CHAPTER – I

INTRODUCTION

CHAPTER- 1

INTRODUCTION

(1.1) INTRODUCTION:

The second order linear differential equations with variable coefficients

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0.$$

when solved by the method of solution in series, gives certain functions, which are not elementary functions (like trigonometrically, logarithmic or exponential). These functions are often called special functions of mathematical physics or more simply the special functions [32.]

The theory of special functions plays a basic role in the formalism of mathematical physics. It covers an extremely wide domain of study (formulated early in the pioneering works of Euler, Gauss, Laplace, Riemann and many others) and continuously refined by new achievements and suggestions.

These functions (polynomials) have been studied since long but a systematic, through and very fast development of the subject has been done during the last five decades. During the decades of eighties the development of larger computing machine has made it possible to study functions with multiple series representations from numerical point of view. Most of the functions of this type are hypergeometic in character and they occur in connection with such matter as statistical distributions, functional equations, characterizations, quantum theory, vibration of beams, conduction of heat, elasticity, telecommunication as well as agriculture and biological sciences. They provide a unique tool for developing simplified yet realistic models of physical problems, thus allowing for analytic solutions and hence a deeper insight into the problem under study.

The specific physical problem can suggest investigating new aspects of the well established theory of special functions as well as introducing new functions and other possible generalizations, which usually exhibit deeper features and there by appear again and again in new role in various fields of mathematics.

Definition:

Orthogonal Polynomials:

A system of real functions $f_n(x)$ (n=0,1,2,....) is said to be orthogonal with weight $\rho_{(x)}$ on the interval [a,b] if

$$\int_a^b \rho_{(x)} f_m(x) f_n(x) dx = 0, \text{if } m \neq n$$

$$\neq 0 \text{ if } m = n \qquad --- \quad (1\cdot1\cdot1)$$

where $\rho(x)$ is a fixed non-negative functions which does not depends on the indices m and n [18].

Example :

The system of functions cosnx(n=0,1,2,.....) is orthogonal with weight 1 on the interval $[0, \pi]$

$$\int_0^\pi \cos mx \cdot \cos nx \cdot dx = 0, \text{if } m \neq n,$$

$$\neq 0, \text{ if } m = n$$

An important class of orthogonal system consists of orthogonal polynomials $P_n(x)$ (n=0,1, 2,.......) , where n is the degree of the polynomials $P_n(x)$. This class contains many special functions commonly encountered in the applications, e.g. Legendre, Hermite, Languerre, Chebyshev and Jacobi polynomials. In addition to the orthogonality property (1.1.1), these functions have many other general properties.

Orthogonal polynomials are of great importance in mathematical physics, approximation theory, the theory of mechanical quadran tures etc.

(1.2) GENERATING FUNCTIONS:

Generating functions play an important role in the investigation of various useful properties of the sequence, which they generate. In 1812, the name generating function was introduced by mathematician Laplace. The theory of generating functions has been developed into various directions and found wide applications in different, branches of Science and Technology. A generating functions may be used to define a set of functions to determine a pure recurrence relation or a differential recurrence relation to evaluate certain integrals etc.

Linear Generating Functions:-

Consider a two variable function F(x , t) which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in t such that ,

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n. \text{-------}(1.2.1)$$

where each member of the coefficient set

$$\{f_n(x)\}_{n=0}^{\infty}$$

is independent of t. Then the expansion (1.2.1) of F(x , t) is said to have generated the set $\{f_n(x)\}$ and F(x , t) is called a linear generating function (or simply, a generating function) for the set $\{f_n(x)\}_{n=1}^{\infty}$ [35]

The definition may be extended slightly to include a generating functions of the type.

$$G(x,t)=\sum_{n=0}^{\infty} C_n g_n(x) t^n \text{ --------(1.2.2)}$$

where the sequence $\{C_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of x and t.

A set functions may have more than one generating function. However, if ,

$$G(x,t)=\sum_{n=0}^{\infty} h_n(x)\cdot t^n$$

Then G (x , t) is unique generator for the set $\{h_n(x)\}$ as the coefficient set.

