CHAPTER – II

BIORTHOGONAL POLYNOMIALS AND GENERATING FUNCTIONS

CHAPTER - 2

Biorthogonal Polynomials and Generating Functions.

(2.1) INTRODUCTION:

The notion of biorthogonality is extremely useful in calculations involving the penetration of gamma rays through matter as well as in determining moments of certain distribution functions.

Recently Andhare and Jagtap [2] constructed a pair of biorthogonal polynomials related to the Konhauser biorthogonal pair $Z_n^\alpha(x\,;\,k)$ and $Y_n^\alpha(x\,;k)$ in the following manner.

$$S_{2n}(x;k,\ell)=\frac{(-1)^n n!(1+n)_n}{(1+\beta)_n} Z_n^\beta(x^{2k};\ell) \text{----}(2.1.1)$$

$$=\frac{(-1)^n n!2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n} Z_n^\beta(x^{2k};\ell)$$

$$=\frac{(-1)^n n!\cdot 2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n}\frac{\Gamma(n\ell+\beta+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{2kj\ell}}{\Gamma(j\ell+\beta+1)}$$

$$=\frac{(-1)^n 2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n}\Gamma(n\ell+\beta+1)\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{2kj\ell}}{\Gamma(\ell j+\beta+1)}$$

Reverting the order of summation

$$=\frac{(-1)^n 2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)n}\Gamma(n\ell+\beta+1)\sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n}{j}\cdot x^{2k\ell(n-j)}}{\Gamma[\ell(n-j)+\beta+1]}$$

$$=\frac{2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n}\Gamma(n\ell+\beta+1)\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{x^{2kn\ell-2kj\ell}}{\Gamma(n\ell-\ell j+\beta+1)}$$

where

$$\beta=\frac{-1}{2k}, k=1,2,3.,....,.n=0,1,2,3 \quad .....\text{and}..\ell=1,3,5......$$

$$.S_{2n+1}(x;k,\ell)=\frac{(-1)^n n!\left(\frac{3}{2}\right)_n 2^{2n+1}}{(1-\beta)_n}x^{\ell}Z_{n}^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$=\frac{(-1)^n n!\left(\frac{3}{2}\right)_n 2^{2n+1}}{(1-\beta)_n}x^{\ell}\,\frac{\Gamma(n\ell-\beta\ell+1)}{n!}\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\cdot\frac{x^{2kj\ell}}{\Gamma(j\ell-\beta\ell+1)} \quad --(2.1.2)$$

$$=\frac{(-1)^n (3/2)_n 2^{2n+1}}{(1-\beta)_n}\Gamma(n\ell-\beta\ell+1)\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\cdot\frac{x^{2kj\ell+\ell}}{\Gamma(j\ell-\beta\ell+1)}$$

Reverting the order of summation

$$=\frac{(-1)^n\left(\frac{3}{2}\right)_n 2^{2n+1}\Gamma(n\ell-\beta\ell+1)}{(1-\beta)_n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\cdot\frac{x^{2k\ell(n-j)+\ell}}{\Gamma(j\ell-\beta\ell+1)}$$

$$=\frac{\left(3/2\right)_n 2^{2n+1}\Gamma(n\ell-\beta\ell+1)}{(1-\beta)_n}\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\cdot\frac{x^{2kn\ell-2kj\ell+\ell}}{\Gamma(n\ell-j\ell-\beta\ell+1)}$$

Combining even and odd case, we get.

$$S_n(x;k,\ell)=\frac{2^n\left(\frac{1+2\epsilon}{2}\right)_{\left[\frac{n}{2}\right]}}{(1+\beta-2\beta\epsilon)_{\left[\frac{n}{2}\right]}}\Gamma\left(\frac{n\ell+2\beta-\epsilon(2\beta+2\beta\ell+\ell)}{2}+1\right)$$

$$\cdot\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\binom{\left[\frac{n}{2}\right]}{j}\frac{x^{2k\left[\frac{n}{2}\right]\ell-2kj\ell+\ell\epsilon}}{\Gamma\left(\frac{n\ell+2\beta-\epsilon(2\beta+2\beta\ell+\ell)}{2}+1-\ell j\right)} \qquad ---(2.1.3)$$

Similarly,

$$T_{2n}(x;k,\ell)=\frac{(-1)^n n!(1+n)_n}{(1+\beta)_n}Y_n^{\beta}(x^{2k};\ell) \qquad ---(2.1.4)$$

$$=\frac{(-1)^n n!2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n}\frac{1}{n!}\sum_{r=0}^{n}\frac{x^{2kr}}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\cdot\left(\frac{s+\beta+1}{\ell}\right)_n$$

$$=\frac{(-1)^n 2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n}\sum_{s=0}^{n}\frac{x^{2k(n-r)}}{(n-r)!}\sum_{s=0}^{n-r}(-1)^s\binom{n-r}{s}\cdot\left(\frac{s+\beta+1}{\ell}\right)_n$$

$$=\frac{(-1)^n 2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)n}\sum_{n=0}^{n}\frac{x^{2kn-2kr}}{(n-r)!}\sum_{s=0}^{n-r}(-1)^s\binom{n-r}{s}\cdot\left(\frac{s+\beta+1}{\ell}\right)_n$$

by reverting the order of summation.

$$T_{2n+1}(x;k,\ell)=\frac{(-1)^n n!2^{2n+1}\left(3/2\right)_n}{(1-\beta)_n}xY_n^{-\beta\ell}(x^{2k};\ell)$$

$$=\frac{(-1)^n n!2^{2n+1}\left(3/2\right)_n}{(1-\beta)_n}x\frac{1}{n!}\sum_{s=0}^{r}\frac{x^{2kr}}{r!}\sum_{s=0}^{n}(-1)^s\binom{r}{s}\cdot\left(\frac{s-\beta\ell+1}{\ell}\right) \qquad ---(2.1.5)$$

$$= \frac{(-1)^n 2^{2n+1}\left(\frac{3}{2}\right)_n}{(1-\beta)_n} x \sum_{r=0}^{n} \frac{x^{2k(n-r)}}{(n-r)!} \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s} \cdot \left(\frac{s-\beta\ell+1}{\ell}\right)_n$$

by reverting the order of summation.

$$= \frac{(-1)^n 2^{2n+1}\left(\frac{3}{2}\right)_n}{(1-\beta)_n} \sum_{r=0}^{n} \frac{x^{2kn-2kr+1}}{(n-r)!} \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s} \cdot \left(\frac{s-\beta\ell+1}{\ell}\right)_n$$

Combining even and odd cases

$$T_n(x;k,\ell) = \frac{(-1)^{\left[\frac{n}{2}\right]} 2^n \left(\frac{1+2\epsilon}{2}\right)_{\left[\frac{n}{2}\right]}}{(1+\beta-2\beta\epsilon)_{\left[\frac{n}{2}\right]}} \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{2k\left[\frac{n}{2}\right]-2kr+\epsilon}}{\left(\left[\frac{n}{2}\right]-r\right)!}$$

$$\sum_{s=0}^{\left[\frac{n}{2}\right]-r} (-1)^s \cdot \binom{\left[\frac{n}{2}\right]-r}{s} \cdot \left(\frac{s+\beta+1-(\beta+\beta\ell)\epsilon}{\ell}\right)_{\left[\frac{n}{2}\right]} \quad ---(2.1.6)$$

(2.2) Biorthogonality :- Let,

$$\rho(x) = w(x) = x^{2k-2}\exp\left(-x^{2k}\right)$$

where k is positive integer.

$$\int_{-\infty}^{\infty} x^{2k-2}\exp(-x^{2k})S_{2n}(x;k,\ell)T_{2n}(x;k,\ell)dx$$

$$= \int_{-\infty}^{\infty} x^{2k-2}\exp.\left(-x^{2k}\right)\frac{(-1)^n n!2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n} z_n^{\beta}\left(x^{2k};\ell\right)$$

$$\frac{(-1)^n n!2^{2n}\left(\frac{1}{2}\right)_n}{(1+\beta)_n} Y_n^{\beta}\left(x^{2k};\ell\right)dx. \qquad ---(2.2.1)$$

$$= \frac{[n!]^2 2^{4n}\left[\left(\frac{1}{2}\right)_n\right]^2}{\left[(1+\beta)_n\right]^2}\int_{-\infty}^{\infty} x^{2k-2}\exp.\left(-x^{2k}\right)z_n^{\beta}\left(x^{2k};\ell\right)Y_n^{\beta}\left(x^{2k};\ell\right)dx$$

$$= \frac{[n!]^2 2^{4n+1}\left[\left(\frac{1}{2}\right)_n\right]^2}{\left[(1+\beta)_n\right]^2}\int_{0}^{\infty} x^{2k-2}\exp\left(-x^{2k}\right)Z_n^{\beta}\left(x^{2k};\ell\right)Y_n^{\beta}\left(x^{2k};\ell\right)\cdot dx$$

Put

$$x^{2k} = u;.x = u^{\frac{1}{2k}}.$$

$$2kx^{2k-1}dx = du$$

$$x^{2k-1}dx = \frac{du}{2k}$$

The limits of integration are when x=0, u=0 and when $x=\infty$, $u=\infty$.

$$= \frac{\left[n!\left(\frac{1}{2}\right)_n\right]^2}{k\left[(1+\beta)_n\right]^2} 2^{4n}{}_0\int^{\infty} u^{-\frac{1}{2k}}\exp(-u)Z_n^{\beta}(u;\ell)Y_n^{\beta}(u;\ell)du$$

$(\because \beta = -\frac{1}{2k})$

$$= \frac{[n!^2]2^{4n}\left[\left(\frac{1}{2}\right)_n\right]^2}{k\left[(1+\beta)_n\right]^2}\int_0^\infty u^\beta \exp(-u) Z_n^\beta(u;\ell) Y_n^\beta(u;\ell)du.$$

$$= \frac{[n!]^2 2^{4n}\left[\left(\frac{1}{2}\right)_n\right]^2}{k\left[(1+\beta)_n\right]^2}\frac{\Gamma(\ell n+\beta+1)}{n!}$$

$$= \frac{n! 2^{4n}\left[\left(\frac{1}{2}\right)_n\right]^2}{\left[(1+\beta)_n\right]^2}\frac{\Gamma(n\ell+\beta+1)}{k}$$

