CHAPTER-III
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III:1 INTRODUCTION

Pawlak [P_1] defined rough set as an ordered pair of lower and upper approximations of a subset of a universal set.

Dubois and Prade [D_1,D_2] replaced the term "a subset" by "a fuzzy subset" of the universal set and introduced the concept of rough fuzzy set, as an ordered pair of lower and upper approximations of the fuzzy set.

In this chapter we shall discuss the properties of rough fuzzy set.

III : 2 ROUGH FUZZY SETS

Let $K = (U, \mathcal{E})$ be an approximation space, where $\mathcal{E} = \{ E_\lambda | \lambda \in \Lambda \}$ is the partition of U. Let R be the equivalence relation induced by $\mathcal{E}$.

Definition (III:2:1) [D_1, D_2] :

For each fuzzy set X of U, the <u>lower approximation</u> of X with respect to R, is the fuzzy set $\underline{R}X : \mathcal{E} \longrightarrow [0,1]$ defined as $\underline{R}X(E_i) = \inf\{ X(x) | w(E_i) = [x]_R \}$; where w is a mapping form $\mathcal{E}$ to U defined as $w(E_i) = \{x \in U | E_i = [x]_R \}$.

Definition (III:2:2) [D_1, D_2] :

For each fuzzy set X of U, the <u>upper approximation</u> of X with respect to R is the fuzzy set $\bar{R}X : E \longrightarrow [0,1]$ defined as

$$\bar{R}X\ (E_i) = \text{Sup}\ \{\ X(x) \mid w(E_i) = [x]_R\ \}.$$

Definition (III:2:3) [D_1, D_2] :

Let ($\underline{R}X$, RX) be a rough fuzzy set corrosponding to a fuzzy set X of U. The <u>fuzzy extension</u> $w(\underline{R}X)$ and $w(\bar{R}X)$ of $\underline{R}X$ and $\bar{R}X$ respectively are fuzzy sets of U, defined as follows

$w(\underline{R}X) : U \longrightarrow [0, 1]$ by

$w(\underline{R}X)\ (X) = \underline{R}X\ (E_i)$, if $x \in w(E_i)$ and

$w(\bar{R}X) : U \longrightarrow [0,1]$ by

$w(\bar{R}X)\ (x) = \bar{R}X\ (E_i)$, if $x \in w(E_i)$.

Proposition (III:2:5) [D_2] :

A fuzzy set X of U is equal to its lower and upper approximations, if and only if X is constant on equivalence classes of R.

Proof :

Suppose that the fuzzy set X of U is constant on equivalence classes of R.

Let $x \in E$

Therefore, $w(\underline{R}X)(x) = \underline{R}X(E_i)$

$$= \inf \{ X(x) \mid w(E_i) = [x]_R \}$$

$$= \inf_{x \in E_i} \{ X(x) \}$$

$$= X(x)$$

Thus, $w(\underline{R}X)(x) = X(x)$.

Similarly $w(\bar{R}X)(x) = X(x)$

Hence fuzzy set X is equal to its lower and upper approximations.

Conversely, suppose that X is equal to its lower and upper approximations.

i.e. $w(\underline{R}X)(x) = w(\bar{R}X)(x) = X(x)$

i.e. $\inf_{x \in Ei} X(x) = \sup_{x \in Ei} X(x) = X(x)$

This shows that X is constant on equivalence classes of R.

Proposition (III:2:6) [D_2] :

Let $K = (U, \acute{E})$ be an approximation space. R is an equivalence relation on U, induced by $\acute{E}$. X and Y are fuzzy sets of U, then following holds.

i) $w(\underline{R}X) \subseteq X \subseteq w(\bar{R}X)$

ii) $w(\bar{R}(X \cup Y) = w(\bar{R}X) \cup w(\bar{R}Y)$

iii) $w(\bar{R}(X \cap Y)) \subseteq w(\bar{R}X) \cap w(\bar{R}Y)$

iv) $w(\underline{R}(X \cap Y)) = w(\underline{R}X) \cap w(\underline{R}Y)$

v) $w(\underline{R}(X \cup Y)) \supseteq w(\underline{R}X) \cap w(\underline{R}Y)$

vi) $w(-\underline{R}X) = w(\bar{R}(-X))$

vii) $w(-RX) = w(\underline{R}(-X))$

viii) $w(\bar{R}(w(\bar{R}X))) = w(\underline{R}(w(\bar{R}X))) = w(\bar{R}X)$

ix) $w(\underline{R}(w(\underline{R}X))) = w(\bar{R}(w(\underline{R}X))) = w(\underline{R}X)$

Proof : Let $x \varepsilon U$ be such that $x \varepsilon E_i \varepsilon É$.

