CHAPTER - II

SYSTEM OF STREAMLINES

2.1 Introduction :

The local behaviour of congruences in the space-time of the magnetofluid is investigated. The vanishing of the divergence of the stress energy tensor is the differential formulation of the law of conservation of energy and 4-momentum. As this equation places constraints on the dynamical evolution of the stress energy tensor it is known as equation of motion for stress energy. Equation of stream-line represents a line which is parallel to the direction of fluid flow at a given instant, while equation of continuity (or continuity equation) is an equation obeyed by any conserved indestructible quantity such as mass, electric charge, thermal energy, electrical energy or quantum mechanical probability, which is essentially a statement that the rate of increase of the quantity in any region equals the total current flowing into the region. Mass conservation law is the notion that mass is neither created nor destroyed, it is violeted by many microscopic phenomena. The local conservation laws are studied in section two while the characteristics of some special flows of the magnetofluid are described in section three.

2.2 Differential identities :

The famous Ricci identities give rise to local conservation laws

$$T^{ab}{}_{;b} = 0 \, ,$$

with the help of Einstein field equations. These laws are also the laws of conservation of energy and momentum. These laws generate the laws of motion of the dynamical system.

The stress energy tensor for the perfect magnetofluid (1.2.25) can be put in the form

$$T^{ab} = Au^a u^b - Bg^{ab} - \mu\, h^a h^b \, , \tag{2.2.1}$$

where

$$A = \rho f + \mu h^2 \, ,$$

and $B = p + \mu\,(1 - \frac{\mu}{2})\, h^2$.

Then the local conservation law $T^{ab}{}_{;b} = 0$, for the magneto-fluid described by (2.2.1) supplies the result.

$$A_{,b}\, u^a u^b + Au^a{}_{;b} u^b + Au^a u^b{}_{;b} - B_{,b}\, g^{ab} -$$

$$- B\, g^{ab}{}_{;b} - \mu h^a{}_{;b} h^b - \mu h^a h^b{}_{;b} - \mu_{,b} h^a h^b = 0 \, ,$$

i.e. $(\rho f + \mu h^2)_{,b}\, u^a u^b + (\rho f + \mu h^2)\, u^a{}_{;b} u^b + (\rho f + \mu h^2) u^a u^b{}_{;b} -$

$$- [\, p + \mu(1 - \tfrac{\mu}{2}) h^2\,]_{,b}\, g^{ab} - \mu h^a{}_{;b} h^b - \mu h^a h^b{}_{;b} = 0 \, .$$

On utilizing (0.3.5) and (0.3.8) in above equation we get

$$(\varrho f + \mu h^2)_{,b}\, u^a u^b + (\varrho f + \mu h^2) [\dot{u}^a + \theta u^a] -$$

$$- [p + \mu (1- \tfrac{\mu}{2}) h^2]_{,b}\, g^{ab} - \mu h^a{}_{;b}\, h^b - \mu h^a h^b{}_{;b} = 0 . \qquad (2.2.2)$$

Equation of Continuity :

By transvecting equation (2.2.2) with u_a we derive the continuity equation for the perfect magnetofluid in the form

$$(\varrho f + \mu h^2)_{,b}\, u^a u_a u^b + (\varrho f + \mu h^2) [u_a \dot{u}^a + \theta u^a u_a] -$$

$$- [p + \mu(1- \tfrac{\mu}{2}) h^2]_{,b}\, g^{ab} u_a - \mu h^a{}_{,b} h^b u_a - \mu h^a u_a h^b{}_{,b} = 0,$$

i.e. $$(\varrho f + \mu h^2)_{,b} u^b + (\varrho f + \mu h^2)\, \theta - [p + \mu(1- \tfrac{\mu}{2}) h^2]_{,b}\, u^b -$$

$$- \mu h^a{}_{;b}\, h^b u_a = \theta. \qquad (2.2.3)$$

[vide (0.3.3),(0.3.12)and (1.3.3)]