We now extend our definition of a generating function to include functions which possess Laurent series expansions. If the set $\{f_n(x)\}$ is defined for $n = 0, \pm 1, \pm 2\dots\dots$, the definition (1.2.2) can be extended in terms of the Laurent series expansion:

$$F(x,t)=\sum_{n=-\infty}^{\infty} \gamma_n f_n(x)\cdot t^n \text{ ------(1.2.3)}$$

where the sequence $\{\gamma_n\}$ is independent of x and t

Bilinear and Bilateral generating Functions:-

If a three variable function f(x, y, t) possesses a formal power series expansion in t such that

$$F(x,y,t)=\sum_{n=0}^{\infty}\gamma_n f_n(x) f_n(y) t^n \text{ ------(1.2.4)}$$

where the sequence $\{\gamma_n\}$ is independent of x , y and t then F(x, y, t) is called a bilinear generating function for the set $\{f_n(x)\}$

Suppose that a three variable function H(x, y, t) has a formal power series expansion in t such that ,

$$H(x,y,t)=\sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^2 \text{ ------(1.2.5)}$$

where the sequence $\{h_n\}$ is independent of x , y and t the sets of functions

$$\{f_n(x)\}_{n=0}^{\infty}, \{g_n(x)\}_{n=0}^{\infty}$$

are different, then H(x, y, t) is called a bilateral generating function for the sets.

$$\{f_n(x)\}_{n=0}^{\infty} \text{..or..} \{g_n(y)\}_{n=0}^{\infty}$$

The above definitions of a bilateral generating functions used earlier by Rainville [30, p.170] and McBride [21, p.19] may be extended to include bilateral generating functions of the type.

$$.H(x,y,t)=\sum_{n=0}^{\infty}\gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y)\cdot t^n \text{ --(1.2.6)}$$

where the sequence $\{\gamma_n\}$ is independent of x , y and t the sets of functions

$$\{f_n(x)\}_{n=0}^{\infty}...\text{and}....\{g_n(y)\}_{n=0}^{\infty}$$

are different and $\alpha(n)$ and $\beta(n)$ are functions of n which are not necessarily equal.

Suppose $G(x_1,x_2,.....x_r\,;\,t)$ is a function of $r+1$ variables which has a formal expansion in powers of t such that

$$.G(x_1 x_{2,}----x_r;t)=\sum_{n=0}^{\infty}C_n g_n(x_1 x_{2,}----x_r)\cdot t^n ------(1.2.7)$$

where the sequence $\{C_n\}$ is independent of the variables $x_1,x_2,.....x_r$ and t. Then we say that $G(x_1,x_2,.....x_r;\,t)$ is a multivariable generating functions for the set $\{g_n(x_1,x_2,.....x_r)\}_{n=1}$ corresponding to non zero coefficient C_n.

A natural extension of the multivariable generating function (1.2.7) is a multiple generating function which may be defined formally by

$$\Psi(x_1,x_2,......x_r;t_1,t_2,....t_r)$$

$$=\sum_{n_1,n_2,--n_r=0}^{\infty}C(n_1,n_2,--n_r)\cdot\phi\,n_1,n_2,....n_r(x_1,x_2,--x_r)t_1^{n_1}t_2^{n_2}......,t_r^{n_r}--(1.2.8)$$

where the multiple sequence $\{C(n_1,\,n_2,....,n_r\}$ is independent of the variables x_1, $x_2,....,x_r$ and $t_1,\,t_2,.....t_r$.

It is not difficult to extend the definitions of bilinear and bilateral generating functions to include multivariable generating functions as (1.2.9) $F(x_1,x_2,\ldots\ldots x_r; y_1,y_2,\ldots\ldots y_r; t)$

$$F(x_1,x_2,\ldots\ldots,x_r;y_1,y_2,\ldots.y_r;t)=\sum_{n=0}^{\infty}\gamma_n f_{\alpha(n)}(x_1,x_2,\ldots\ldots x_r)f_{\beta(n)}(y_1,y_2,\ldots.y_r)\cdot t^n \quad \text{---(1.2.9)}$$

and

$H(x_1,x_2,\ldots\ldots,x_r; y_1,y_2,\ldots\ldots,y_s; t)$

$$=\sum_{n=0}^{\infty} h_n f_{\alpha(n)}(x_1,x_2,---x_r)g_{\beta(n)}(y_1,y_2,---y_r)t^n \quad \text{--(1.2.10)}$$

respectively.

A multi-variables generating function $G(x_1,x_2,\ldots\ldots x_r\ t)$ given by (1.2.7) is said to be a multi-linear generating functions if,

$$g_n(x_1,x_2,\ldots\ldots x_r)= f\alpha_1(n)(x_1)\ f\alpha_2(n)(x_2)\ldots\ldots f_{\alpha_r}(n)(x_r) \quad \text{——(1.2.11)}$$

where $\alpha_1(n)$, $\alpha_2(n),\ldots\ldots\alpha_r(n)$; are functions of n which are not necessarily equal. More generally, if the functions occurring on the right hand side of (1.2.11) are all different, the multivariable generating functions (1.2.7) will be called a multilateral generating function.