Now

$$\int_{-\infty}^{\infty} x^{2k-2}\exp\left(-x^{2k}\right)\cdot S_{2n+1}(x;k,\ell)\cdot T_{2n+1}(x;k,\ell)\cdot dx$$

$$= \int_{-\infty}^{\infty} x^{2k-2}\exp\left(-x^{2k}\right)\cdot \frac{(-1)^n n!\left(\frac{3}{2}\right)_n 2^{2n+1}}{(1-\beta)_n} x^\ell Z_n^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$\frac{(-1)^n n!\left(\frac{3}{2}\right)_n 2^{2n+1}}{(1-\beta)_n} x Y_n^{-\beta\ell}\left(x^{2k};\ell\right)\cdot dx. \qquad --- \quad (2\cdot 2\cdot 2)$$

$$= \frac{[n!]^2\left[\left(\frac{3}{2}\right)_n\right]^2 2^{4n+2}}{\left[(1-\beta)_n\right]^2} 2\int_0^\infty x^{2k-2}\exp\left(-x^{2k}\right)\cdot \left(x^2\right)^{\left(\frac{\ell+1}{2}\right)} Z_n^{-\beta\ell}\left(x^{2k};\ell\right) Y_n^{-\beta\ell}\left(x^{2k};\ell\right)\cdot dx$$

$$Put..x^{2k} = u... \Rightarrow u^{1/k} \Rightarrow x = u^{1/2k}$$

$$2kx^{2k-1}dx = du$$

$$x^{2k-1}dx = \frac{du}{2k}$$

The.. limits..of.. integration..are..

when ..$x = 0, u = 0$

when $x = \infty, u = \infty.$

$$= \frac{[n!]^2 \left[\left(\frac{3}{2}\right)_n\right]^2 2^{4n+3}}{[(1-\beta)_n]^2} \int_0^\infty u^{-\frac{1}{2k}} \exp\cdot(-u)(u)^{\left(\frac{\ell+1}{2k}\right)} Z_n^{-\beta\ell}(u;\ell)\cdot Y_n^{-\beta\ell}(u;\ell)\frac{du}{2k}$$

$$= \frac{[n!]^2 \left[\left(3/2\right)_n\right]^2 2^{4n+2}}{k[(1-\beta)_n]^2} {}_0\int^\infty u^{\ell/2k} \exp\cdot(-u)\cdot Z_n^{-\beta\ell}(u;\ell)\cdot Y_n^{-\beta\ell}(u;\ell)du$$

$$= \frac{[n!]^2 \left[\left(3/2\right)_n\right]^2 2^{4n+2}}{k[(1-\beta)_n]^2} {}_0\int^\infty \left(u^{-1/2k}\right)^{-\ell} \exp(-u)\cdot Z_n^{-\beta\ell}(u;\ell)\cdot Y_n^{-\beta\ell}(u;\ell)\cdot du$$

$$= \frac{[n!]^2 \left[\left(3/2\right)_n\right]^2 2^{4n+2}}{k[(1-\beta)_n]^2} {}_0\int^\infty u^{-\beta\ell} \exp(-u)\cdot Z_n^{-\beta\ell}(u;\ell)\cdot Y_n^{-\beta\ell}(u;\ell)\cdot du$$

$$= \frac{[n!]^2 \left[\left(3/2\right)_n\right]^2 2^{4n+2}}{k[(1-\beta)_n]^2} \frac{\Gamma(n\ell - \beta\ell + 1)}{n!}$$

$$= \frac{n! \left[\left(3/2\right)_n\right]^2 2^{4n+2}}{[(1-\beta)_n]^2} \frac{\Gamma(n\ell - \beta\ell + 1)}{k}$$

Combining (2.2.1) and (2.2.2) we get

$$\int_{-\infty}^{\infty} x^{2k-2} \exp\left(-x^{2k}\right) \cdot S_n(x;k,\ell) \cdot T_n(x;k,\ell) dx$$

$$= \frac{\left[\frac{n}{2}\right]! \cdot 2^{2n}}{k} \left[\frac{\left(\frac{1+2\epsilon}{2}\right)_{\left[\frac{n}{2}\right]}}{\left(1+\beta-2\beta\epsilon\right)_{\left[\frac{n}{2}\right]}} \right]^2 \Gamma\left(\frac{n\ell+2\beta-\epsilon(2\beta+2\beta\ell+\ell)}{2}+1 \right) \text{---} (2.2.3)$$

Thus we obtain.

$$\int_{-\infty}^{\infty} x^{2k-2} \exp\left(-x^{2k}\right) \cdot S_n(x;k,\ell) \cdot T_m(x;k,\ell) dx$$

$$= \frac{\left[\frac{n}{2}\right]! \cdot 2^{2n}}{k} \cdot \left[\frac{\left(\frac{1+2\epsilon}{2}\right)_{\left[\frac{n}{2}\right]}}{\left(1+\beta-2\beta\epsilon\right)_{\left[\frac{n}{2}\right]}} \right]^2 \Gamma\left(\frac{n\ell+2\beta-\epsilon(2\beta+2\beta\ell+\ell)}{2}+1 \right) \cdot \delta m, n. \text{--} (2.2.4)$$

where $\delta m, n$ is the usual Kronecker's delta.

(2.3) Generating Functions.

[A] Generating Function Related to Konauser Biorthogonal Parynomial of First Kind:

By definition

$$S_{2n}(x;k,\ell)=\frac{(-1)^n n!2^{2n}\left(\frac{1}{2}\right)_n Z_n^{\beta}\left(x^{2k};\ell\right)}{(1+\beta)_n}$$

Again recall the generating function obtained by Karande and Thakare[15]

$$\sum_{n=0}^{\infty}\frac{(c)_n z_n^{\alpha}(x:k)\cdot t^n}{(1+\alpha)_{kn}}=(1-t)^{-c}{}_1F_k\left[\begin{matrix} c; & \frac{-tx^k}{k^k(1-t)} \\ \Delta(k,1+\alpha); & \end{matrix}\right]. \quad ---(2.3.1)$$

Replace α by β , x by x^{2k} and k by ℓ in (2.3.1)

$$\sum_{n=0}^{\infty}\frac{(c)_n Z_n^{\beta}\left(x^{2k};\ell\right)\cdot t^n}{(1+\beta)_{n\ell}}=(1-t)^{-c}{}_1F_{\ell}\left[\begin{matrix} c; & -tx^{2k\ell} \\ \Delta(\ell;1+\beta); & \ell^{\ell}(1-t)\end{matrix}\right] \quad ---(2.3.2)$$

Now

$$\sum_{n=0}^{\infty}\frac{(c)_n(1+\beta)_n s_{2n}(x;k,\ell)\cdot t^{2n}}{\left(\frac{1}{2}\right)_n(1+\beta)_{n\ell}\, n!}$$

$$=\sum_{n=0}^{\infty}\frac{(c)_n(1+\beta)_n(-1)^n n!2^{2n}\left(\frac{1}{2}\right)_n Z_n^{\beta}\left(x^{2k};\ell\right)\cdot t^{2n}}{(1+\beta)_n\left(\frac{1}{2}\right)_n(1+\beta)_{n\ell}\, n!,} \qquad ,\text{by}\cdot(2.1.1)$$

$$=\sum_{n=0}^{\infty}\frac{(c)_n Z_n^{\beta}\left(x^{2k};\ell\right)\cdot\left(-4t^2\right)^n}{(1+\beta)_{kn}} \qquad ---(2.3.3)$$

By applying (2.3.2) to equation (2.3.3) we get

$$\sum_{n=0}^{\infty} \frac{(c)_n (1+\beta)_n s_{2n}(x;k,\ell)\cdot t^{2n}}{(1/2)_n (1+\beta)_{n\ell}\, n!}$$

$$= \left(1+4t^2\right)^{-c} {}_1F_\ell\left[\begin{matrix} c; \\ \Delta(\ell,1+\beta); \end{matrix}\, \frac{4t^2 x^{2k\ell}}{\ell^\ell\left(1+4t^2\right)}\right] \qquad ---(2.3.4)$$

Let $\theta = t\cdot\frac{d}{dt}$

$$\left(1+\theta\right)\sum_{n=0}^{\infty} \frac{(c)_n (1-\beta)_n s_{2n+1}(x;k,\ell)\cdot t^{2n}}{\left(1-\beta\ell\right)_{n\ell} \left(3/2\right)_n\, n!}$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n (1+2n)\cdot(1-\beta)_n s_{2n+1}(x;k,\ell) t^{2n}}{\left(1-\beta\ell\right)_{n\ell} \left(3/2\right)_n\, n!}$$

Multiplying by t on both the side , we get

$$t\left(1+\theta\right)\sum_{n=0}^{\infty} \frac{(c)_n s_{2n+1}(x;k,\ell)\cdot t^{2n} (1-\beta)_n}{\left(1-\beta\ell\right)_{n\ell} \left(3/2\right)_n\, n!}$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n (1-\beta)_n (1+2n) s_{2n+1}(x;k,\ell)\cdot t^{2n+1}}{\left(1-\beta\ell\right)_{n\ell} \left(3/2\right)_n\, n!}$$

i.e.

$$\cdot\sum_{n=0}^{\infty}\frac{(c)_n(1-\beta)_n(1+2n)\cdot s_{2n+1}(x;k,\ell)\cdot t^{2n+1}}{(1-\beta\ell)_{n\ell}(3/2)_n\, n!}$$

$$=t(1+\theta)\sum_{n=0}^{\infty}\frac{(c)_n(1-\beta)_n s_{2n+1}(x;k,\ell)\cdot t^{2n}}{(1-\beta\ell)_{n\ell}(3/2)_n\, n!} \quad ---(2.3.5)$$

By applying (2.1.2) on right hand side of the above equation.

$$=t\cdot(1+\theta)\sum_{n=0}^{\infty}\frac{(c)_n(1-\beta)_n(-1)^n n!\cdot(3/2)_n 2^{2n+1}x^{\ell}Z_n^{-\beta\ell}(x^{2k};\ell)\cdot t^{2n}}{(1-\beta)_n(1-\beta\ell)_{n\ell}(3/2)_n\, n!}$$

$$=2t\cdot x^{\ell}(1+\theta)\sum_{n=0}^{\infty}\frac{(c)_n Z_n^{-\beta\ell}(x^{2k};\ell)\cdot(-4t^2)}{(1-\beta\ell)_{n\ell}}$$

Replace x by x^{2k} , α by - $\beta\ell$ and k by ℓ in equation (2.3.2)

$$\cdot\sum_{n=0}^{\infty}\frac{(c)_n Z_n^{-\beta\ell}(x^{2k};\ell)\cdot t^n}{(1-\beta\ell)_{n\ell}}=(1-t)^{-c}{}_1F_{\ell}\left[\begin{matrix}c; & \dfrac{-tx^{2k\ell}}{\ell^{\ell}(1-t)}\\ \Delta(\ell,1-\beta\ell); & \end{matrix}\right]--(2.3.6)$$