From (1.2.20) and (1.2.23) we have

$$\varrho f = r + p .$$

Hence equation (2.2.3) becomes

$$[r+p+\mu h^2 - p - \mu(1- \tfrac{\mu}{2}) h^2]_{,b}\, u^b + [r+p+\mu h^2]\, \theta - \mu h^a{}_{;b} h^b u_a = 0 ,$$

i.e. $$[r + \frac{\mu^2 h^2}{2}]_{,b}\, u^b + [r+p]\theta + \mu h^2 \theta - \mu h^a{}_{;b} h^b u_a = 0 ,$$

which on making use of (0.3.5) and (1.3.3) takes the form

$$(r + \frac{\mu^2 h^2}{2})_{,b}\, u^b + (r+p)\, \theta + \mu h^2 u^b{}_{;b} + \mu\, u_{a;b}\, h^a h^b = 0 .$$

This with the usage of (1.3.8,a) can be reduced to

$$(r + \frac{\mu^2 h^2}{2})_{,b} u^b + (r+p)\, \theta + (- \tfrac{\mu}{2} h^2{}_{,b} u^b) = 0 ,$$

i.e. $\left[r + \frac{\mu^2 h^2}{2} - \frac{\mu h^2}{2} \right]_{,b} u^b + (r + p)\,\theta = 0,$

i.e. $\left[r - \frac{\mu}{2}(1 - \mu)\, h^2 \right]_{,b} u^b + (r+p)\theta = 0 \, .$ (2.2.4)

This is the equation of continuity for the magnetofluid.

Equation of continuity in Ricci rotation coefficients :

In the form of Ricci rotation coefficients the equation of continuity (2.2.4) can be put as

$$\left[r - \frac{\mu}{2}(1-\mu)h^2 \right]_{,b} u^b + (r+p) \left[-\gamma_{411} - \gamma_{422} - \gamma_{433} \right] = 0 \, ,$$

i.e. $\left[r - \frac{\mu}{2}(1-\mu)h^2 \right]_{,b} u^b - (r+p)(\gamma_{411} + \gamma_{422} + \gamma_{433}) = 0.$ (2.2.5)

Effect of thermodynamical variables on equation of continuity :

From the thermodynamical equations (1.2.20) and (1.2.21) we have

$$r = \rho\,(1 + i - p/\rho\,) \, ,$$

i.e. $dr = d\left[\rho\,(1+i - p/\rho\,) \right] ,$

i.e. $dr = d\rho\,(1+i - p/\rho\,) + \rho\, d(1+i-p/\rho).$ (2.2.6,a)

On account of (1.2.21) this equation becomes

$$dr = d\rho\,(1+\epsilon\,) + \rho\, di - \rho d\,(p/\rho),$$

i.e. $dr = (1+\epsilon)d\rho + \rho di - \rho \left[\frac{\rho dp - p d\rho}{\rho^2} \right] ,$

i.e. $dr = (1+\epsilon)d\rho + \rho di - dp + \frac{p}{\rho}\, d\rho \, ,$

i.e. $dr = (1+\epsilon+ p/\varrho)\, d\varrho + \varrho di - dp$,

i.e. $dr = (1+\epsilon + p/\varrho)\, d\varrho + \varrho\, (di - \frac{dp}{\varrho})$.

It follows from (1.?.23) and (1.3.4) that

$$dr = f d\varrho + \varrho\, TdS ,$$

i.e. $r_{,a} = f\varrho_{,a} + \varrho\, T\, S_{,a}$. (2.2.6,b)

Hence the equation (2.2.4) i.e. continuity equation becomes

$$[\, (f\varrho_{,a}+ \varrho TS_{,a})-(\tfrac{1}{2}\, \mu\, (1- \tfrac{\mu}{*})\, h^2)_{;a}\,]\, u^a+ \varrho f\, \Theta = 0 ,$$

i.e. $[\, \varrho TS_{,a} - (\frac{1}{2}\, \mu(1-\frac{\mu}{*})h^2)_{;a}\,]\, u^a + f\varrho_{,a}u^a +$

$$\varrho\, f\, u^a{}_{;a} = 0 ,$$

i.e. $[\varrho TS_{,a}- \frac{1}{2}\, \mu\, (1- \frac{\mu}{*})\, h^2{}_{,a}]u^a+ f(\varrho u^a)_{;a}= 0$. (2.?.6)

