CHAPTER-IV

FUZZY ROUGH FUZZY SETS

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IV:1 INTRODUCTION

In previous chapter we canvassed the notion of rough fuzzy sets introduced by Dubois and Prade [D_1,D_2]. Dubois and Prade [D_1,D_2] have approached the problem of combining the concept of roughness and fuzziness in a different way also; which we are going to discuss here. The new concept so emerged is described by him as a fuzzy rough set. However, as we shall see in the following discussion it is not merely a fuzzification of a rough set, but an entirely new concept. Here, they consider fuzzy equivalence relations; fuzzy partition and obtain rough set corresponding to a fuzzy set. This rough set may be more appropriately called fuzzy rough fuzzy set. However, Dubois and Prade designate it as a fuzzy rough set. Since the term fuzzy rough set is already used in Chapter II for a different concept we shall use the term fuzzy rough fuzzy set.

We begin with some basic notions.

IV:2 FUZZY RELATION; FUZZY PARTATION

Definition (IV:2:1) [D_3;M]

A <u>fuzzy relation</u> R on U is a fuzzy set

$R : UXU \longrightarrow [0,1]$.

Note (IV:2:2)

Throughout this chapter we consider a bianary operation * on [0,1] with the following properties :

(i) $a * (b * c) = (a * b) * c$

(Assocativity)

(ii) $a * b = b * a$

(Commutativity)

(iii) $a \le b$ implies $a * c \le b * c$

(Monotonicity)

(iv) $a * 1 = a$

(Existence of identity)

where, a,b,c are elements of [0,1].

Definition IV:2:3 [D_1;D_2;D_3]

A fuzzy relation R on U is a <u>* similarity relation on U</u> if

(i) $R(x,x) = 1$

(Reflexivity)

(ii) $R(x,y) = R(y,x)$

(Commutativity)

(iii) $R(x,y) \ge R(x,z) * R(z,y)$

(*-transitivity)

If * = min, then the above relation is called a <u>similarity</u> relation on U [Z_1].

Example (IV:2:4) [B]

Let $R_1, R_2, \ldots, R_n$ be equivalence relations on U. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be real number's such that $\alpha_i > 0$ and $\Sigma\, \alpha_i = 1$. Define

$$R(x,y) = \sum_{i=1}^{n} \alpha_i \; R_i(x,y).$$

Clearly R is a fuzzy relation on U. Let T_m be a bianary operation on [0,1], defined by

$$a\; T_m\; b = \max \{ 0,\; a+b-1 \}.$$

Since each R_i is an equivalence relation and $\Sigma\alpha_i = 1$,
R is reflexive and symmetric relation.
We claim that R is T_m - transitive relation.
To justify the claim we are to show that $\forall\; x,z \in U$,

$$R(x,z) \geq R(x,y)\; T_m\; R(y,z), \quad \forall\; y \in U \qquad \text{(i)}$$

If $R(x,y) + R(y,z)-1 \leq 0$, then $R(x,y)\; T_m\; R(y,z) = 0$ and the equality (i) holds trivially.

Therefore, consider $R(x,y) + R(y,z)-1 > 0$.
By definition, we have

$$R(x,y) + R(y,z)-1 = \Sigma\; \alpha_i (\; R_i(x,y) + \; R_i(y,z)-1\;) \qquad \text{(ii)}$$

Obviousely, coefficient of α_i in the above equation is either +1 or -1 or 0.

Let without the loss of generality it be +1 for $i = 1,2, \ldots, K$ only.

But then for $i = 1,2,\ldots,K$;

$R_i(x,y) = 1$ and $R_i(y,z) = 1$ and hence $R_i(x,z) = 1$.

Therefore,

$$
\begin{aligned}
R(x,z) &= \sum_{i=1}^{n} \alpha_i \; R_i(x,z) \\
&\geq \sum_{i=1}^{K} \alpha_i \\
&\geq \Sigma\, \alpha_i(R_i(x,y) + R_i(y,z)-1) \\
&= R(x,y) + R(y,z)-1 \\
&= R(x,y)\ T_m\ R(y,z).
\end{aligned}
$$

Hence R is T_m similarity relation on U.