Applying (2.3.6) to right hand side of equation (2.3.5) we get,

$$\sum_{n=0}^{\infty}\frac{(c)_n s_{2n+1}(x;k,\ell)\cdot t^{2n+1}(1+2n)\cdot(1-\beta)_n}{(1-\beta\ell)_{n\ell}(3/2)_n\, n!}$$

$$=2tx^{\ell}(1+\theta)(1+4t^2)^{-c}{}_1F_{\ell}\left[\begin{matrix}c; & \dfrac{4t^2x^{2k\ell}}{\ell^{\ell}(1+4t^2)}\\ \Delta(\ell,1-\beta\ell); & \end{matrix}\right]$$

$$=2tx^{\ell}(1+4t^2)^{-c}{}_1F_{\ell}\left[\begin{matrix}c; & \dfrac{4t^2x^{2k\ell}}{\ell^{\ell}(1+4t^2)}\\ \Delta(\ell,1-\beta\ell); & \end{matrix}\right]$$

$$+2tx^{\ell}(-c)(1+4t^2)^{-c-1}(8t^2)\cdot{}_1F_{\ell}\left[\begin{matrix} c; \\ \Delta(\ell,1-\beta\ell); \end{matrix} \frac{4t^2x^{2k\ell}}{\ell^{\ell}(1+4t^2)}\right]$$

$$+2tx^{\ell}(1+4t^2)^{-c}\theta\cdot{}_1F_{\ell}\left[\begin{matrix} c; \\ \Delta(\ell,1-\beta\ell); \end{matrix} \frac{4t^2x^{2k\ell}}{\ell^{\ell}(1+4t^2)}\right]$$

$$=\frac{2tx^{\ell}(1+4t^2-8ct^2)}{(1+4t^2)^{c+1}}\,{}_1F_{\ell}\left[\begin{matrix} c; \\ \Delta(\ell,1-\beta\ell); \end{matrix} \frac{4t^2x^{2k\ell}}{\ell^{\ell}(1+4t^2)}\right]$$

$$+2tx^{\ell}(1+4t^2)^{-c}\theta\,{}_1F_{\ell}\left[\begin{matrix} c; \\ \Delta(\ell,1-\beta\ell); \end{matrix} \frac{4t^2x^{2k\ell}}{\ell^{\ell}(1+4t^2)}\right]---(2.3.7)$$

Now

$$\theta\,{}_1F_{\ell}\left[\begin{matrix} c; \\ \Delta(\ell,1-\beta\ell); \end{matrix} \frac{4t^2x^{2k\ell}}{\ell^{\ell}(1+4t^2)}\right]$$

$$=\theta\sum_{n=0}^{\infty}\frac{(c)_n\left[\frac{4x^{2k\ell}}{\ell^{\ell}}\right]^n\frac{t^{2n}}{(1+4t^2)^n}}{\left(\frac{1-\beta\ell}{\ell}\right)_n\left(\frac{1-\beta\ell+1}{\ell}\right)_n---\left(\frac{\ell-\beta\ell}{\ell}\right)_n n!}$$

$$=\theta\sum_{n=0}^{\infty}\frac{(c)_n\left[\frac{4x^{2k\ell}}{\ell^{\ell}}\right]^n\frac{t^{2n}}{(1+4t^2)^n}}{\left(\frac{1-\beta\ell}{\ell}\right)_n\left(\frac{2-\beta\ell}{\ell}\right)_n---\left(\frac{\ell-\beta\ell}{\ell}\right)_n n!}$$

$$=\sum_{n=0}^{\infty}\frac{(c)_n\left[\frac{4x^{2k\ell}}{\ell^{\ell}}\right]^n\theta\cdot\left\{\frac{t^{2n}}{(1+4t^2)^n}\right\}}{\left(\frac{1-\beta\ell}{\ell}\right)_n\left(\frac{2-\beta\ell}{\ell}\right)_n---\left(\frac{\ell-\beta\ell}{\ell}\right)_n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(c)_n \left[\frac{4x^{2k\ell}}{\ell^\ell}\right]^n \left\{ \frac{(1+4t^2)^n 2nt^{2n} - t^{2n} n (1+4t^2)^{n-1} 8t^2}{\left[(1+4t^2)^n\right]^2} \right\}}{\left(\frac{1-\beta\ell}{\ell}\right)_n \left(\frac{2-\beta\ell}{\ell}\right)_n --- \left(\frac{\ell-\beta\ell}{\ell}\right)_n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(c)_n \left[\frac{4x^{2k\ell}}{\ell^\ell}\right]^n \frac{2n \cdot t^{2n}}{(1+4t^2)^n} \cdot \left\{1 - \frac{4t^2}{1+4t^2}\right\}}{\left(\frac{1-\beta\ell}{\ell}\right)_n \left(\frac{2-\beta\ell}{\ell}\right)_n --- \left(\frac{\ell-\beta\ell}{\ell}\right)_n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(c)_n \left[\frac{4x^{2k\ell}}{\ell^\ell}\right]^n \frac{2t^{2n}}{(1+4t^2)^{n+1}}}{\left(\frac{1-\beta\ell}{\ell}\right)_n \left(\frac{2-\beta\ell}{\ell}\right)_n --- \left(\frac{\ell-\beta\ell}{\ell}\right)_n (n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(c)_{n+1} \left[\frac{4x^{2k\ell}}{\ell^\ell}\right]^{n+1} 2t^{2n+2}}{\left(\frac{1-\beta\ell}{\ell}\right)_{n+1} \left(\frac{2-\beta\ell}{\ell}\right)_{n+1} --- \left(\frac{\ell-\beta\ell}{\ell}\right)_{n+1} n!(1+4t)^{n+2}}$$

$$= \frac{8c \cdot t^2 x^{2k\ell}}{\ell^\ell (1+4t^2)^2} \sum_{n=0}^{\infty} \frac{(c+1)_n \left[\frac{4t^2 x^{2k\ell}}{\ell^\ell (1+4t^2)}\right]^n}{\left(\frac{1-\beta\ell}{\ell}\right)_{n+1} \left(\frac{2-\beta\ell}{\ell}\right)_{n+1} --- \left(\frac{\ell-\beta\ell}{\ell}\right)_{n+1} n!} \quad ---(2.3.8)$$

But

$$\left(\frac{1-\beta\ell}{\ell}\right)_{n+1} = \left(\frac{1-\beta\ell}{\ell}\right) \cdot \left(\frac{1-\beta\ell}{\ell}+1\right) --- \left(\frac{1-\beta\ell}{\ell}+n+1-1\right) = \left(\frac{1-\beta\ell}{\ell}\right) \cdot \left(\frac{1-\beta\ell}{\ell}+1\right)_n$$

$$\left(\frac{2-\beta\ell}{\ell}\right)_{n+1} = \left(\frac{2-\beta\ell}{\ell}\right) \cdot \left(\frac{2-\beta\ell}{\ell}+1\right) --- \left(\frac{2-\beta\ell}{\ell}+n+1-1\right) = \left(\frac{2-\beta\ell}{\ell}\right) \cdot \left(\frac{2-\beta\ell}{\ell}+1\right)_n$$

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$$\left(\frac{\lambda-\beta\ell}{\ell}\right)_{n+1}=\left(\frac{\lambda-\beta\ell}{\ell}\right)\cdot\left(\frac{\lambda-\beta}{\ell}+1\right)---\left(\frac{\lambda-\beta\ell}{\ell}+n+1-1\right)=\left(\frac{\lambda-\beta\ell}{\ell}\right)\cdot\left(\frac{\lambda-\beta\ell}{\ell}+1\right)_n$$

$$\therefore\left(\frac{1-\beta\ell}{\ell}\right)_{n+1}\left(\frac{2-\beta\ell}{\ell}\right)_{n+1}---\left(\frac{1-\beta\ell}{\ell}\right)_{n+1}$$

$$=\left(\frac{1-\beta\ell}{\ell}\right)\cdot\left(\frac{2-\beta\ell}{\ell}\right)---\left(\frac{\ell-\beta\ell}{\ell}\right)\cdot\left(\frac{1-\beta\ell+1}{\ell}\right)_n\left(\frac{2-\beta\ell+1}{\ell}\right)_n---\left(\frac{\ell-\beta\ell+1}{\ell}\right)_n$$

$$=\frac{(1-\beta\ell)_\ell}{\ell^\ell}\Delta(\ell,1-\beta\ell+\ell)$$

Using above values in (2.3.8) we get,

$$\theta_1F_\ell\left[\begin{matrix}c;\\ \Delta(\ell;1-\beta\ell\cdot);\end{matrix}\frac{4t^2x^{2k\ell}}{\ell^\ell(1+4t^2)}\right]$$

$$=\frac{8ct^2x^{2k\ell}}{\ell^\ell(1+4t^2)^2}\sum_{n=0}^{\infty}\frac{(c+1)_n\left[\frac{4t^2x^{2k\ell}}{\ell^\ell(1+4t^2)}\right]^n}{\frac{(1-\beta\ell)_\ell}{\ell^\ell}\Delta(\ell,1-\beta\ell+\ell)\cdot n!}.$$

$$=\frac{8ct^2x^{2k\ell}}{(1-\beta\ell)_\ell(1+4t^2)^2}{}_1F_\ell\left[\begin{matrix}c+1;\\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\frac{4t^2x^{2k\ell}}{\ell^\ell(1+4t^2)}\right]---(2.3.9)$$

Using (2.3.9) in equation (2.3.7) we get,

$$\sum_{n=0}^{\infty}\frac{(c)_n(1-\beta)_n(1+2n)\cdot S_{2n+1}(x;k,\ell)\cdot t^{2n+1}}{(1-\beta\ell)_{n\ell}(3/2)_n n!}$$

$$=\frac{2tx^\ell(1+4t^2-8ct^2)}{(1+4t^2)^{c+1}}{}_1F_\ell\left[\begin{matrix}c;\\ \Delta(\ell,1-\beta\ell);\end{matrix}\frac{4t^2x^{2k\ell}}{\ell^\ell(1+4t^2)}\right]$$

$$+\frac{2tx^{\ell}\left(8ct^{2}x^{2k\ell}\right)}{(1-\beta\ell)_{\ell}\left(1+4t^{2}\right)^{c+2}}\,{}_{1}F_{\ell}\left[\begin{matrix}c+1; \\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\ \frac{4t^{2}x^{2k\ell}}{\ell^{\ell}\left(1+4t^{2}\right)}\right]$$

$$=\frac{2tx^{\ell}\left(1+4t^{2}-8ct^{2}\right)}{\left(1+4t^{2}\right)^{c+1}}\,{}_{1}F_{\ell}\left[\begin{matrix}c; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\ \frac{4t^{2}x^{2k\ell}}{\ell^{\ell}\left(1+4t^{2}\right)}\right]$$

$$+\frac{16ct^{3}x^{2k\ell+\ell}}{(1-\beta\ell)_{\ell}\left(1+4t^{2}\right)^{c+2}}\,{}_{1}F_{\ell}\left[\begin{matrix}c+1; \\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\ \frac{4t^{2}x^{2k\ell}}{\ell^{\ell}\left(1+4t^{2}\right)}\right]----(2.3.10)$$

We write,

$$\sum_{n=0}^{\infty}\frac{(c)_{n}(1+\beta)_{n}S_{2n}(x;k,\ell)t^{2n}}{\left(\frac{1}{2}\right)_{n}(1+\beta)_{n\ell}\,n!}+\sum_{n=0}^{\infty}\frac{(c)_{n}(1-\beta)_{n}(1+2n)\cdot S_{2n+1}(x;k,\ell)t^{2n+1}}{(1-\beta\ell)_{n\ell}\left(\frac{3}{2}\right)_{n}n!}$$

$$=\sum_{n=0}^{\infty}\frac{(c)_{\left[\frac{n}{2}\right]}(1+2n\in)\cdot(1+\beta-2\beta\in)_{\left[\frac{n}{2}\right]}S_{n}(x;k,\ell)t^{n}}{\left(\frac{1}{2}+\in\right)_{\left[\frac{n}{2}\right]}(1+\beta-\beta(\ell+1)\in)_{\ell\left[\frac{n}{2}\right]}\left[\frac{n}{2}\right]!}\qquad ---(2.3.11)$$