(i)
$$
\begin{aligned}
w(\underline{R}X)(x) &= \underline{R}X(E_i) \\
&= \inf \{ X(y) \mid w(E_i) = [y]_R \} \\
&= \inf_{y \varepsilon E_i} X(y) \\
&\leq X(x) \\
&\leq \sup_{y \varepsilon E_i} X(y) \\
&= \bar{R}X(E_i) \\
&= w(\bar{R}X)(x)
\end{aligned}
$$

Thus, $w(\underline{R}X) \subseteq X \subseteq w(\bar{R}X)$.

(ii)
$$
\begin{aligned}
w(\bar{R}(X \cup Y))(x) &= \bar{R}(X \cup Y)(E_i) \\
&= \sup_{x \varepsilon E_i} \{(X \cup Y)(x)\} \\
&= \sup_{x \varepsilon E_i} \{\max\{ X(x), y(x) \}\} \\
&= \max \left[\sup_{x \varepsilon E_i} X(x), \sup_{x \varepsilon E_i} Y(x) \right] \\
&= \max \{\bar{R}X(E_i), \bar{R}Y(E_i)\} \\
&= \max \{w(\bar{R}X)(x), w(\bar{R}Y)(x)\} \\
&= (w(\bar{R}X) \cup w(\bar{R}Y))(x)
\end{aligned}
$$

Thus, $w(\bar{R}(X \cup Y)) = w(\bar{R}X) \cup w(\bar{R}Y)$.

(iii) $w(\bar{R}(X \cap Y))(x) = \bar{R}(X \cap Y)(E_i)$

$$= \sup_{x \varepsilon E_i} \{ (X \cap Y)(x) \}$$

$$= \sup_{x \varepsilon E_i} \{ \min\{ X(x), y(x) \}\}$$

$$\leq \min \left[\sup_{x \varepsilon E_i} X(x), \sup_{x \varepsilon E_i} Y(x) \right]$$

$$= \min \{ \bar{R}X(E_i), \bar{R}Y(E_i) \}$$

$$= \min \{ w(\bar{R}X)(x), w(\bar{R}Y)(x) \}$$

$$= (w(\bar{R}X) \cap w(\bar{R}Y))(x)$$

Thus, $w(\bar{R}(X \cap Y)) \subseteq w(\bar{R}X) \cap w(\bar{R}Y)$.

(iv) $w(\underline{R}(X \cap Y))(x) = \underline{R}(X \cap Y)(E_i)$

$$= \inf_{x \varepsilon E_i} \{ (X \cap Y)(x) \}$$

$$= \inf_{x \varepsilon E_i} \{ \min\{ X(x), y(x) \}\}$$

$$= \min \left[\inf_{x \varepsilon E_i} X(x), \inf_{x \varepsilon E_i} Y(x) \right]$$

$$= \min \{ \underline{R}X(E_i), \underline{R}Y(E_i) \}$$

$$= \min \{ w(\underline{R}X)(x), w(\underline{R}Y)(x) \}$$

$$= (w(\underline{R}X) \cap w(\underline{R}Y))(x)$$

Thus, $w(\underline{R}(X \cap Y)) = w(\underline{R}X) \cap w(\underline{R}Y)$.