Definition (IV:2:5) [D_1,H]

Let U be a universe and R be a *-similarity relation on U. Then for any $x \in U$, a <u>fuzzy class $[x]_R$</u> is a fuzzy set $[x]_R : U \longrightarrow [0,1]$ defined as

$$[x]_R(y) = R(x,y),\ \forall\ y \in U.$$

Remark (IV:2:6)

If R is the crisp equivalence relation, then fuzzy class is an equivalence class induced by R.

Definition (IV:2:7) [D_1,D_2,H]

A fuzzy set X of U is called <u>fuzzy equivalence class</u> of a *-similarity relation R on U, if,

(i) There is $x \in U$ such that $X(x) = 1$

(X is normalized)

(ii) $X(x) * R(x,y) \leq X(y)$.

(Extensionality condation)

(iii) $X(x) * X(y) \leq R(x,y)$

(Singlton condation)

Proposition (IV:2:8) [D_1,D_2]

Every fuzzy class is a fuzzy equivalence class.

Proof

For $x \in U$ and a *-similarity relation R, consider the fuzzy class $[x]_R$.

We claim that $[x]_R$ is fuzzy equivalence class.

(i) Since R is reflexive, $x \in U$; $R(x,x) = 1$.

Therefore, $[x]_R(x) = 1$

Thus X is normalized.

(ii) Let $y,z \in U$

$$[x]_R(z) * R(z,y) = R(x,z) * R(z,y)$$
$$\leq R(x,y), \text{ since R is *-similarity relation}$$
$$= [x]_R(y)$$

Hence Extensionality condation holds.

(iii)Let $y, z \in U$

Therefore,

$[x]_R(y) * [x]_R(z) = R(x,y) * R(x,z)$

$\leq R(y,z)$, since R is *-similarity relation.

Hence Singlton condation holds.

Thus, $[x]_R$ is a fuzzy equivalence class of *-similarity relation R.

Proposition (IV:2:9) [H]

Let X_1, X_2 be two equivalence classes be such that $X_1(x) = X_2(x) = 1$, for some $x \in U$, Then $X_1 = X_2$.

Proof

Let $z \in U$. Then

$$
\begin{aligned}
X_1(z) &= X_1(z) * 1 \\
&= X_1(z) * X_1(x) * X_2(x) \\
&\leq \sup_y \{ X1(z) * X1(y) * X2(y) \} \\
&\leq X_1(z)
\end{aligned}
$$

$$
\begin{aligned}
\text{Thus, } X1(z) &= \sup_y \{ X1(z) * X1(y) * X2(y) \} \\
&\leq \sup_y \{ R(z,y) * X2(y) \} \\
&\leq \sup_y \{ X2(z) \} \\
&= X_2(z)
\end{aligned}
$$

Therefore, $X_1(z) \leq X_2(z)$

Similarly, we can prove $X_2(z) \leq X_1(z)$

Hence, $X_1 = X_2$.

Corollary (IV:2:10)

If X_1, X_2 are fuzzy equivalence classes such that $X_1 \subseteq X_2$ under the assumption of Prop. (IV:2:9), then $X_1 = X_2$.

Proof

$X_1 \subseteq X_2$ implies $X_1(x) \leq X_2(x) \ \forall x \in U$.

But, $\exists x \in U$ such that $X_1(x) = 1$.

Therefore, $X_1(x) = X_2(x) = 1$

Hence the result.

Proposition (IV:2:11) [D_1]

For any $x, y \in U$, $x \neq y$ and $R(x,y) = 1$, fuzzy classes $[x]_R$ and $[y]_R$ are equal.