(1.3) Brief Survey of the work done the other Researchers.

In brief we shall discuss here the general context of bi-orthogonal polynomial sets with particular reference to the known result obtained by others.

Let $r(x)$ and $s(x)$ be real polynomials in x of degree $h > 0$ and $k > 0$ respectively. Let $R_m(x)$ and $S_n(x)$ denote polynomials of degree m and n in $r(x)$ and

$s(x)$ respectively then $R_m(x)$ and $S_n(x)$ are polynomials of degree mh and nh in x. The polynomials $r(x)$ and $s(x)$ are called basic polynomials.

The real valued function $p(x)$ of real variable is x is an admissible weight function on the finite or infinite interval (a, b) if all the moments

$$I_{i,j} = {}_a\int^b p(x)\{r(x)\}^i\{s(x)\}^j \cdot dx \cdot, \quad \forall\, i,j = 0,1,2,\ldots\ldots \qquad ---(1.3.1)$$

exists, with

$$I_{0,0} = {}_a\int^b p(x)dx \neq 0.$$

Definition:

Biorthogonal Polynomials:

The polynomials sets $\{R_m(x)\}$ and $\{S_n(x)\}$ are biorthogonal over the interval (a, b) with respect to the admissible weight functions $\rho(x)$ over the interval (a, b) and basic polynomials $r(x)$ and $s(x)$ provided orthogonality conditions.

$$\left.\begin{aligned} {}_a\int^b p(x)\{R_m(x)\}\{s_n(x)\}dx &= 0, \text{if } m,n = 0,1,2,---, m \neq n \\ &\neq 0, \text{ if } m = n \qquad \text{---}(1.3.2) \end{aligned}\right]$$

are satisfied.

In 1951, Spencer and Fano [33] used a particular pair of biorthogonal polynomial sets in order to calculate penetration of gamma rays through matter. They did not establish any general properties for this particular pair of biorthogonal polynomial sets but they obtained the property of biorthogonality of polynomials in x and polynomials in x^2 with respect to weight function $x^\alpha e^{-x}$ where α is non negative integer over the interval $(0, \infty)$.

Preiser [29] in 1962 acquainted with the work of Spencer and Fano,[33] established the third order differential equation of the type,

$$.A(x)Y_n^{\prime\prime\prime} + B(x)Y_n^{\prime\prime} + C(x)Y_n^{\prime} = \lambda_n Y_n \text{ - - - - - } (1.3.3)$$

exists such that it has biorthogonal polynomials solutions of degree n in x^m and such that the reduced adjoint differential equations of

$$((p(x)A(x)Z_n)^{\prime\prime\prime} + (P(x)B(x)Z_n)^{\prime\prime} - (P(x)C(x)Z_n)^{\prime} = \lambda_n P(x)Z_n . \text{ - - } (1.3.4)$$

has biorthogonal polynomial solutions of degree n in x, n=0,1,2.... In (1.3.3) and

(1.3.4) the prime denote differentiation with respect to x; p(x) is a weight function having three continuous derivatives, A(x), B(x) C(x) are functions of x independent of n and λ_n is a parameter independent of x.

Preiser also established the existence of pure recurrence relations connecting four successive polynomials and obtained generating functions for the polynomials in x^2 and gave integral representations for his pair of biorthogonal polynomial sets.

Unaware of the work of Spencer – Fano [33] and Preiser [29], I. M. Sheffer, proposed the application of notion of biorthogonality to polynomial sets, the determination of the conditions under which pairs of biorthogonals set exists and the comparison of the properties of biorthogonal polynomials with properties of orthogonal polynomials. With these directions and under the guidance of I. M. Sheffer, Konhauser [16] in 1965 obtained may more general properties of birothogonal polynomial sets.

In order to apply his general investigations of biorthogonal sets, Konhauser [17], 1967 considered two biorthogonal polynomials sets $\{Z_n^{\alpha}(x; k)\}$ and $\{Y^{\alpha}_n(x;k)\}$

of degree n in x^k and x respectively, where x is real, k is positive integer and $\alpha > -1$. For k=1 both these polynomials set reducesto the generalized Laguerre polynomials $L^{\alpha}_n(x)$; for k =2 we get the polynomials considered by Preiser [29].