Mass conservation law :

If the magnetofluid satisfies mass conservation law i.e. law of conservation of Baryons

$$(\, \varrho\, u^a)_{;a} = 0 , \qquad (2.2.7)$$

then the equation (2.2.6) takes the form

$$\varrho TS_{,a}\, u^a = \tfrac{1}{2}\, \mu(\, 1 - \tfrac{\mu}{*}\,)\, h^2{}_{,a}u^a. \qquad (2.?.8)$$

Hence we condlude that for the magnetofluid obeying mass conservation law the flow is adiabatic if and only if $[\, \frac{1}{2}\, \mu(1- \frac{\mu}{*})\, h^2\,]$ is conserved along the four velocity u^a .

Equations of streamlines :

The equations of streamlines for the magnetofluid are obtained by substituting the value of $(\varrho f)\,\theta$ from the equation of continuity (2.2.4) into equation (2.2.2). Thus we get

$$(\varrho f+\mu h^2)_{,b}u^a u^b + (\varrho f+\mu h^2)\,\dot{u}^a + \varrho f\theta\, u^a + \mu h^2\theta u^a -$$
$$- [\, p + \mu\,(1 - \tfrac{\mu}{2}\,)\, h^2\,]_{,b}g^{ab} - \mu\,(h^a h^b)_{;b} = 0\,.$$

It follows from equation (2.2.4) and (1.3.8a) that

$$(\varrho f + \mu h^2)_{,b}\, u^a u^b + (\varrho f + \mu h^2)\,\dot{u}^a - [\, r - \tfrac{\mu}{2}\,(1-\mu)\, h^2]_{,b}u^a u^b -$$
$$- \mu h_c u^c_{\;;b}\, h^b u^a - \tfrac{\mu h^2}{2}{}_{,b}\, u^a u^b - [p + \mu(1-\tfrac{\mu}{2})h^2]_{,b}\, g^{ab} - \mu(h^a h^b)_{;b} = 0\,,$$

$$\text{i.e. } (\varrho f+\mu h^2 - r + \tfrac{\mu}{2}\,(1-\mu)\, h^2 - \tfrac{\mu h^2}{2}\,]_{,b}\, u^a u^b - (\varrho f+\mu h^2)\dot{u}^a -$$

$$-\mu\, u_{b;c}h^b h^c u^a - [\, p + \mu(1-\tfrac{\mu}{2})\, h^2]_{,b}\, g^{ab} - \mu\,(h^a h^b)_{;b} = 0\,,$$

$$\text{i.e. } [\, p + \mu\,(1-\tfrac{\mu}{2})\, h^2\,]_{,b}\,(u^a u^b - g^{ab}\,) - (\varrho f+\mu h^2)\,\dot{u}^a -$$

$$- \mu\, u_{b;c}h^b h^c u^a - \mu\,(\, h^a h^b\,)_{;b} = 0\,,$$

$$\text{i.e. } [\, p + \mu\,(1-\tfrac{\mu}{2})\, h^2\,]_{,b}\, p^{ab} - \mu(h^a h^b)_{;b} -$$

$$-(\varrho f+\mu h^2)\, u^a_{\;;b}u^b - \mu\, u_{b;c}h^b h^c u^a = 0\,. \tag{2.2.9}$$

The equations (2.2.9) are the equations of streamlines.

Theorem : The magnetic field lines are divergence free if and only if $\left(p + \frac{\mu}{2} (1-\mu) h^2 \right)$ conserves along the magnetic lines.