Proof

Let $z \in U$. Then

$$
\begin{aligned}
[x]_R(z) &= R(x,z) \\
&\geq R(x,y) * R(y,z) \\
&= R(x,y), \text{ since } R(x,y) = 1 \\
&= [y]_R(z)
\end{aligned}
$$

Thus $[x]_R \geq [y]_R$

Similarly we can prove $[y]_R \geq [x]_R$

Hence $[x]_R = [y]_R$.

Proposition (IV:2:12)[D_2]

Let X be a fuzzy equivalence class with respect to *-similarity relation R on U. Then x ε U such that

$X = [x]_R$.

Proof

Since X is normalized, there exist x_o ε U such that $X(x_o) = 1$

Consider the fuzzy class $[x_o]_R$.

By Prop.(IV:2:8), $[x_o]_R$ is fuzzy equivalence class.
Hence X and $[x_o]_R$ are two fuzzy equivalence classes such that

$$X(x_o) = 1 = [x_o]_R\,(x_o)$$

Therefore by proposition (IV:2:9)

$$X = [x_o]$$

Hence the proof.

Proposition (IV:2:8) and (IV:2:9) leads to the following.

Proposition (IV:2:13)[D_2]

The set of fuzzy equivalence classes is the set $\{ [x]_R \mid x \in U \}$, for any *-similarity relation R.

Definition (IV:2:14)

Let $\Phi = \{ X_1, X_2, \ldots, X_n \}$ be a family of normal fuzzy sets and $n < |U|$. Then Φ is said to be <u>fuzzy partition</u> on U, if

(i) $\inf_{x} \left[\max_{i=1,n} X_i(x) \right] > 0$

(Φ covers U)

(ii) $\min\{ X_i(x), X_j(x) \} < 1, \forall i,j;\ i \neq j$

(i.e. all X_i's are disjoint)

Note (IV:2:15)

In [D_1,D_2] fuzzy partition is defined as follows.

(i) $\inf_{x} \left[\max_{i=1,n} X_i(x) \right] > 0$

(ii) $\sup_{x} \{ \min(X_i(x), X_j(x)) \} \ \forall i,j;\ i \neq j$

However, we shall use Def.(IV:2:14) which is moregeneral.

Proposition (IV:2:16)

Let U be the universe. Then a *-similarity relation R on U induces a fuzzy partition on U.

Proof

Let $\Phi = \{ [x]_R \mid x \in U \}$

Our claim is that Φ is fuzzy partition on U.

(i) We have for any $y \in U$

$$\cup\{ [x]_R \mid x \in U \}(y) = \sup_x \{ [x]_R(y) \} = 1 > 0$$

Therefore,

$$\inf \{ \max_x \{ [x]_R(y) \mid x \in U \}\} > 0.$$

(ii) Let $[x]_R \neq [y]_R$

suppose that for some $z \in U$

$$\min \{ [x]_R(z), [y]_R(z) \} = 1$$

Therefore,

$$[x]_R(z) = [y]_R(z) = 1$$

But then by Prop.(IV:2:9),

$$[x]_R = [y]_R$$

This is a contradiction

Hence $\min\{ [x]_R(z), [y]_R(z) \} < 1$.

Thus Φ be a fuzzy partitioning of U.

IV:3 FUZZY ROUGH FUZZY SETS

Definition (IV:3:1)[D_1,D_2]

Let U be the universe, $\Phi = \{ X_1,\ldots,X_n \}$, $n < |U|$ be a fuzzy partition of U. Let X be a fuzzy set of U. The <u>lower approximation</u> of X with respect to Φ, is the fuzzy set

$$\underline{P}X : \Phi \longrightarrow [0,1] \text{ defined as follows :}$$

$$\underline{P}X(X_i) = \inf \{ \max(1-X_i(x); X(x)) \}$$

Definition (IV:3:2)[D_1,D_2]

Let U be the universe. $\Phi = \{X_1,\ldots,X_n\}$, $n < |U|$ be a fuzzy partition of U. Let X be a fuzzy set of U. The <u>upper approximation</u> of X with respect to Φ, is the fuzzy set $\bar{P}X$: $\Phi \longrightarrow [0,1]$ defined as,

$$\bar{P}X(X_i) = \sup_x \{ \min (X_i(x)\ X(x)) \}$$

Definition (IV:3:3) [D_1,D_2]

Let U be the universe, $\Phi = \{ X_i, X_2, \ldots, X_n \}, n < |U|$ be a partition of U. A <u>fuzzy rough fuzzy set</u> is a pair ($\underline{P}X$, $\bar{P}X$).