In view of the identity (2.3.11) equations (2.3.4) and(2.3.10) have the following generating functions

$$\sum_{n=0}^{\infty}\frac{(c)_{\left[\frac{n}{2}\right]}(1+2n\in)\cdot(1+\beta-2\beta\in)_{\left[\frac{n}{2}\right]}\cdot S_{n}(x;k;\ell)t^{n}}{\left(\frac{1}{2}+\in\right)_{\left[\frac{n}{2}\right]}(1+\beta-\beta(\ell+1)\in)_{\ell\left[\frac{n}{2}\right]}\left[\frac{n}{2}\right]!}$$

$$=(1+4t^{2})^{-c}\,{}_{1}F_{\ell}\left[\begin{matrix}c; \\ \Delta(\ell;1+\beta\ell)\end{matrix}\ \frac{4t^{2}x^{2k\ell}}{\ell^{\ell}(1+4t^{2})}\right]$$

$$+\frac{2tx^{\ell}\left(1+4t^2-8ct^2\right)}{\left(1+4t^2\right)^{c+1}}\,{}_1F_{\ell}\left[\begin{matrix}c; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\ \frac{4t^2x^{2k\ell}}{\ell^{\ell}\left(1+4t^2\right)}\right]$$

$$+\frac{16ct^3x^{2k\ell+\ell}}{(1-\beta\ell)_{\ell}\left(1+4t^2\right)^{c+2}}\,{}_1F_{\ell}\left[\begin{matrix}c+1; \\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\ \frac{4t^2x^{2k\ell}}{\ell^{\ell}\left(1+4t^2\right)}\right]. \quad ---(2.3.12)$$

For k = 1 and ℓ = k equation (2.3.12) reduces to

$$\sum_{n=0}^{\infty}\frac{(c)_{\left[\frac{n}{2}\right]}(1+2n\in)(1+\beta-2\beta\in)_{\left[\frac{n}{2}\right]}S_n(x;k)t^n}{\left(\frac{1}{2}+\in\right)_{\left[\frac{n}{2}\right]}(1+\beta-\beta(k+1)\in)_{k\left[\frac{n}{2}\right]}\left[\frac{n}{2}\right]!}$$

$$=\left(1+4t^2\right)^{-c}\,{}_1F_k\left[\begin{matrix}c; \\ \Delta(k,1+\beta);\end{matrix}\ Z\right]$$

$$+\frac{2x^{k}\left(1+4t^2-8ct^2\right)}{\left(1+4t^2\right)^{c+1}}\,{}_1F_k\left[\begin{matrix}c; \\ \Delta(k,1-\beta k);\end{matrix}\ Z\right]$$

$$+\frac{16ct^3x^{3k}}{(1-\beta k)_k\left(1+4t^2\right)^{c+2}}\,{}_1F_k\left[\begin{matrix}c+1; \\ \Delta(k,1-\beta k+k);\end{matrix}\ Z\right] \quad ---(2.3.13)$$

where $Z=\dfrac{4t^2x^{2k}}{k^k\left(1+4t^2\right)}$

For k = 1 and β= -1/2 equation (2.3.13) reduces to generating function for Hermite polynomials first obtained by Brafman [3, p. 949, (33)]

(B) Generating Function Related to Konhauser Biarthogonal Polynomials of Second Kind :-

By definition of Tn (x;k,ℓ)

$$T_{2m+2n}(x;k,\ell)=\frac{(-1)^{m+n}(m+n)!2^{2m+2n}\left(\frac{1}{2}\right)_{m+n}}{(1+\beta)_{m+n}}Y^{\beta}_{m+n}(x^{2k};\ell)$$

$$=\frac{(-1)^{m+n}(m+n)!2^{2m+2n}\left(\frac{1}{2}\right)_{m}\cdot(m+1/2)_n}{(1+\beta)_m\cdot(1+\beta+m)_n}Y^{\beta}_{m+n}(x^{2k};\ell)$$

$$\sum_{n=0}^{\infty}\frac{(1+\beta+m)_n}{\left(m+\frac{1}{2}\right)_n}\frac{T_{2m+2n}(x;k,\ell)t^{2n}}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(m+n)!\cdot 2^{2m+2n}\left(\frac{1}{2}\right)_m}{(1+\beta)_m}Y^{\beta}_{m+n}(x^{2k};\ell)\frac{t^{2n}}{n.!}$$

$$=\frac{(-1)^m 2^{2m}\left(\frac{1}{2}\right)_m m!}{(1+\beta)_m}\sum_{n=0}^{\infty}\binom{m+n}{n}Y^{\beta}_{m+n}(x^{2k},\ell)(-4t^2)^n$$

$$=\frac{(-1)^m 2^{2m}\left(\frac{1}{2}\right)_m m!}{(1+\beta)_m}(1+4t^2)^{-(\beta+m\ell+1)/\ell}$$

$$\exp.\left\{x^{2k}\left[1-(1+4t^2)^{\frac{-1}{\ell}}\right]\right\}Y^{\beta}_m\left(x^{2k}(1+4t^2)^{\frac{-1}{\ell}};\ell\right)$$

$$=(1+4t^2)^{-(\beta+m\ell+1)/\ell}\exp.\left\{x^{2k}\left[1-(1+4t^2)^{\frac{-1}{\ell}}\right]\right\}.T_{2m}\left(x(1+4t^2)^{\frac{-1}{2k\ell}};k,\ell\right)\text{---}(2.3.14)$$

By definition of T(x;k,ℓ)

$$T_{2m+2n+1}(x;k,\ell)=\frac{(-1)^{m+n}(m+n)!2^{2m+2n+1}}{(1-\beta)_{(m+n)}}(3/2)_{(m+n)}\cdot x\cdot Y_{m+n}^{-\beta\ell}(x^{2k};\ell)$$

$$=\frac{(-1)^{m+n}(m+n)!2^{2m+2n+1}(3/2)_m(3/2+m)_n}{(1-\beta)_m\cdot(1-\beta+m)_n}\cdot x\cdot Y_{m+n}^{-\beta\ell}(x^{2k};\ell)$$

Now,

$$\frac{(1-\beta+m)_n}{(3/2+m)_n}T_{2m+2n+1}(x;k,\ell)=\frac{(-1)^{m+n}(m+n)!2^{2m+2n+1}(3/2)_m}{(1-\beta)_m}\cdot x\cdot Y_{m+n}^{-\beta\ell}(x^{2k};\ell);$$

$$\sum_{n=0}^{\infty}\frac{(1-\beta+m)_n}{(3/2+m)_n}T_{2m+2n+1}(x;k,\ell)\frac{t^{2n+1}}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(m+n)!2^{2m+2n+1}t^{2n+1}}{(1-\beta)_m n!}(3/2)_m xY_{m+n}^{-\beta\ell}(x^{2k};\ell);.$$

$$=\frac{(-1)^m 2^{2m+1}(3/2)_m m!(xt)}{(1-\beta)_m}\sum_{n=0}^{\infty}\binom{m+n}{n}Y_{m+n}^{-\beta\ell}(x^{2k};\ell)(-4t^2)^n$$

$$=(-1)^m 2^{2m+1}(3/2)_m m!(tx).(1+4t^2)^{-(-\beta\ell+m\ell+1)/\ell}$$

$$\exp\left\{x^{2k}[1-(1+4t^2)^{\frac{-1}{\ell}}]\right\}Y_m^{\beta\ell}(x^{2k}(1+4t^2)^{\frac{-1}{\ell}};\ell)$$

$$=\frac{t(1+4t^2)^{-(-\beta\ell+m\ell+1)/\ell}\exp\left\{x^{2k}[1-(1+4t^2)^{\frac{-1}{\ell}}]\right\}.T_{2m+1}(x(1+4t^2)^{-1/2k\ell};k,\ell)}{(1+4t^2)^{-1/2k\ell}}$$

$$=t(1+4t^2)^{-(-\beta\ell+m\ell-\frac{1}{2k}+1)/\ell}\exp\left\{x^{2k}[1-(1+4t^2)^{\frac{-1}{\ell}}]\right\}.T_{2m+1}(x(1+4t^2)^{-1/2k\ell};k,\ell)\quad ---(2.3.15)$$

Combining (2.3.14) and (2.3.15) we get,

$$\sum_{n=0}^{\infty} \frac{(1+m+\beta-2\beta\in)_{\left[\frac{n}{2}\right]}}{(m+1/2+\in)_{\left[\frac{n}{2}\right]}} T_{2m+n} \frac{t^n}{\left[\frac{n}{2}\right]!}$$

$$=(1+4t^2)^{-(m\ell+1)/\ell} \exp.\left\{x^{2k}[1-(1+4t^2)^{\frac{-1}{\ell}}]\right\}$$

$$\left\{(1+4t^2)^{-\frac{\beta}{\ell}} T_{2m}(x(1+4t^2)^{-\frac{1}{2k\ell}};k,\ell)+t(1+4t^2)^{(\beta\ell+\frac{1}{2k})/\ell} T_{2m+1}(x(1+4t^2)^{-\frac{1}{2k\ell}};k,\ell)\right\}$$

---(2.3.16)

Particular cases :-

(I) For k = l = 1, β = -1/2, we obtain following the generating relation for the Hermite Polynomial

$$\sum_{n=0}^{\infty} \frac{H_{2m+n}(x)t^n}{\left[\frac{n}{2}\right]!} = (1+4t^2)^{-(m+1)} \exp.\left\{\frac{4x^2t^2}{(1+4t^2)}\right\}.$$

$$\cdot\left[(1+4t^2)^{\frac{1}{2}} H_{2m}(x(1+4t^2)^{\frac{-1}{2}})+tH_{2m+1}(x(1+4t^2)^{\frac{-1}{2}})\right]$$

$$=(1+4t^2)^{-(m+1)} \exp.\left\{\frac{4x^2t^2}{(1+4t^2)}\right\}.$$

$$\cdot\left[(1+4t^2)^{\frac{1}{2}} H_{2m}(x(1+4t^2)^{\frac{-1}{2}})+tH_{2m+1}(x(1+4t^2)^{\frac{-1}{2}})\right] \qquad ---(2.3.17)$$

This is generating relation obtained by Thakare and Madhekar [37, p.1035.]