Proof : The equation (2.2.2) is

$$(\rho f + \mu h^2)_{,b}\, u^a u^b + (\rho f + \mu h^2) \left[\dot{u}^a + \theta u^a \right] -$$

$$- \left[p + \mu \left(1 - \frac{\mu}{2}\right) h^2 \right]_{,b} g^{ab} - \mu h^a{}_{,b} h^b - \mu h^a h^b{}_{;b} = 0 .$$

On transvecting this with h_a we have

$$\left[\rho f + \mu h^2 \right] u^a{}_{;b}\, u^b h_a - \left[p + \mu \left(1 - \frac{\mu}{2}\right) h^2 \right]_{,b} h^b -$$

$$- \mu h_a h^a{}_{;b} h^b + \mu h^2 h^b{}_{;b} = 0 .$$

From (0.3.8) and (1.3.3) we get

$$\left[\rho f + \mu h^2 \right] \dot{u}^a h_a - \left[p + \mu \left(1 - \frac{\mu}{2}\right) h^2 \right]_{,b} h^b +$$

$$+ \frac{1}{2} \mu h^2{}_{,b} h^b + \mu h^2 h^b{}_{;b} = 0 .$$

Employing Maxwell equations this reduces to

$$- (\rho f + \mu h^2)\, h^b{}_{;b} - \left[p + \mu \left(1 - \frac{\mu}{2}\right) h^2 \right]_{,b} h^b +$$

$$+ \frac{1}{2} \mu h^2{}_{,b} h^b + \mu h^2 h^b{}_{;b} = 0 ,$$

i.e. $- (\rho f)\, h^b{}_{;b} - \left[p + \frac{\mu}{2} (1 - \mu) h^2 \right]_{,b} h^b = 0$,

i.e. $\left[p + \frac{\mu}{2} (1-\mu) h^2 \right]_{,b} h^b = (\rho f)\, h^b{}_{;b}$. (2.2.10)

This result justifies the statement of the theorem.

<u>Note</u> : From (1.3.6) and (2.2.10) we derive

$$[\, p + \tfrac{\mu}{2}\,(1-\mu)h^2 \,]_{,b}\, h^b = -(\rho f)\, \dot{u}_a h^a. \qquad (2.2.11)$$

By introducing the value of $\dot{u}_a$ in Ricci rotation coefficients the equation (2.2.11) becomes

$$[p + \tfrac{\mu}{2}\,(1-\mu)h^2]_{,b}h^b = -[\rho f][\gamma_{414} V_a + \gamma_{424} W_a + \gamma_{434} N_a]h^a. \qquad (2.2.12)$$

Thus we infer from (2.2.12) that <u>the four acceleration is orthogonal to magnetic field if and only if the quantity ($p + \frac{\mu}{2}(1-\mu)h^2$) is invariant along the magnetic lines.</u>

2.3 <u>The special flows of the magnetofluid</u>

For the magnetofluid we have obtained the results

$$\dot{u}_a h^a = -\, h^b{}_{;b}\,, \qquad (1.3.6)$$

$$\sigma_{ab}h^a h^b + \tfrac{2}{3}\,\theta\, h^2 + \tfrac{1}{2}\, h^2{}_{,b}u^b = 0\,, \qquad (1.3.8)$$

$$[\, r - \tfrac{\mu}{2}\,(1-\mu)h^2 \,]_{,b}u^b + (r+p)\theta = 0\,, \qquad (2.2.4)$$

and

$$[\, p + \tfrac{\mu}{2}\,(1-\mu)\, h^2 \,]_{,b}h^b = -(\rho f)\, \dot{u}_a h^a\,. \qquad (2.2.11)$$

<u>Geodesic flow</u> :

From (0.5.1) and (1.3.6) we have

$$h^b{}_{;b} = 0\,. \qquad (2.3.1)$$

Also we have from (0.5.1) and (2.2.11)

$$[\, p + \tfrac{\mu}{2}\,(1-\mu)h^2 \,]_{,b}\, h^b = 0\,. \qquad (2.3.2)$$

Essentially expanding flow :