Definition (IV:3:4) [D_1,D_2]

Let R be a *-similarity relation on U and let $U/R = \{ [x]_R \mid x \varepsilon U \} = \Phi$ be a fuzzy partitioning of U. The <u>lower approximation</u> of X with respect to R is the fuzzy set

$$w(\underline{P}X) : U \longrightarrow [0,1] \text{ defined as follows}$$

$$w(\bar{P}X)(x) = \inf_x \{ \max(X(y),\ 1-R(x,y)) \} \quad \forall\ x\ \varepsilon\ U.$$

where w is a mapping from U/R to U defined as

$$w([x]_R) = \{ y \mid [x]_R = [y]_R \} = \{ y \mid y\ \varepsilon\ [x]_R \}.$$

Definition (IV:3:5) [D_1,D_2]

Let $U/R = \{ [x]_R | x\ \varepsilon\ U \} = \Phi$, be a partitioning of U and X be a fuzzy set of U. The <u>upper approximation</u> of X with respect to R is the fuzzy set,

$w(\bar{P}X) : U \longrightarrow [0,1]$ defined as,

$$w(\bar{P}X)(x) = \sup_{x} \{ \min(X(y), R(x,y)) \}$$

Definition (IV:3:6) [D_1,D_2]

A <u>fuzzy rough fuzzy set</u> is a pair $(w(\underline{P}X), w(\bar{P}X))$.

Proposition (IV:3:7)

Let R be a crisp equivalence relation, then the fuzzy rough fuzzy set defined by Def.(IV:3:3) agrees with the rough fuzzy set defined by Def.(III:2:3).

Proof

Let $\Phi_1 = \{ [x]_R | x \varepsilon U \}$ be a (crisp) partitioning of U, then by Def.(III:2:3) $(\underline{R}X, \bar{R}X)$ be a rough fuzzy set where,

$$\underline{R}X([x]_R) = \inf_{x}\{ X(y) | [x]_R = [y]_R \} \qquad (i)$$

and

$$\bar{R}X([x]_R) = \sup_{x}\{ X(y) | [x]_R = [y]_R \} \qquad (ii)$$

Let, $\Phi_2 = \{\mathcal{X}_{[x]_R} | x \varepsilon U \}$ be a fuzzy partitioning induced by (characteristics function of) R, then by Def.(IV:3:3) $(\underline{P}X, \bar{P}X)$ be a fuzzy rough fuzzy set where

$$\underline{P}X(\mathcal{X}_{[x]_R}) = \inf_{y} \{ \max (X(y), 1-\mathcal{X}_{[x]_R}(y) \} \qquad (iii)$$

and

$$\bar{P}X(\mathcal{X}_{[x]_R}) = \sup_{y} \{ \min (X(y), \mathcal{X}_{[x]_R}(y) \} \qquad (iv)$$

Our claim is that,

$$\underline{P}X(\chi_{[x]_R}) = \underline{R}X([x]_R) \quad \text{and}$$

$$\bar{P}X(\chi_{[x]_R}) = \bar{R}X([x]_R)$$

Consider, $\underline{P}X(\chi_{[x]_R}) = \inf_y\{\max(X(y), 1-\chi_{[x]_R}(y))\}$, by (III)

$$= \inf_y\{ X(y) \mid y \in [X]_R \}$$

$$= \inf_y\{ X(y) \mid [x]_R = [y]_R \}$$

$$= \underline{R}X([x]_R), \quad \text{by (I)}$$

Similarly,

$$\bar{P}X(\chi_{[x]_R}) = \sup_y\{ \min(X(y), \chi_{[x]_R}(y)) \}, \quad \text{by (IV)}$$

$$= \sup_y \{ X(y) \mid y \in [x]_R \}$$

$$= \sup_y \{ X(y) \mid [x]_R = [y]_R \}$$

$$= \bar{R}X([x]_R)$$

Proposition (IV:3:7)