(v) $w(\underline{R}(X \cup Y))(x) = \underline{R}(X \cup Y)(E_i)$

$$= \inf_{x \varepsilon E_i} \{ (X \cup Y)(x) \}$$

$$= \inf_{x \, \varepsilon \, E_i} \{ \max\{ X(x), \ y(x) \}\}$$

$$\geq \max \left[\inf_{x \, \varepsilon \, E_i} X(x), \ \inf_{x \, \varepsilon \, E_i} Y(x) \right]$$

$$= \max \{ \underline{R}X(E_i), \ \underline{R}Y(E_i) \}$$

$$= \max \{ w(\underline{R}X)(x), \ w(\underline{R}Y)(x) \}$$

$$= (w(\underline{R}X) \cup w(\underline{R}Y))(x)$$

Thus, $w(\underline{R}(X \cup Y)) \supseteq w(\underline{R}X) \cup w(\underline{R}Y)$.

(vi) $w(-\underline{R}X)(x) = -\underline{R}X(E_i)$

$$= 1-\underline{R}X(E_i)$$

$$= 1- \inf_{x \, \varepsilon \, E_i} X(x)$$

$$= \sup_{x \, \varepsilon \, E_i} \{ 1-X(x) \}$$

$$= \bar{R}(1-X) \ (E_i)$$

$$= \bar{R}(-X) \ (E_i)$$

$$= w(\bar{R}(-X)) \ (x)$$

Thus, $w(-\underline{R}X) = w(\bar{R}(-X))$.

(vii) Similarly we can prove,

$w(-\bar{R}X) = w(\underline{R}(-X))$

(viii) Since $w(\bar{R}X)(x) = \bar{R}X(E_i)$; $w(\bar{R}X)$ is a fuzzy set of U with constant membership on equivalence classes of U and by proposition (III:2:5),

$w(\bar{R}X)$ is equal to its lower and upper approximations.

i.e. $w(\underline{R}(w(\bar{R}X))) = w(\bar{R}(w(\bar{R}X))) = w(\bar{R}X)$.

(ix) Similarly we can prove,

$$w(\underline{R}(w(\underline{R}X))) = w(\bar{R}(w(\underline{R}X))) = w(\underline{R}X).$$

The following proposition shows that the concepts of rough fuzzy set and rough set agree for crisp sets.

Proposition (III:2:7)

Let Z be a non-empty crisp subset of U, then

$\underline{R}(\chi_Z) = \chi_{\underline{R}Z}$ and

$\bar{R}(\chi_Z) = \chi_{\bar{R}Z}$.

Proof

Let Z be a non-empty crisp subset of U. Let $x \varepsilon U$ be such that $x \varepsilon E_i \varepsilon \acute{E}$.

Consider, $w(\underline{R}(\chi_Z))(x) = \underline{R}(\chi_Z)(E_i)$

$$= \inf_{x \varepsilon E_i} \chi_Z(x)$$

$$= \begin{cases} 1, & \text{if } x \varepsilon \{ y \varepsilon U \mid [y]_R \subseteq Z \} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{if } x \varepsilon \underline{R} Z \\ 0, & \text{otherwise} \end{cases}$$

$$= \chi_{\underline{R}Z}(x)$$

Thus, $\underline{R}\chi_Z = \chi_{\underline{R}Z}$

Similarly,

Consider, $w(\bar{R}(\chi_Z))(x) = \bar{R}(\chi_Z)(E_i)$

$$= \sup_{x \varepsilon E_i} \chi_Z(x)$$

$$= \begin{cases} 1, \text{if } x \varepsilon \{y \varepsilon U \mid [y]_R \cap Z \neq \Phi\} \\ 0, \quad \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, \text{ if } x \varepsilon \bar{R} Z \\ 0, \quad \text{otherwise} \end{cases}$$

$$= \chi_{\bar{R}Z}(x)$$

Thus, $\bar{R}\chi_Z = \chi_{\bar{R}Z}$.

III:3 SUMMARY

In this chapter we discuss the concept of rough fuzzy set. Dubois and Prade [D_1,D_2] used the fuzzy set instead of subset of U and proceeded on the lines of Pawlak [P_1] to obtain the rough fuzzy set. They showed that a rough fuzzy set have similar properties.

As we have shown in proposition (III:2:7), the treatment of Dubois and Prade [D_1,D_2] is satisfactory in the sense that it agrees with that of Pawlak [P_1] when a fuzzy set is a characteristic function of a crisp set.