(II) For k = 0 equation (2.3.17) reduces to a generating function for Hermite polynomial given by Doestsch, [13. P.590 (7)] (See also Szego [36, Problem 24 P. 380]),

$$\sum_{n=0}^{\infty} \frac{H_n(x)t^n}{\left[\frac{n}{2}\right]!} = (1+4t^2)^{-1} \exp.\left\{\frac{4x^2t^2}{1+4t^2}\right\}\left[(1+4t^2)^{1/2}\ 1 + .2xt((1+4t^2)^{-1/2}\right]$$

$$= (1+4t^2)^{-3/2} \exp\left\{\frac{4x^2t^2}{1+4t^2}\right\}.(1+4t^2+2xt) \qquad ---(2.3.18)$$

C] Generating Functions Related to Konhauser Biorthogonal Polynomials:

The generating function for $Z^{\alpha}_n(x;k)$ obtained by Srivastava [34 p, p.490] (see also Karande and Thakare [14])

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;k)t^n}{(1+\alpha)_{kn}} = e^t {}_0F_k\left[\begin{matrix} --; \\ \Delta(k;1+\alpha); \end{matrix} -\left(\frac{x}{k}\right)^k t\right] \qquad ---(2.3.19)$$

Put $\alpha=\beta$, $k=l$ and replace x by x^{2k}, we get

$$\sum_{n=0}^{\infty} \frac{Z_n^{\beta}(x^{2k};\ell)t^n}{(1+\beta)_{n\ell}} = e^t {}_0F_\ell\left[\begin{matrix} --; \\ \Delta(\ell;1+\beta); \end{matrix} -\left(\frac{x^{2k}}{\ell}\right)^{\ell} t\right] \qquad ---(2.3.20)$$

By definition of $S_{2n}(x;k,l)$

$$S_{2n}(x;k,\ell) = \frac{(-1)^n n! 2^{2n} (1/2)_n Z_n^{\beta}(x^{2k};\ell)}{(1+\beta)_{n\ell}}$$

Now

$$\sum_{n=0}^{\infty}\frac{(1+\beta)_n S_{2n}(x;k,\ell)t^{2n}}{(1/2)_n(1+\beta)_{n\ell}\,n!}$$

$$=\sum_{n=0}^{\infty}\frac{(1+\beta)_n(-1)^n n!2^{2n}(1/2)_n Z_n^{\beta}(x^{2k};\ell)t^{2n}}{(1/2)_n(1+\beta)_n(1+\beta)_{n\ell}\,n!}$$

$$=\sum_{n=0}^{\infty}\frac{Z_n^{\beta}(x^{2k};\ell)(-4t^2)^n}{(1+\beta)_{n\ell}} \qquad ---(2.3.21)$$

Using equation (2.3.20) in equation (2.3.21)

$$\sum_{n=0}^{\infty}\frac{(1+\beta)_n S_{2n}(x;k,\ell)t^{2n}}{(1/2)_n(1+\beta)_{n\ell}\,n!}=e^{-4t^2}\,{}_0F_{\ell}\left[\begin{matrix}--; \\ \Delta(\ell;1+\beta);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{k}\right)^{\ell}\right]--(2.3.22)$$

Replacing α by $-\beta l$, k by ℓ and x by x^{2k} in equation (2.3.19)

$$\sum_{n=0}^{\infty}\frac{Z_n^{-\beta\ell}(x^{2k};\ell)t^n}{(1-\beta\ell)_{n\ell}}=e^{t}\,{}_0F_{\ell}\left[\begin{matrix}--; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; -\left(\frac{x^{2k}}{\ell}\right)^{\ell}t\right] \qquad ---(2.3.23)$$

By definition of $S_{2n+1}(x;k,l)$

$$S_{2n+1}(x;k,\ell)=\frac{(-1)^n(3/2)_n n!2^{2n+1}x^{\ell}\; Z_n^{-\beta\ell}(x^{2k};\ell)}{(1-\beta)_n}$$

Let $\theta=t\dfrac{d}{dt}$

$$(1+\theta)\sum_{n=0}^{\infty}\frac{(1-\beta)_n S_{2n+1}(x;k,\ell)t^{2n}}{(3/2)_n(1-\beta\ell)_{n\ell}\,n!}$$

$$= \sum_{n=0}^{\infty} \frac{(1-\beta)_n (1+2n).S_{2n+1}(x;k,\ell)t^{2n}}{(3/2)_n (1-\beta\ell)_{n\ell}\, n!}$$

i.e.

$$\sum_{n=0}^{\infty} \frac{(1+2n)(1-\beta)_n S_{2n+1}(x;k,\ell)t^{2n}}{(3/2)_n (1-\beta\ell)_{n\ell}\, n!}$$

$$= (1+\theta)\sum_{n=0}^{\infty} \frac{(1-\beta)_n S_{2n+1}(x;k,\ell)t^{2n}}{(3/2)_n (1-\beta\ell)_{n\ell}\, n!}$$

$$= (1+\theta)\sum_{n=0}^{\infty} \frac{(1-\beta)_n (-1)^n (3/2)_n n!.2^{2n+1} x^{\ell} Z_n^{-\beta\ell}(x^{2k};\ell)t^{2n}}{(1-\beta)_n (3/2)_n (1-\beta\ell)_{n\ell}\, n!}$$

$$= (1+\theta)\sum_{n=0}^{\infty} \frac{2x^{\ell} Z_n^{-\beta\ell}(x^{2k};\ell)(-4t^2)^n}{(1-\beta\ell)_{n\ell}}$$

$$= 2x^{\ell}(1+\theta)\sum_{n=0}^{\infty} \frac{Z_n^{-\beta\ell}(x^{2k};\ell)(-4t^2)^n}{(1-\beta\ell)_{n\ell}} \quad ---(2.3.24)$$

Multiplying above equation by t on both the sides.

$$\sum_{n=0}^{\infty} \frac{(1+2n)(1-\beta)_n S_{2n+1}(x;k,\ell)t^{2n+1}}{(3/2)_n (1-\beta\ell)_{n\ell}\, n!}$$

$$= 2tx^{\ell}(1+\theta)\sum_{n=0}^{\infty} \frac{Z_n^{-\beta\ell}(x^{2k};\ell)(-4t^2)^n}{(1-\beta\ell)_{n\ell}}$$

By using equation (2.3.23)

$$=2tx^{\ell}(1+\theta)e^{-4t^2}{}_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]$$

$$=2tx^{\ell}e^{-4t^2}{}_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]+2tx^{\ell}\theta\cdot\left\{e^{-4t^2}{}_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell)\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]\cdot\right\}$$

$$=2tx^{\ell}e^{-4t^2}{}_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]-16t^3\cdot x^{\ell}e^{-4t^2}{}_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]$$

$$+2t\cdot x^{\ell}e^{-4t^2}\theta_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]$$

$$=2t\cdot x^{\ell}(1-8t^2)e^{-4t^2}{}_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]+2tx^{\ell}e^{-4t^2}\theta_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right] \quad -(2.3.25)$$

Now

$$\theta\,{}_0F_{\ell}\left[\begin{matrix}\rightarrow; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\; 4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]$$

$$=\theta\sum_{n=0}^{\infty}\frac{\left[4\cdot\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]^{n}t^{2n}}{n!\left(\frac{1-\beta\ell}{\ell}\right)_n\left(\frac{1-\beta\ell+1}{\ell}\right)_n\cdots\left(\frac{1-\beta\ell+\ell-1}{\ell}\right)_n}.$$

$$=\theta\sum_{n=0}^{\infty}\frac{4^n\left(\frac{x^{2k}}{\ell}\right)^{n\ell}t^{2n}}{n!\left(\frac{1-\beta\ell}{\ell}\right)_n\left(\frac{2-\beta\ell}{\ell}\right)_n\cdots\left(\frac{\ell-\beta\ell}{\ell}\right)_n}$$

$$= \sum_{n=1}^{\infty} \frac{4^n \left(\frac{x^{2k}}{\ell}\right)^{n\ell} 2n\, t^{2n}}{n! \left(\frac{1-\beta\ell}{\ell}\right)_n \left(\frac{2-\beta\ell}{\ell}\right)_n \cdots \left(\frac{\ell-\beta\ell}{\ell}\right)_n}$$

$$= \sum_{n=1}^{\infty} \frac{4^n \left(\frac{x^{2k}}{\ell}\right)^{n\ell} 2.t^{2n}}{(n-1)! \left(\frac{1-\beta\ell}{\ell}\right)_n \left(\frac{2-\beta\ell}{\ell}\right)_n \cdots \left(\frac{\ell-\beta\ell}{\ell}\right)_n}$$

$$= \sum_{n=0}^{\infty} \frac{2(4t^2)^{n+1} \left(\frac{x^{2k}}{\ell}\right)^{\ell(n+1)}}{n! \left(\frac{1-\beta\ell}{\ell}\right)_{n+1} \left(\frac{2-\beta\ell}{\ell}\right)_{n+1} \cdots \left(\frac{\ell-\beta\ell}{\ell}\right)_{n+1}}$$

$$= 2(4t^2)\left(\frac{x^{2k}}{\ell}\right)^{\ell} \sum_{n=0}^{\infty} \frac{\left[4t^2 \left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]^n}{n! \left(\frac{1-\beta\ell}{\ell}\right)_{n+1} \left(\frac{2-\beta\ell}{\ell}\right)_{n+1} \cdots \left(\frac{\ell-\beta\ell}{\ell}\right)_{n+1}} \quad ---(2.3.26)$$

But

$$\left(\frac{1-\beta\ell}{\ell}\right)_{n+1} = \left(\frac{1-\beta\ell}{\ell}\right)\left(\frac{1-\beta\ell}{\ell}+1\right)\cdots\left(\frac{1-\beta\ell}{\ell}+n+1-1\right) = \left(\frac{1-\beta\ell}{\ell}\right)\left(\frac{\ell-\beta\ell}{\ell}+1\right)_n$$

$$\left(\frac{2-\beta\ell}{\ell}\right)_{n+1}=\left(\frac{2-\beta\ell}{\ell}\right)\left(\frac{2-\beta\ell}{\ell}+1\right)\ldots\left(\frac{2-\beta\ell}{\ell}+n+1-1\right)=\left(\frac{2-\beta\ell}{\ell}\right)\left(\frac{2-\beta\ell}{\ell}+1\right)_n$$

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$$\left(\frac{\ell-\beta\ell}{\ell}\right)_{n+1}=\left(\frac{\ell-\beta\ell}{\ell}\right)\left(\frac{\ell-\beta\ell}{\ell}+1\right)\ldots\left(\frac{\ell-\beta\ell}{\ell}+n+1-1\right)=\left(\frac{\ell-\beta\ell}{\ell}\right)\left(\frac{\ell-\beta\ell}{\ell}+1\right)_n$$

$$\therefore\left(\frac{1-\beta\ell}{\ell}\right)_{n+1}\left(\frac{2-\beta\ell}{\ell}\right)_{n+1}---\left(\frac{\ell-\beta\ell}{\ell}\right)_{n+1}.$$

$$=\left(\frac{1-\beta\ell}{\ell}\right)\left(\frac{2-\beta\ell}{\ell}\right)---\left(\frac{\ell-\beta\ell}{\ell}\right)\left(\frac{1-\beta\ell}{\ell}+1\right)_n\left(\frac{2-\beta\ell}{\ell}+1\right)_n\ldots\left(\frac{\ell-\beta\ell}{\ell}+1\right)_n$$

$$=\frac{(1-\beta\ell)(2-\beta\ell)---(\ell-\beta\ell)}{\ell^{\ell}}\left(\frac{1-\beta\ell}{\ell}+1\right)_n\left(\frac{2-\beta\ell}{\ell}+\ell\right)_n\ldots\ldots\left(\frac{\ell-\beta\ell}{\ell}+1\right)_n$$

$$=\frac{(1-\beta\ell)_{\ell}}{\ell^{\ell}}\left(\frac{1-\beta\ell}{\ell}+1\right)_n\left(\frac{2-\beta\ell}{\ell}+1\right)_n\ldots\left(\frac{\ell-\beta\ell}{\ell}+1\right)_n$$