Let Φ be a fuzzy partition of U be such that $\Phi = U/R$, where R be a similarly relation on U. Let $x \in U$ be such that $[x]_R = X_i \in U$. Then

$$w(\underline{P}X)(x) = \underline{P}X([x]_R) \quad \text{and}$$

$$w(\bar{P}X)(x) = \bar{P}X([x]_R)$$

Proof

$$w(\underline{P}X)(x) = \inf_{y} \{ \max(X(y),\ 1-R(x,y)) \}$$

$$= \inf_{y} \{ \max(X(y),\ 1-[x]_R(y)) \}$$

$$= \inf \{ \max(X(y),\ 1-X_i(y)) \}$$

$$= \underline{P}X\ (X_i)$$

$$= \underline{P}X\ ([x]_R)$$

Similarly,

$$w(\bar{P}X)(x) = \bar{P}X([x]_R)$$

Proposition (IV:3:8)

Let R be a similarity relation on U be such that $U/R = \Phi$; fuzzy partitioning of U. Then followings are holds.

i) $w(\bar{P}(X \cup Y)) = w(\bar{P}X) \cup w(\bar{P}Y)$

ii) $w(\bar{P}(X \cap Y)) \subseteq w(\bar{P}X) \cap w(\bar{P}Y)$

iii) $w(\underline{P}(X \cap Y)) = w(\underline{P}X) \cap w(\underline{P}Y)$

iv) $w(\underline{P}(X \cup Y)) \supseteq w(\underline{P}X) \cup w(\underline{P}Y)$

v) $w(\bar{P}(-X)) = -w(\underline{P}X)$

vi) $w(\underline{P}(-X)) = -w(\bar{P}X)$

vii) $w(\underline{p}x) \subseteq X \subseteq w(\bar{P}X)$

viii) $w(\bar{P}\bar{P}X) \supseteq w(\bar{P}X)$ and $w(\underline{P}\underline{P}X) \subseteq w(\underline{P}X)$

Proof Let $x \in U$

(i) Consider

$$w(\bar{P}(X \cup Y))(x) = \sup_y \{ \min((X \cup Y)(y); R(x,y) \}$$

$$= \sup_y \{ \min(\max(X(y), Y(y)); R(x,y)) \}$$

$$= \sup_y \{ \max(\min(X(y), R(x,y)); \max(Y(y), R(x,y))) \}$$

$$= \max(\sup_y \{ \min(X(y), R(x,y)) \}; \sup_y \{ \min(Y(y), R(x,y)) \})$$

$$= \max(w(PX)(x), w(PY)(x))$$

$$= (w(\bar{P}X) \cup w(\bar{P}Y))(x)$$

Thus,

$$w(\bar{P}(X \cup Y)) = w(\bar{P}X) \cup w(\bar{P}Y)$$

(ii)

$$w(\bar{P}(X \cap Y))(x) = \sup_y \{ \min((X \cap Y)(y); R(x,y) \}$$

$$= \sup_y \{ \min(\min(X(y), Y(y)); R(x,y)) \}$$

$$= \sup_y \{ \min(\min(X(y), R(x,y)); \min(Y(y), R(x,y))) \}$$

$$\leq \min(\sup_y \{ \min(X(y), R(x,y)) \}; \sup_y \{ \min(Y(y), R(x,y)) \})$$

$$= \min(w(\bar{P}X)(x), w(\bar{P}Y)(x))$$

$$= (w(\bar{P}X) \cap w(\bar{P}Y))(x)$$

Thus,

$$w(\bar{P}(X \cap Y)) \subseteq w(\bar{P}X) \cap w(\bar{P}Y)$$

$$\text{(iii)}\quad w(\underline{P}(X \cap Y))(x) = \inf_y \{ \max((X \cap Y)(y);\ 1-R(x,y) \}$$