It follows from equations (0.5.2) and (1.3.6)

$$h^b{}_{;b} = 0 \ , \qquad (2.3.1)$$

from (0.5.2) and (1.3.8)

$$\tfrac{2}{3}\,\theta\, h^2 + \tfrac{1}{2}\, h^2{}_{,b}\, u^b = 0 \ ,$$

i.e. $\tfrac{2}{3}\,\theta\, h^2 = -\tfrac{1}{2}\, h^2{}_{,b} u^b$,

i.e. $$\tfrac{2}{3}\,(\underset{411}{\gamma} + \underset{422}{\gamma} + \underset{433}{\gamma})h^2 = \tfrac{1}{2}h^2{}_{,b}u^b, \qquad (2.3.3)$$

and from (0.5.2) and (2.2.11)

$$[\, p + \tfrac{\mu}{2}\,(1-\mu)\, h^2 \,]_{,b}\, h^b = 0 \ . \qquad (2.3.2)$$

Killing flow :

(0.5.3) together with (1.3.6), (1.3.8), (2.2.4) and (2.2.11) give the following results

$$h^b{}_{;b} = 0 \ , \qquad (2.3.1)$$

$$h^2{}_{,b}u^b = 0 \ , \qquad (2.3.4)$$

$$r_{,b}u^b = 0 \ , \qquad (2.3.5)$$

and $$p_{,b}h^b = 0 \ . \qquad (2.3.6)$$

Born rigid flow :

Utilizing (0.5.4), (1.3.8) and (2.2.4) we obtain

$$h^2{}_{,b}u^b = 0 \ , \qquad (2.3.4)$$

and $$r_{,b}u^b = 0 \ . \qquad (2.3.5)$$

Harmonic flow :

From (0.5.5), (1.3.6), (1.3.8), (2.2.4) and (2.2.11) we get

$$h^{b}{}_{;b} = 0 \ , \tag{2.3.1}$$

$$\sigma_{ab}\, h^{a}h^{b} = -\tfrac{1}{2}\, h^{2}{}_{,b}u^{b} \ ,$$

i.e. $[\gamma_{412}\,(V_aW_b + W_aV_b) + \gamma_{423}\,(W_aN_b + N_aW_b) +$

$$+\gamma_{431}(N_aV_b + V_aN_b)\,]h^{a}h^{b} = -\tfrac{1}{2}\, h^{2}{}_{,b}u^{b} \ , \tag{2.3.7}$$

$$[\, r - \tfrac{\mu}{2}\,(1-\mu)\, h^{2}\,]_{,b}u^{b} = 0 \tag{2.3.8}$$

and

$$[\, p + \tfrac{\mu}{2}\,(1-\mu)\, h^{2}\,]_{,b}h^{b} = 0 \ . \tag{2.3.2}$$

Boost flow :

We have from (0.5.6), (1.3.8) and (2.2.4)

$$h^{2}{}_{,b}\, u^{b} = 0 \ , \tag{2.3.4}$$

and

$$r_{,b}u^{b} = 0 \ . \tag{2.3.5}$$

Steady state magnetofluid :

Equation (0.5.7) with (1.3.8) and (2.2.4) provide the results

$$h^{2}{}_{,b}u^{b} = 0 \ , \tag{2.3.4}$$

and

$$r_{,b}u^{b} = 0 \ . \tag{2.3.5}$$

Thus for the special kinds of flows we admit the following results :

1) For geodesic flow, essentially expanding flow, killing flow, and harmonic flow the magnetic field is divergence free.

2) In killing flow, Born rigid flow, boost flow and steady state magnetofluid the magnitude of the magnetic field is invariant.

3) In harmonic flow the quantity $[\, r - \frac{\mu}{2}(1-\mu)\, h^2 \,]$ remains unchanged.

4) Flow remains essentially expanding if the magnitude of the magnetic field is not invariant.

5) For harmonic flow, the projection of shear tensor along h^a ~~and h^b~~ vanishes if and only if the magnitude of the magnetic vector remains constant along four-velocity.

6) The energy density r is ~~fixed~~ invariant in killing flow, Born rigid flow, boost flow and steady state magnetofluid.

7) The isotropic pressure p for the killing flow is invariant along magnetic lines, while for geodesic flow, essentially expanding flow and harmonic flow the quantity $[\, p + \frac{\mu}{2}(1-\mu)\, h^2 \,]$ does not vary along the magnetic lines.