Due to above relation equation (2.3.26) becomes.

$$\theta_o F_{\ell}\left[\begin{matrix}-; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^{\ell}4t^2\right]$$

$$=\frac{8t^2x^{2k\ell}}{\ell^{\ell}}\sum_{n=0}^{\infty}\frac{\ell^{\ell}\left[4t^2\left(\frac{x^{2k}}{\ell}\right)^{\ell}\right]^n}{n!(1-\beta\ell)_{\ell}\left(\frac{1-\beta\ell}{\ell}+1\right)_n\left(\frac{2-\beta\ell}{\ell}+1\right)_n--\left(\frac{\ell-\beta\ell}{\ell}+1\right)_n}.$$

$$=\frac{8t^2x^{2k\ell}}{(1-\beta\ell)_\ell}\sum_{n=0}^{\infty}\frac{\left[4t^2\left(\frac{x^{2k}}{\ell}\right)^\ell\right]^n}{\left(\frac{1-\beta\ell}{\ell}+1\right)_n\left(\frac{2-\beta\ell}{\ell}+1\right)_n--\left(\frac{\ell-\beta\ell}{\ell}+1\right)_n n!}$$

$$=\frac{8t^2x^{2k\ell}}{(1-\beta\ell)_\ell}\sum_{n=0}^{\infty}\frac{\left[4t^2\left(\frac{x^{2k}}{\ell}\right)^\ell\right]^n}{\left(\frac{1-\beta\ell+\ell}{\ell}\right)_n\quad\cdots\quad\left(\frac{\ell-\beta\ell+\ell}{\ell}\right)_n n!}$$

$$=\frac{8t^2x^{2k\ell}}{(1-\beta\ell)_\ell}\cdot{}_0F_\ell\left[\begin{matrix}--; \\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^\ell 4t^2\right] \quad ---(2.3.27)$$

Using equation (2.3.27) in equation (2.3.21)

$$\sum_{n=0}^{\infty}\frac{(1+2n)(1-\beta)_n S_{2n+1}(x;k;\ell)t^{2n+1}}{(3/2)_n(1-\beta)_{n\ell}\, n!}$$

$$=2tx^\ell(1-8t^2)\cdot e^{-4t^2}{}_0F_\ell\left[\begin{matrix}--; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^\ell 4t^2\right]$$

$$+\frac{2tx^\ell e^{-4t^2}8t^2x^{2k\ell}}{(1-\beta\ell)_\ell}\cdot{}_0F_\ell\left[\begin{matrix}--; \\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^\ell 4t^2\right]$$

$$=2tx^\ell(1-8t^2)\cdot e^{-4t^2}{}_0F_\ell\left[\begin{matrix}--; \\ \Delta(\ell,1-\beta\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^\ell 4t^2\right]$$

$$+\frac{16t^3x^{(2k+1)\ell}e^{-4t^2}}{(1-\beta\ell)_\ell}\cdot{}_0F_\ell\left[\begin{matrix}--; \\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^\ell 4t^2\right]---(2.3.28)$$

We fruitfully combine (2.3.22) and (2.3.28) we get generating function for $S_n(x;k;\ell)$

$$\sum_{n=0}^{\infty}\frac{(1+\beta-2\beta\in)_{\left[\frac{n}{2}\right]}S_n(x;k,\ell)t^n(1+2n\in)}{\left(\frac{1}{2}+\in\right)_{\left[\frac{n}{2}\right]}(1+\beta-\in\beta(\ell+1))_{\ell\left[\frac{n}{2}\right]}\left[\frac{n}{2}\right]!}$$

$$=e^{-4t^2}\cdot{}_0F_\ell\left[\begin{matrix}-;\\ \Delta(\ell,1+\beta);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^{\ell}4t^2\right]$$

$$+2t(1-8t^2)x^{\ell}e^{-4t^2}\cdot{}_0F_\ell\left[\begin{matrix}-;\\ \Delta(\ell,1-\beta\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^{\ell}4t^2\right]$$

$$+\frac{16\cdot t^3(x^{2k\ell})^{\ell}e^{-4t^2}x^{\ell}}{(1-\beta\ell)_\ell}\cdot{}_0F_\ell\left[\begin{matrix}-;\\ \Delta(\ell,1-\beta\ell+\ell);\end{matrix}\left(\frac{x^{2x}}{\ell}\right)^{\ell}4t^2\right]$$

For β = -1/2, ℓ = k and k = 1 in equation (2.3.28) reduces to the result obtained by Andhare and Jagtap [1,p,83]

Particular case :- Put k = 1, ℓ = 1 and β = - 1/2 in equations (2.3.22) and (2.3.28) and combining the resulting equations we obtain the following generating function for Hermite polynomials.

$$\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{[n!]^2}=e^{-4t2}oF_\ell\left[\begin{matrix}-;\\ 1/2;\end{matrix}4t^2x^2\right]+2tx(1-8t^2)e^{-4t^2}oF_\ell\left[\begin{matrix}-;\\ 3/2;\end{matrix}4t^2x^2\right]+$$

$$+\frac{32}{2}2t^3x^3e^{-4t2}oF_\ell\left[\begin{matrix}-;\\ 5/2;\end{matrix}4t^2x^2\right] \qquad ---(2.3.29)$$

(2.4) Bilateral Generating Functions:-

Let

$$M_{n,q}^{p,\mu}(y,y_2,\ldots y_N;z)=\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{1}{(n-qj)!}c_j\Omega_{m+pj}(y_1y_2\ldots y_N)\cdot z^j \qquad ---(2.4.1)$$

and

$$\Lambda_{m,q}(x;y_1,y_2,\ldots y_N;t)=\sum_{n=0}^{\infty}C_n T_{m+qn}(x;k)\Omega_{m+pn}(y_1y_2\ldots y_N)t^n \qquad ---(2.4.2)$$

where μ is arbitrary complex number, p, q are positive integers and Ω_μ $(y_1,y_2, \ldots, y_N)$ is nonvanishing functions of N variables $y_1,y_2,\text{----}\ y_N$; $N\geq 1$, $c_j\neq o$, $j=1,2,3,\ldots$

Consider

$$\sum_{n=0}^{\infty}\frac{T_{2m+2n}(x;k,\ell)M_{n,q}^{p,\mu}(y_1,y_2,---y_N;z)t^{2n}(1+\beta)_{m+n}}{\left(\frac{1}{2}\right)_{m+n}}$$

$$=\sum_{n=0}^{\infty}T_{2m+2n}(x;k,\ell)\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{1}{(n-qj)!}c_j\Omega_{\mu+pj}(y_1,y_2,--y_N)z^jt^{2n}\frac{(1+\beta)_{m+n}}{\left(\frac{1}{2}\right)_{m+n}}---(2.4.3)$$

Replacing n by $n+qj$

$$= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} T_{2m+2n+2qj}(x;k,\ell) \frac{1}{n!} c_j \Omega_{\mu+pj}(y_1, y_2, --y_N) z^j t^{2n+2qj} \frac{(1+\beta)_{m+n+qj}}{(\frac{1}{2})_{m+n+qj}}$$

$$= \sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, --y_N)(zt^{2q})^j \sum_{n=0}^{\infty} T_{2m+2n+2qj}(x;k,\ell) \frac{t^{2n}(1+\beta)_{m+n+qj}}{n!(\frac{1}{2})_{m+n+qj}}$$

$$= \sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, --y_N)(zt^{2q})^j \cdot$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{m+n+qj} 2^{2m+2n+2qj}(m+n+qj)!(1+\beta)_{m+n+qj}(\frac{1}{2})_{m+n+qj}}{(1+\beta)_{m+n+qj}(\frac{1}{2})_{m+n+qj}} \cdot Y^{\beta}_{m+n+qj}(x^{2k};\ell) \frac{t^{2n}}{n!}$$

by....using..definitions..of. $T_{2n}(x;k,\ell)$

$$= \sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, ---y_N)(z.t^{2q})^j$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{m+qj}(m+qj)!}{(1+\beta)_{m+n+qj}} \binom{m+n+qj}{n} 2^{2m+2qj} \cdot Y^{\beta}_{m+n+qj}(x^{2k};\ell)(-4t^2)^n$$

$$= \sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, ---y_N)(m+qj)!(-1)^{m+qj} 2^{2m+2qj}(zt^{2q})^j$$

$$\sum_{n=0}^{\infty} \binom{m+n+qj}{n} \cdot Y^{\beta}_{m+n+qj}(x^{2k};\ell) \cdot (-4t^2)^n \qquad ---(2.4.3)$$

From the generating function [26] (see also Srivastava [34]).

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y^{\alpha}_{m+n}(x^2;k) t^n = (1-t)^{-(\alpha+mk+1)/k} \exp.\left\{x^2\left[1-(1-t)^{\frac{-1}{k}}\right]\right\} \cdot$$

$$\cdot Y^{\alpha}_{m}\left(x^2(1-t^2)^{-1/k};k\right) \qquad ---(2.4.4)$$

where m is any non negative integer .