In fact Konhauser [17] stated that

$$Z_n^{\alpha}(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \text{------}(1.3.5)$$

and

$$Y_n^{\alpha}(x;k) = \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left[\frac{e^{-xt}(t+1)^{\alpha+kn}}{(t^{k-1}+kt^{k-2}+\dots+k)^{n+1}} \right] \Bigg|_{t=0} \text{------}(1.3.6)$$

we call (1.3.5) as Konhauser biorthogonal set of first kind and (1.3.6) as Konhauser biorthogonal set of second kind.

The field of study of biorthogonal polynomial sets is quite new and hence it will be worthwhile to review some of the work accomplished in this very interesting area. Prabhakar [25] has obtained generating functions of integrals and recurrence relations for the Konhauser biorthogonal sets of the first kind $Z_n^{\alpha}(x ; k)$. .At a letter stage Prabhakar deduced a generating functions, Rodrigue's formula by using contour integral representation for the polynomials $Y_n^{\alpha}(x; k)$. An application of these results be obtained two finite sums involving $Z_n^{\alpha}(x; k)$.

Using Langrange's theorem, Carlitz [5, 6, 7] obtained a generating function and an explicit polynomial expression for the polynomials $Y_n^{\alpha}(x; k)$. Also he shown that $Y_n^{\alpha}(x; k)$ can be identified with the polynomials studied recently by Chatterjea [8,9].

Karande and Thakare [14] have constructed a much more general generating functions in the hypergeometric form. Some recurrence relations, bilinear generating functions for $Z_n^{\alpha}(x; k)$. They introduced the polynomials related to the Konhauser biorthogonal polynomials of the second kind and obtained some of their properties.

Patil [23,24] have obtained some operational formula for $Y_n^{\alpha}(x; k)$ in terms of the differential operators $tx^k + x^{k+1} D$ where $D = d/dx$. By using operational technique they obtained pure recurrence relation, generating functions and multilinear generating function for the Konhauser biorthogonal polynomial, sets.

The question of constructing a pair of biorthogonal polynomials suggested by the Jacobi polynomials and the Hermite polynomials remained open for several years. Prabhakar and Tomar [28] introduced a biorthogonal pair $\{U_n(x; k)\}$ $\{V_n(x;k)\}$ of polynomial sets which is analogously suggested by the orthogonal set of Legendre polynomials $P_n(x)$. Prabhakar and Kashyap [27] have discussed the polynomials.

$$U_n^{\alpha}(x;k) = \frac{1}{\left(\frac{1}{k}\right)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+j}{k}\right)_n \left(\frac{1-x}{2}\right)^j \text{----(1.3.7)}$$

$$.V_n^{\alpha}(x;k) = \frac{1}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} (1+\alpha+kj) \left(\frac{1-x}{2}\right)^{kj} \text{-------(1.3.8)}$$

and shown that they form a pair of biorthogonal polynomials over (-1, 1) with respect to the weight function $\{ [(1-x)/2)]^{\alpha} \}$. For $k=1$ both $U_n^{\alpha}(x; k)$ and $V_n^{\alpha}(x; k)$ reduces to $P_n^{\alpha,0}(x; k)$

Madhekar and Thakare [19, 20] succeed in completely setting the question of constructing a pair of biorthogonal polynomials suggested by the Jacobi polynomials. They have defined the following pair of biorthogonal polynomials $J_n(\alpha, \beta, k; x)$ and $K_n(\alpha, \beta, k; x)$ which are suggested by Jacobi polynomials.

$$J_n(\alpha,\beta,k;x)=\frac{(1+\alpha)_{kn}}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}}\left(\frac{1-x}{2}\right)^{kj} \quad \text{---}(1.3.9)$$

$$K_n(\alpha,\beta,k;x)=\sum_{r=0}^{n}\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s}\frac{(1+\beta)_n}{n!\,r!\,(1+\beta)_{n-r}}\left(\frac{s+\alpha+1}{k}\right)_n\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r} \quad \text{---}(1.3.10)$$

In $J_n(\alpha, \beta; k; x)$ has the following hypergeometric form

$$J_n(\alpha,\beta,K;X)=\frac{(1+\alpha)_{kn}}{n!}\,{}_{k+1}F_k\left[\begin{matrix}-n,\Delta(k,1+\alpha+\beta+n);\\ \Delta(k,1+\alpha);\end{matrix}\left(\frac{1-x}{2}\right)^k\right] \quad \text{--}(1.3.11)$$

where $\Delta(m, \delta)$ is sequence of m parameters..

