_{ab} - \frac{1}{2} R\, g_{ab} = T_{ab} .$$

Remark : The Einstein field equations are the most revolutionary equations. They form the unifying bridge between geometry and dynamics of the space time.

b) Maxwell equations :

According to Maugin (1972), the Maxwell equations for the magnetofluid are

$$(u^a h^b - u^b h^a)_{;b} = 0 , \qquad (1.3.2)$$

where h^a is the space-like magnetic field vector with

$$u_a h^a = 0, \quad h_a h^a = - h^2 . \tag{1.3.3}$$

c) Thermodynamical equations :

The relations connecting thermodynamical variables are (Lichnerowicz, 1967)

$$TdS - di + \frac{dp}{\rho} = 0 , \tag{1.3.4}$$

where T is the proper temperature,

S is the entropy,

i is the enthalpy,

ρ is the proper material density.

Consequences of Maxwell equations through Ricci rotation coefficients :

The equation (1.3.2) can be written as

$$u^a{}_{;b}\, h^b + u^a h^b{}_{;b} - u^b{}_{;b} h^a - u^b h^a{}_{;b} = 0 . \tag{1.3.5}$$

I) Contracting (1.3.5) with u_a and using (0.3.4) we get

$$h^b{}_{;b} - h^a{}_{;b}\, u_a h^b = 0 ,$$

$$\text{i.e.} \quad h^b{}_{;b} = h^a{}_{;b}\, u_a u^b .$$

Which on using (1.3.3) supplies

$$h^b{}_{;b} = - u_{a;b}\, u^b h^a .$$

Consequently

$$\dot{u}_a h^a = - h^b{}_{;b} . \tag{1.3.6}$$

Thus magnetic field vector h^a is divergence free if and only if the four acclaration is orthogonal to the magnetic field vector h^a

By the use of Ricci rotation coefficients the equation (1.3.6) can be rewritten as

$$[\underset{414}{\gamma} Va + \underset{424}{\gamma} Wa + \underset{434}{\gamma} Na]\, h^a = - h^b{}_{;b} \,, \tag{1.3.7}$$

$$\text{i.e.} \quad \underset{414}{\gamma} V_a h^a + \underset{424}{\gamma} W_a h^a + \underset{434}{\gamma} N_a h^a = - h^b{}_{;b}$$

Thus for divergence free magnetic field, ~~along one of the space-time tetrad,~~ we have

$$\underset{414}{\gamma} = 0 \,, \quad \underset{424}{\gamma} = 0, \quad \underset{434}{\gamma} = 0 \,.$$

II) Transvecting (1.3.5) with h_a and using (1.3.3) we get

$$u^a{}_{;b}\, h_a h^b + h^2 . u^b{}_{;b} + \frac{1}{2} h^2{}_{,b}\, u^b = 0 \tag{1.3.8,a}$$

From (0.3.5) and (0.3.15) it follows that

$$[\sigma_{ab} + W_{ab} + \frac{1}{3} \theta P_{ab} + \dot{u}_a u_b] h^a h^b + h^2 \theta + \frac{1}{2} h^2{}_{,b}\, u^b = 0 \,.$$

This equation with (0.3.10) and (1.3.3) gives

$$\sigma_{ab} h^a h^b + \frac{1}{3} \theta [g_{ab} - u_a u_b] h^a h^b + h^2 \theta + \frac{1}{2} h^2{}_{,b}\, u^b = 0 \,,$$

$$\text{i.e.} \quad \sigma_{ab} h^a h^b - \frac{1}{3} \theta h^2 + \frac{1}{2} h^2{}_{,b}\, u^b + h^2 \theta = 0 \,,$$

$$\text{i.e.} \quad \sigma_{ab} h^a h^b + \frac{2}{3} \theta h^2 + \frac{1}{2} h^2{}_{,b}\, u^b = 0 \,. \tag{1.3.8}$$

In terms of Ricci rotation coefficients the equation (1.3.8) takes the form

$$[(\gamma_{411} + \frac{1}{3}\Theta) V_a V_b + (\gamma_{422} + \frac{1}{3}\Theta) W_a W_b + (\gamma_{433} + \frac{1}{3}\Theta) N_a N_b +$$

$$+ \frac{1}{2} (\gamma_{412} + \gamma_{421}) (V_a W_b + W_a V_b) + \frac{1}{2} (\gamma_{423} + \gamma_{432}) (W_a N_b + N_a W_b) +$$

$$+ \frac{1}{2} (\gamma_{431} + \gamma_{413}) (N_a V_b + V_a N_b)] h^a h^b - \frac{2}{3} (\gamma_{411} + \gamma_{422} + \gamma_{433}) h^2 +$$