Using equation (2.4.4) in right hand side of equation (2.4.3) we obtain.

$$\sum_{n=0}^{\infty} T_{2m+2n}(x;k,\ell) M_{n,q}^{p,\mu}(y_1, y_2, --- y_N; z) \cdot t^{2n} \frac{(1+\beta)_{m+n}}{\left(\frac{1}{2}\right)_{m+n}}$$

$$= \sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, --- y_N) \cdot (-1)^{m+qj} (m+qj)! \cdot 2^{2m+2qj}$$

$$(zt^{2q})^j (1+4t^2)^{-[\beta+(m+qj)\ell+1]/\ell} \exp\left\{x^{2k}\left[1-(1+4t^2)^{-1/\ell}\right]\right\} Y_{m+qj}^{\beta}\left(x^{2k}(1+4t^2)^{-1/\ell};\ell\right)$$

$$= \exp.\left\{x^{2k}\left[1-(1+4t^2)^{-1/\ell}\right]\right\}(1+4t^2)^{-(\beta+m\ell+1)/\ell}$$

$$\sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, --- y_N) \cdot (-1)^{m+qj} (m+qj)! \cdot 2^{2m+2qj}$$

$$\left[z\left(\frac{t^2}{1+4t^2}\right)^q\right]^j Y_{m+qj}^{\beta} \cdot \left(x^{2k}(1+4t^2)^{-1/\ell};\ell\right)$$

$$= \exp\left\{x^{2k}\left[1-(1+4t^2)^{-1/\ell}\right]\right\}(1+4t^2)^{-(\beta+m\ell+1)/\ell}$$

$$\sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, --- y_N)(-1)^{m+qj}(m+qj)!\, 2^{2m+2qj}$$

$$\frac{\left(\frac{1}{2}\right)_{m+qj}}{(1+\beta)_{m+qj}} Y_{m+qj}^{\beta}\left(x^{2k}(1+4t^2)^{-1/\ell};\ell\right) \cdot \left[Z\left(\frac{t^2}{1+4t^2}\right)^q\right]^j \frac{(1+\beta)_{m+qj}}{\left(\frac{1}{2}\right)_{m+qj}}$$

$$= \exp\left\{x^{2k}\left[1-(1+4t^2)^{-1/\ell}\right]\right\}(1+4t^2)^{-(\beta+m\ell+1)/\ell} \cdot \sum_{j=0}^{\infty} \frac{(1+\beta)_{m+qj}}{(1/2)_{m+qj}} c_j\, \Omega_{\mu+pj}(y_1, y_2, ..., y_N;)$$

$$T_{2m+2qj} \cdot (x \cdot (1+4t^2)^{-1/2k\ell}; k, \ell)\left[z\left(\frac{t^2}{1+4t^2}\right)^q\right]^j \qquad ---(2.4.5)$$

This is bilateral generating function for even case of second set $\{T_n(x; k, \ell)\}$

Similarly we obtain bilateral generating function for the odd case of second set $\{T_n(x; k, \ell)\}$ as follows.

$$\sum_{n=0}^{\infty} T_{2m+2n+1}(x;k,\ell)\frac{(1-\beta)_{m+n}}{(3/2)_{m+n}} M_{m,q}^{p,\mu}(y_1,y_2,\ldots\ldots y_N;z)t^{2n+1}$$

$$= \sum_{j=0}^{\infty} c_j.\Omega_{\mu+pj}(y_1,y_2,\ldots\ldots y_N)(xt)(zt^{2q})^j.(-1)^{m+qj}.$$

$$\cdot\ 2^{2m+2qj+1}(m+qj)! \sum_{n=0}^{\infty}\binom{m+n+qj}{n} Y_{m+n+qj}^{-\beta\ell}(x^{2k};\ell)(-4t^2)^n \qquad ---(2.4.6)$$

Using the generating function (2.4.4) in right hand side of the equation (2.4.6) we get

$$\sum_{n=0}^{\infty} T_{2m+2n+1}(x;k,\ell)\frac{(1-\beta)_{m+n}}{(3/2)_{m+n}} M_{m,q}^{p,\mu}(y_1,y_2,\ldots\ y_N;z)t^{2n+1}$$

$$= t(1+4t^2)^{-(-\beta\ell-\frac{1}{2k}+m\ell+1)/\ell}.\exp.\left\{x^{2k}\left[1-(1+4t^2)^{-1/\ell}\right]\right\}$$

$$\sum_{j=0}^{\infty} c_j.\Omega_{\mu+pj}(y_1,y_2,\ldots,y_N)\frac{(1-\beta)_{m+qj}}{(3/2)_{m+qj}} T_{2m+2qj+1}(x(1+4t^2)^{-1/2k\ell};k,\ell)\left[z\left(\frac{t^2}{1+4t^2}\right)^q\right]^j$$

----(2.4.7)

Combining equations (2.4.5) and (2.4.7) we obtain bilateral generating function.

$$\sum_{n=0}^{\infty} \frac{T_{2m+n}(x;k,\ell) M^{p,\mu}_{\left[\frac{n}{2}\right],q}(y_1, y_2, \ldots\ldots y_N; z)(1+\beta-2\beta\in)_{m+\left[\frac{n}{2}\right]} t^n}{(1/2+\in)_{m+\left[\frac{n}{2}\right]}}.$$

$$= \exp.\left\{x^{2k}\left[1-(1+4t^2)^{-1/\ell}\right]\right\} \sum_{j=0}^{\infty} c_j . \Omega_{.\mu+pj}(y_1, y_2, \ldots\ldots y_N)\left[z\left(\frac{t^2}{1+4t^2}\right)^q\right]^j$$

$$\left\{\left[\frac{(1+4t^2)^{-(-\beta+m\ell+1)/\ell} . (1+\beta)_{m+qj}}{(1/2)_{m+qj}} T_{2m+2qj} \cdot (x(1+4t^2)^{-1/2k\ell}; k, \ell)\right]\right.$$

$$\left. + \frac{t(1+4t^2)^{-\left(\beta\ell-\frac{1}{2k}+m\ell+1\right)/\ell}(1-\beta)_{\mu+qj}}{\left(\frac{3}{2}\right)_{m+qj}} T_{2m+2qj+1}\left(x(1+4t^2)^{-1/2k\ell}; k, \ell\right)\right\} \ldots (2.4.8)$$

Put $\beta = -1/2$, $k = 1$, $l = k$ in equation (2.4.5) we get

$$\sum_{n=0}^{\infty} T_{2m+2n}(x,k) M^{p\mu}_{n,q}(y_1, y_2, -- y_N; z) t^{2n}$$

$$= \exp\left\{x^2\left[1-(1+4t^2)^{\frac{-1}{k}}\right]\right\}(1+4t^2)^{-\left(mk+\frac{1}{2}\right)/k}$$

$$\sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, -- y_N) T_{2m+2qj}\left(x(1+4t^2)^{-1/2k}; k\right)\left[z\left(\frac{t^2}{1+4t^2}\right)^q\right]^j --- (2.4.9)$$

By definition of (2.4.2), equation (2.4.9) becomes,

$$\sum_{n=0}^{\infty} T_{2m+2n}(x;k) M^{p,\mu}_{n,q}(y_1, y_2, -- y_N; z) t^{2n}$$

$$= (1+4t^2)^{-\left(mk+\frac{1}{2}\right)/k} \exp.\left\{x^2\left[1-(1+4t^2)^{\frac{-1}{k}}\right]\right\}$$

$$\Lambda_{2m,2q}\left[x(1+4t^2)^{-1/2k}; y_1, y_2, -- y_N; z\left(\frac{t^2}{1+4t^2}\right)^q\right] --- (2.4.10)$$

Similarly putting $\beta = -1/2$, $l = k$, and $k = 1$ in equation in (2.4.7)

$$\sum_{n=0}^{\infty} T_{2m+2n+1}(x;k) M_{n,q}^{p,\mu}(y_1, y_2, -- y_N; z) t^{2n+1}$$

$$= t\left(1+4t^2\right)^{-\left(mk+\frac{k}{2}+\frac{1}{2}\right)/k} \exp.\left\{x^2\left[1-\left(1+4t^2\right)^{\frac{-1}{k}}\right]\right\}.$$

$$\cdot \sum_{j=0}^{\infty} c_j \Omega_{\mu+pj}(y_1, y_2, -- y_N) T_{2m+2qj+1}\left(x\left(1+4t^2\right)^{-1/2k}; k\right) \cdot \left[z\left(\frac{t^2}{1+4t^2}\right)^q\right]^j$$

$$= t\left(1+4t^2\right)^{-\left(mk+\frac{k}{2}+\frac{1}{2}\right)/k} \exp.\left\{x^2\left[1-\left(1+4t^2\right)^{\frac{-1}{k}}\right]\right\}$$

$$\cdot \Lambda_{2m+1,2q}\left[x\left(1+4t^2\right)^{-1/2k}; y_1, y_2, -- y_N; z\left(\frac{t^2}{1+4t^2}\right)^q\right] --- (2.4.11)$$

Combing (2.4.10) and (2.4.11) we obtain bilateral generating function.

$$\sum_{n=0}^{\infty} T_{2m+n}(x;k) M^{p,\mu}_{\left[\frac{n}{2}\right],q}(y_1,y_2,--y_N;z)t^n$$

$$=\left(1+4t^2\right)^{-(mk+(k+1)/2)/k} \exp.\left\{x^2\left[1-\left(1+4t^2\right)^{-1/k}\right]\right\}$$

$$\ldots\left\{\left(1+4t^2\right)^{1/2} \Lambda_{2m,2q}\left[x\left(1+4t^2\right)^{-1/2k};y_1,y_2,--y_N;z\left(\frac{t^2}{1+4t^2}\right)^q\right]+\right.$$

$$\left.+t\Lambda_{2m+1,2q}\left[x\left(1+4t^2\right)^{-1/2k};y_1,y_2,\ldots,y_N;z\left(\frac{t^2}{1+4t^2}\right)^q\right]\right\}---(2.4.12)$$

Particular Cases:-

Case(I): For k = 1, equation (2.4.12) gives the bilateral generating function for Hermite polynomials

$$\sum_{n=0}^{\infty} H_{2m+n}(x) M^{p,\mu}_{\left[\frac{n}{2}\right],q}(y_1,y_2,\_\_y_N;z)t^n$$

$$=\left(1+4t^2\right)^{-(m+1)} \exp.\left\{\frac{4x^2t^2}{1+4t^2}\right\}\left\{\left(1+4t^2\right)^{1/2}\right.$$

$$\Lambda'_{2m,2q}\left[x\left(1+4t^2\right)^{-1/2};y_1,y_2,--y_N;z\left(\frac{t^2}{1+4t^2}\right)^q\right]+$$

$$\left.+t\Lambda'_{2m+1,2q}\left[x\left(1+4t^2\right)^{-1/2};y_1,y_2,--y_N;z\left(\frac{t^2}{1+4t^2}\right)^q\right]\right\}---(2.4.13)$$

where

$$\Lambda'_{m,q}[x;y_1,y_2---y_N;t]=\sum_{n=0}^{\infty}c_n H_{m+qn}(x)\Omega_{\mu+pn}(y_1,y_2,--y_N)t^n$$

Case: (II)

By Substituting $m=0, q=1, c_j\Omega_{\mu+pj}(y_1,y_2,--y_N)=\frac{\delta j}{j.!}$

We obtain the following bilateral generating function due to Thakare and Madhekar. [39,p.36 (4.6)]

$$\sum_{n=0}^{\infty}\frac{T_n(x;k)\sigma_n(z)t^n}{\left(\frac{n}{2}\right)!}=(1+4t^2)^{-(k+1)/2k}\exp.\left\{x^2\left[1-(1+4t^2)^{-1/k}\right]\right\}.G\left(X;\frac{zt^2}{1+4t^2}\right)$$

---(2.4.14)

where,

$$\sigma_n(z)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{\left[\frac{n}{2}\right]}{j}\delta_j z^j$$

$X=x(1+4t^2)^{-1/2k}$. .and,

$$G(x;z)=\sum_{j=0}^{\infty}\delta_n\left[(1+4t^2)^{1/2}T_{2n}(x;k)+T_{2n+1}(x;k)\frac{z^n}{n!}\right] \qquad ---(2.4.15)$$

Case: (III) For $Z = 0$, $\delta_0 = 1$ in equation (2.4.15) we get the generating function obtained by Thakare. and Madhekar.