$$= \inf_y \{ \max(\min(X(y),Y(y));\ 1-R(x,y)) \}$$

$$= \inf_y \{ \min(\max(X(y),1-R(x,y));\ \max(Y(y),\ 1-R(x,y))) \}$$

$$= \min(\inf_y \{ \max(X(y),1-R(x,y)) \};\ \inf_y \{ \max(Y(y),\ 1-R(x,y)) \})$$

$$= \min(w(\underline{P}X)(x),\ w(\underline{P}Y)(x))$$

$$= (w(\underline{P}X) \cap w(\underline{P}Y))\ (x)$$

Thus,

$$w(\underline{P}(X \cap Y)) = w(\underline{P}X) \cap w(\underline{P}Y)$$

$$\text{(iv)}\quad w(\underline{P}(X \cup Y))(x) = \inf_y \{ \max((X \cup Y)(y);\ 1-R(x,y) \}$$

$$= \inf_y \{ \max(\max(X(y),Y(y));\ 1-R(x,y)) \}$$

$$= \inf_y \{ \max(\max(X(y),1-R(x,y));\ \max(Y(y),\ 1-R(x,y))) \}$$

$$\geq \max(\inf_y \{ \max(X(y),1-R(x,y)) \};\ \inf_y \{ \max(Y(y),\ 1-R(x,y)) \})$$

$$= \max(w(\underline{P}X)(x),\ w(\underline{P}Y)(x))$$

$$= (w(\underline{P}X) \cup w(\underline{P}Y))\ (x)$$

Thus,

$$w(\underline{P}(X \cup Y)) \supseteq w(\underline{P}X) \cup w(\underline{P}Y)$$

(v) $w\,(\overline{P}(-X))(x) = \sup_{y}\{\min(-X(y),\ R(x,y)\ \}$

$$= \sup_{y}\{\ 1-\max(X(y),\ 1-R(x,y))\ \}$$

$$= 1-\inf_{y}\{\ \max(X(y),\ 1-R(x,y))\ \}$$

$$= 1-\ w(\underline{P}X)\ (x)$$

$$= \ -\ w(\underline{P}X)\ (x)$$

Thus,

$$w(\overline{P}(-X)) = -w(\underline{P}X)$$

(vi) Similarly we can prove

$$w(\underline{P}(-X)) = -w(\overline{P}X)$$

(vii) Consider $w(\overline{P}X)(x) = \sup_{y}\{\ \min(X(y),\ R(x,y))\ \}$

$$= \max\{\ X(x),\ \sup_{x \neq y}\{X(y),\ R(x,y)\ \}\}$$

$$\geq\ X(x)$$

Thus $w(\overline{P}X) = X$ (I)

We have,

$-X \subseteq w(\overline{P}(-X))$, Since by (V)

$= -w(\underline{P}X)$

Hence,

$w(\underline{P}X) \subseteq X$ (II)

By (I) and (II)

$$w(\underline{P}X) \subseteq X \subseteq w(\overline{P}X)$$

(viii) We have,

$$w(\bar{P}\bar{P}X) = w(\bar{P}(w(\bar{P}X))) \text{ and}$$

$$w(\underline{P}\underline{P}X) = w(\underline{P}(w(\underline{P}X)))$$

To prove (viii), replace X by $w(\underline{P}X)$ and w(PX) respectively in inequality (vii).

IV:4 SUMMARY

We have shown in this chapter that the fuzzy rough set defined by Dubois and Prade [D_1,D_2] is nothing but the fuzzy rough fuzzy set. Dubois and Prade used fuzzy relation; fuzzy partition, in the defination of rough set and obtain fuzzy rough fuzzy set corresponding to a given fuzzy set.