$$+ \frac{1}{2} h^2_{,b} u^b = 0 .$$

On introducing the value of Θ in this equation we get

$$\{ \frac{1}{3} [(2 \gamma_{411} - \gamma_{422} - \gamma_{433}) V_a V_b + (- \gamma_{411} + 2\gamma_{422} - \gamma_{433}) W_a W_b +$$

$$+ (-\gamma_{411} - \gamma_{422} + 2\gamma_{433}) N_a N_b] + \frac{1}{2} [(\gamma_{412} + \gamma_{421}) (V_a W_b + W_a V_b) +$$

$$+ (\gamma_{423} + \gamma_{432}) (W_a N_b + N_a W_b) + (\gamma_{431} + \gamma_{413}) (N_a V_b + V_a N_b)] \} h^a h^b -$$

$$- \frac{2}{3} (\gamma_{411} + \gamma_{422} + \gamma_{433}) h^2 + \frac{1}{2} h^2_{,b} u^b = 0 . \qquad (1.3.9)$$

1.4 Space symmetries :

We innumerate below some important spaces existing in literature.

a) Einstein space : (Petrov, 1969)

The Einstein space is defined as the "Riemannian manifold" of any dimention and metric signature satisfying the conditions

$$R_{ab} = C g_{ab} , \qquad (1.4.1)$$

where C is a constant.

Riemannian manifolds satisfying (1.4.1) originate in the theory of Lie groups and almost all the present variants of unified field theory. Every Riemann space of constant curvature is an Einstein space (Petrov 1969).

Note : The contraction of equation (1.4.1) yields

$$g^{ab} R_{ab} = C\, g^{ab} g_{ab} ,$$

i.e. $R = C\, \delta^{a}_{\cdot a}$,

i.e. $R = 4\, C$,

i.e. $C = \frac{1}{4} R$. (1.4.2)

Thus (1.4.1) becomes

$$R_{ab} = \frac{1}{4} R\, g_{ab} , \qquad (1.4.3)$$

i.e.

$$T_{ab} = \frac{1}{4} T\, g_{ab} . \qquad \text{[vide (1.3.1)]} \qquad (1.4.4)$$

Hence $T_{ab}u^{a}u^{b}$ gives

$$T_{ab}u^{a}u^{b} = \frac{1}{4} T\, g_{ab}\, u^{a}u^{b} , \qquad (1.4.5)$$

i.e.

$$T_{ab}\, u^{a}u^{b} = \frac{1}{4} T . \qquad (1.4.6)$$

From (1.4.6) we conclude that the condition (1.2.34) is not satisfied. Hence Einstein space is incompatible with magnetofluid space-time .

b) C-space :

The Weyl conformal curvature tensor has the defining expression (Ellis, 1971)

$$C_{abcd} = R_{abcd} - g_{a[d}R_{c]b} - g_{b[c}R_{d]a} + \frac{R}{3} g_{a[d}g_{c]b} \,. \qquad (1.4.7)$$

This Weyl tensor satisfies the relations

$$C_{abcd} = C_{[ab][cd]} \,, \qquad (1.4.8)$$

$$C^{a}{}_{.bcd} = 0 \,, \qquad (1.4.9)$$

$$C_{a[bcd]} = 0 \,. \qquad (1.4.10)$$

Remark : 1) This Weyl tensor has ten independent components.

2) We observe from (1.4.7)

$R_{abcd} = 0 \Rightarrow C_{abcd} = 0$, but

$C_{abcd} = 0 \not\Rightarrow R_{abcd} = 0$.

Thus the flat space-time implies conformally flat space-time but not the converse.

Definition : The space in which divergence of the Weyl tensor vanishes is called C-space. (Szekers 1964), C-space is the Riemannian space governed by the relations

$$C^{a}{}_{bcd;a} = 0 \,. \qquad (1.4.11)$$

c) P-space :

According to Petrov the space matter tensor P_{abcd} is

$$P_{abcd} = R_{abcd} + S_{abcd} \pm \psi\, g_{abcd} \,, \qquad (1.4.12)$$

where the fourth rank tensors have the expressions.

$$S_{abcd} = \frac{1}{2} [g_{ac}T_{bd} - g_{ad}T_{bc} + g_{bd}T_{ac} - g_{bc} T_{ad}] , \qquad (1.4.13)$$

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc} , \qquad (1.4.14)$$

and ψ is a scalar.

This space matter tensor P_{abcd} satisfies all the algebraic properties of the curvature tensor R_{abcd} and it is more general than the Weyl tensor C_{abcd} [Petrov 1969] .

We have

$$P_{abcd} = - P_{bacd} = - P_{abdc} = P_{cdab} , \qquad (1.4.15)$$

$$P_{a[bcd]} = 0 . \qquad (1.4.16)$$

<u>Definition</u> : The Riemannian space in which

$$P^{a}{}_{bcd;a} = 0 , \qquad (1.4.17)$$

is known as P-space (Shaha 1974).