$$\frac{\delta}{m},\frac{\delta+1}{m},\ldots\ldots\ldots\frac{\delta+m-1}{m},m\geq 1.$$

For k=1, both, $J_n(\alpha, \beta, k; x)$ and $K_n(\alpha, \beta, k; x)$ reduces to the Jacobi polynomials $P_n^{\alpha,\beta}(x)$ Thakare and Madhekar [37,38] obtained the generating functions recurrence relations, multilinear generating relations, bilateral generating functions etc. Parashar [22] also considered the biorthogonal polynomials $J_n^{\alpha,\beta}(x;k)$ and he obtained the recurrence relations, expansion formula by using a forward difference operator.

$$\Delta_k f(\alpha) = f(\alpha+k) - f(\alpha).$$

The study of biorthogonality of pair of polynomials sets received impetus after the publication of the work of Konhauser [16,17]. Unacquainted with the work of Didon [12] and Deruyts [11]. Chai [10] suggested hypergeometric polynomials $Z_n(x\,;k)$ of degree n in x^k that satisfy the following condition,

$$\int_0^1 x^{\alpha}(1-x)^{\beta} Z_n(x,k)x^i dx = 0,\ i = 1,2,3,\ldots\ldots, n-1 \quad ---(1.3.12)$$

$$\neq 0,\ i = n$$

where $Z_n(x;k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}} x^{kj}$

and $\alpha > -1$, k and β are nonnegative integers.

Madhekar and Thakare [19] constructed a pair biorthogonal polynomials with respect to the weight function $(1-x)^{\alpha}$ $(1-x)^{\beta}$, ($\alpha > -1$, $\beta > -1$) over the interval (-1,1).

The pair of biorthogonal polynomials corresponding to the Laguerre weight function could be given as follows (see Konhauser [17] and Carlitz [5])

$$Z_n^{\alpha}(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \quad ---(1.1.13)$$

$$Y_n^{\alpha}(x;k) = \frac{1}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \cdot \left(\frac{s+\alpha+1}{k}\right)_n \quad ----(1.3.14)$$

These polynomials are called as Konhauser polynomials. The Konhauser polynomial $Z_n^{\alpha}(u;k)$ and $Y_n^{\alpha}(u;k)$ satisfy biorthogonal condition.

$$ {}_0\int^{\infty} u^{\alpha} \exp(-u) Z_n^{\alpha}(u;k) Y_n^{-\alpha}(u;k) du = \frac{\Gamma(kn+\alpha+1)}{n!} \delta_{n,m} \quad ---(1.3.15) $$

where $\delta_{n,m}$ is the usual Kronecker's delta.

$$ \delta\, n,m = 1 \,, \text{if } n = m, $$

$$ = 0 \,, \text{if } n \neq m $$

Thakare and Madhekar [37, p.1033.] introduced the polynomials $S_n(x;k)$ and $T_n(x\,;\,k)$ that are related to he Konhauser biorthogonal pair $Z_n^{\alpha}(x;k)$ and $Y_n^{\alpha}(x;k)$ in the following compact form

$$ S_n(x;k) = 2^n \Gamma\left(\frac{kn+k-k\epsilon}{2} + \epsilon\right) \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{\left[\frac{n}{2}\right]}{j} \frac{x^{nk-2kj}}{\Gamma\left(\frac{nk+1+\epsilon}{2} - kj\right)} \quad --(1.3.16) $$

$$ .T_n(x;k) = (-1)^{\left[\frac{n}{2}\right]} 2^n \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r}}{\left(\left[\frac{n}{2}\right]-r\right)!} \sum_{s=0}^{\left[\frac{n}{2}\right]} (-1)^s \binom{\left[\frac{n}{2}\right]-r}{s} \cdot \left(\frac{s+(k+1)\epsilon/2 + 1/2}{k}\right)_{\left[\frac{n}{2}\right]} \quad --(1.3.17) $$

where ϵ is 0 or 1 according to n is even or odd integer. Through out this dissertation ϵ will always have this meaning. Also through out [n] denotes the greatest integer less than or equal to n and k to be positive odd integer in view the of existence theorem of Konhauser [16, p.253]. We note that $S_n(x\,;\,k)$ and $T_n(x\,;\,k)$ are polynomials of degree m and n in x^k and x respectively. For $k = 1$ above relations

gives relationship between the Laguerre's polynomials and the Hermite polynomials (see Szego [36]. Since for k=1 both $S_n(x\,;\,k)$ and $T_n(x\,;\,k)$ reduces to Hermite polynomials and both reduced to the Laguerre's polynomials.

The biorthogonality of the sets $\{s_n(x\,;k)\}$ and $\{T_n(x\,;k)\}$ can be obtained by taking $\alpha = -1/2, \alpha = k/2$ in the biorthogonality relationship for Konhauser polynomials

$$\int_{-\infty}^{\infty} \exp(-x^2) S_n(x;k) T_m(x;k) dx$$

$$= 2^{2n}\,\Gamma(\in + (kn + k - k\in)/2) \cdot [n/2]!\,\delta_{n,m} \qquad ---(1.3.18)$$

where $\delta_{m,n}$ is the usual Kronecker's delta.

The separate direct proof of above (1.3.18) biorthogonal condition is also possible by using the identity of Carlitz [5, p.429] in the form

$$(-j)_m = \sum_{r=0}^{m} \binom{kj+c+m-r}{m-r} \sum_{s=0}^{m-r} (-1)^s \cdot \binom{m-r}{s} \cdot \left(\frac{s+c+1}{k}\right)_m \qquad --(1.3.19)$$

Thakare and Madhekar [38] introduced a pair of biorthogonal polynomial sequence $\{S_n^{\mu}(x\,;k)\}$ and $T_n^{\mu}(x\,;k)$ with respect to Szego-Hermite weight function

$$|x|^{2\mu} \exp(-x^2), \mu > -\frac{1}{2}$$

$$S_n^{\mu}(x;k) = 2^n\,\Gamma((kn + k - k\in)/2 + \mu + \in) \cdot \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{\left[\frac{n}{2}\right]}{j} \frac{x^{nk-2kj}}{\Gamma\left(\frac{k\cdot n+1+\in}{2} - kj + \mu\right)} - (1.3.20)$$

$$T_n^{\mu}(x;k) = (-1)^{\left[\frac{n}{2}\right]} 2^n \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r}}{\left(\left[\frac{n}{2}\right]-r\right)!} \sum_{s=0}^{\left[\frac{n}{2}\right]-r} (-1)^s \binom{\left[\frac{n}{2}\right]-r}{s}$$

$$\cdot \left(\frac{(2s + (k+1)\in + 2\mu + 1)}{2k}\right)_{\left[\frac{n}{2}\right]} \qquad ---(1.3.21)$$

They have obtained following biorthogonal condition.

$$\int_{-\infty}^{\infty} |x|^{2\mu} \exp(-x^2) S_n^{\mu}(x;k) T_n^{\mu}(x;k) dx$$

$$= 2^{2n} \left[\frac{n}{2}\right]! \Gamma(\mu + \in + (kn + k - k \in)/2) \delta_{n,m} \quad ---(1.3.22)$$

The proof of (1.3.22) is possible independently by using the identity (1.3.19) where $\in$, [n], $\delta_{n,m}$ and k have their unusual meaning.

Thakare and Madhekar [38, 1987] obtained, generating functions, mixed recurrence relations for both these sets. For k=1 , both the above sets get reduced to the orthogonal polynomials introduced by Szego.

The work of Konhauser, Thakare and Madhekar motivated me to study the biorthogonal polynomials in special function theory.

(1.4) Notations and Basic formulae:

(i) $$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0; a \neq 0 \\ a.(a+1)(a+2),---(a+n-1), & \text{if } n = 1,2,3... \end{cases}$$

(ii) $$(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k}, 0 \leq k \leq n$$

(iii) $$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & k > n \end{cases}$$

(iv) $$\frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}, a \neq 0, \ \pm 1, \ \pm 2, ...$$

(v) $$(1+n)_n = 2^{2n}\left(\frac{1}{2}\right)_n$$

(vi) $$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \cdot \sum_{k=0}^{n} A(k,n-k)$$

(vii) $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty} \cdot \sum_{k=0}^{\infty} B(k,n+k)$$

(viii) $$\binom{m}{n} = {}^m C_n = \frac{m!}{n!(m-n)!} = \frac{(-1)^n (-m)_n}{n!}, \quad 0 \leq n \leq m.$$

(ix) $$\Delta(m,\delta) = \frac{\delta}{m}, \frac{\delta+1}{m}, \frac{\delta+2}{m}, --- \frac{\delta+m-1}{m}, \quad m \geq 1$$

(x) $$(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n$$