$$\sum_{n=0}^{\infty} T_n(x;k)\frac{t^n}{\left[\frac{n}{2}\right]!} = Y\left(1+4t^2\right)^{-(k+2)/2k}\left[2xt+\left(1+4t^2\right)^{(k+1)/2k}\right] \quad ---(2.4.16)$$

where $Y = \exp\left\{x^2\left[1-\left(1+4t^2\right)^{-1/k}\right]\right\}$

Case: (IV) For $k = 1$,equation (2.4.16) reduces to the generating function for Hermite polynomials given by Doetsch[13],

$$\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{\left[\frac{n}{2}\right]!} = \left(1+4t^2\right)^{-3/2}\left(1+2xt+4t^2\right)\exp.\left(\frac{4x^2t^2}{1+4t^2}\right) \quad ---(2.4.17)$$

(2.5) Recurrence Relations:-

(i) From Konhauser [17,p.305] Doetsch [13] we have

$$xDZ_n^{\alpha}(x;k)=knZ_n^{\alpha}(x;k)-k(kn-k+\alpha+1)_k Z_{n-1}^{\alpha}(x;k) \qquad ---(2.5.1)$$

Replace x by x^{2k}; α by β, k by ℓ.

$$\frac{x^{2k}}{2kx^{2k-1}}DZ_n^{\beta}(x^{2k};\ell)=n\ell.Z_n^{\beta}(x^{2k};\ell)-\ell(n\ell-\ell+\beta+1)_\ell Z_{n-1}^{\beta}(x^{2k};\ell)$$

$$xDZ_n^{\beta}(x^{2k};\ell)=2kn\ell.Z_n^{\beta}(x^{2k};\ell)-2k\ell(n\ell-\ell+\beta+1)_\ell Z_{n-1}^{\beta}(x^{2k};\ell)$$

multiplying by $\frac{(-1)^n n!(1+n)_n}{(1+\beta)_n}$

$$\frac{(-1)^n n!(1+n)_n}{(1+\beta)_n}xDZ_n^{\beta}(x^{2k};\ell)=\frac{2kn\ell(-1)^n n!(1+n)_n Z_n^{\beta}(x^{2k};\ell)}{(1+\beta)_n}$$

$$\frac{-2k\ell(n\ell-\ell+\beta+1)_\ell(-1)^n n!(1+n)_n}{(1+\beta)_n}Z_{n-1}^{\beta}(x^{2k};\ell)$$

$$xDS_{2n}(x;k,\ell)=2kn\ell S_{2n}(x;k,\ell)+2kn\ell\frac{(n\ell-\ell+\beta+1)_\ell 2(2n-1)}{(\beta+n)}S_{2n-2}(x;k,\ell)$$

$$=2kn\ell S_{2n}(x;k,\ell)+\frac{4n(2n-1)k\ell(n\ell-\ell+\beta+1)_\ell}{(\beta+n)}S_{2n-2}(x;k,\ell) \qquad ---(2.5.2)$$

Replace x by x^{2k} and α by $-\beta\ell$,and k by ℓ in equation (2.5.1),

$$\frac{x^{2k}}{2kx^{2k-1}} DZ_n^{-\beta\ell}\left(x^{2k};\ell\right) = n\ell Z_n^{-\beta\ell}\left(x^{2k};\ell\right) - \ell(n\ell-\ell-\beta\ell+1)_\ell Z_{n-1}^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$\therefore \frac{(-1)^n n!\left(3/2\right)_n 2^{2n+1} x x^\ell}{(1-\beta)_n 2k} DZ_n^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$= \frac{(-1)^n n!\left(3/2\right)_n 2^{2n+1} x^\ell}{(1-\beta)_n} n\ell Z_n^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$-\frac{(-1)^n n!\left(3/2\right)_n 2^{2n+1} x^\ell}{(1-\beta)_n} \ell(n\ell-\ell-\beta\ell+1)_\ell Z_{n-1}^{-\beta\ell}\left(x^{2k};\ell\right)$$

But.

$$..D\left\{x^\ell Z_n^{-\beta\ell}\left(x^{2k};\ell\right)\right\} = x^\ell DZ_n^{-\beta\ell}\left(x^{2k};\ell\right) + \ell x^{\ell-1} Z_n^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$x^\ell DZ_n^{-\beta\ell}\left(x^{2k};\ell\right) = D\left[x^\ell Z_n^{-\beta\ell}\left(x^{2k};\ell\right)\right] - \ell x^{\ell-1} Z_n^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$\frac{x}{2k} DS_{2n+1}(x;k,\ell) - \frac{\ell}{2k} S_{2n+1}(x;k,\ell) = n\ell S_{2n+1}(x;k,\ell)$$

$$+\frac{n\ell(2n+1)}{(n-\beta)} 2(n\ell-\ell-\beta\ell+1)_\ell S_{2n-1}(x;k,\ell)$$

$$xDS_{2n-1}(x;k,\ell) = (\ell+2kn\ell)S_{2n+1}(x;k,\ell)$$

$$+\frac{4n\ell(2n+1)(n\ell-\ell-\beta\ell+1)_\ell}{(n-\beta)} S_{2n-1}(x;k,\ell) ---- (2.5.3)$$

Combining (2.5.2) and (2.5.3)

$$xDS_n(x;k,\ell)=\left(2k\ell\left[\tfrac{n}{2}\right]+\ell\in\right)S_n(x;k,\ell)$$

$$+\frac{4k\ell\left[\frac{n}{2}\right]\left(\ell\left[\frac{n}{2}\right]-\ell+\beta+1-(\beta\ell+\beta)\in\right)_\ell(2n-1+\in)S_{n-2}(x;k,\ell)}{\left(\left[n/2\right]+\beta-2\beta\in\right)}\text{-----}(2.5.4)$$

(II) From Konhauser [17, P. 308, equation (16)] We have recurrence relation

$$k(n+1)Y_{n+1}^{\alpha}(x;k)=xDY_n^{\alpha}(x;k)+(kn+\alpha+1-x)Y_{n+1}^{\alpha}(x;k)\ldots(2.5.5)$$

Replace..α.by.β, k.by.ℓ, x.by.x^{2k}

$$\ell(n+1)Y_{n+1}^{\beta}\left(x^{2k};\ell\right)=\frac{x^{2k}}{2kx^{2k-1}}DY_n^{\beta}\left(x^{2k};\ell\right)+\left(n\ell+\beta+\ell+1-x^{2k}\right)Y_n^{\beta}\left(x^{2k};\ell\right)$$

Multiplying..by..n!..$(-1)^{n+1}(2+n)_{n+1}$

$$\ell(n+1)!(2+n)_{n+1}(-1)^{n+1}Y_{n+1}^{\beta}(x;\ell)=n!(-1)^{n+1}(2+n)_{n+1}\frac{x}{2k}DY_n^{\beta}\left(x^{2k};\ell\right)$$

$$+(-1)^{n+1}n!\left(n\ell+\beta+1-x^{2k}\right)(2+n)_{n+1}Y_n^{\beta}\left(x^{2k};\ell\right)$$

$$\ell T_{2n+2}(x;k;\ell)=-\frac{x}{2k}(-1)^n n!(2+n)_{n+1}DY_n^{\beta}\left(x^{2k};\ell\right)$$

$$-(-1)^n n!\left(n\ell+\beta+1-x^{2k}\right)(2+n)_{n+1}Y_n^{\beta}\left(x^{2k};\ell\right)$$

But $(2+n)_{n+1}=\dfrac{\Gamma(2+n+n+1)}{\Gamma(2+n)}=\dfrac{(2+n+n)}{(1+n)}\dfrac{\Gamma(2+n+n)}{\Gamma(1+n)}$

$$=2(2n+1)\frac{\Gamma(1+n+n)}{\Gamma(1+n)}=2(2n+1)(1+n)_n$$

$$\ell T_{2n+2}(x;k,\ell) = \frac{-x}{2k} 2(2n+1)(1+n)_n(-1)^n n! D Y_n^{\beta}(x^{2k};\ell)..$$

$$-2(2n+1)(n\ell+\beta+1-x^{2k})(-1)^n n!(1+n)_n Y_n^{\beta}(x^{2k};\ell)$$

$$= -\frac{x}{k}(2n+1)DT_{2n}(x;k,\ell) - 2(2n+1)(n\ell+\beta+1-x^{2k})T_{2n}(x;k,\ell) \quad ---(2.5.6)$$

Replace α by $-\beta\ell$, k by ℓ and x by x^{2k} in (2.5.5)

$$\ell(n+1)Y_{n+1}^{-\beta\ell}(x^{2k};\ell) = \frac{x^{2k}}{2k.2x^{2k-1}} DY_n^{-\beta\ell}(x^{2k};\ell) + (n\ell-\beta\ell+1-x^{2k})Y_n^{-\beta\ell}(x^{2k};\ell)$$

Multiplying above equation by $\frac{(-1)^{n+1} n! 2^{2n+3}(3/2)_{n+1} x}{(1-\beta)_{n+1}}$

$$\frac{x\ell(-1)^{n+1}(n+1)!\, 2^{2n+3}(3/2)_{n+1}}{(1-\beta)_{n+1}} Y_{n+1}^{-\beta\ell}(x^{2k};\ell)$$

$$= \frac{x^2}{2k} \frac{(-1)^{n+1} n!\, 2^{2n+3}(3/2)_{n+1}}{(1-\beta)_{n+1}} DY_n^{-\beta\ell}(x^{2k};\ell)$$

$$+ \frac{(n\ell-\beta\ell+1-x^{2k})}{(1-\beta)_{n+1}} \frac{(-1)^{n+1} n! 2^{2n+3}(3/2)_{n+1}}{} xY_{n+1}^{-\beta\ell}(x^{2k};\ell)$$

But,

$$D\left[xY_n^{-\beta\ell}(x^{2k};\ell)\right] = xDY_n^{-\beta\ell}(x^{2k};\ell) + Y_n^{-\beta\ell}(x^{2k};\ell)$$

$$xDY_n^{-\beta\ell}(x^{2k};\ell) = D\left[xY_n^{-\beta\ell}(x^{2k};\ell)\right] - Y_n^{-\beta\ell}(x^{2k};\ell)$$

and

$$(3/2)_{n+1}=\frac{\Gamma(3/2+n+1)}{\Gamma 3/2}=(3/2+n)\frac{\Gamma(3/2+n)}{\Gamma 3/2}=(3/2+n)(3/2)_n$$

$$(1-\beta)_{n+1}=\frac{\Gamma(1-\beta+n+1)}{\Gamma(1-\beta)}=\frac{(1-\beta+n)\Gamma(1-\beta+n)}{\Gamma(1-\beta)}$$

$$=(1-\beta+n)(1-\beta)_n$$

$$\ell T_{2n+3}(x;k,\ell)=\frac{x}{2k}\frac{(-4)(-1)^n 2^{2n+1}\left(\frac{3+2n}{2}\right)(3/2)_n}{(1-\beta+n)(1-\beta)_n}\{DxY_{n+1}^{-\beta\ell}(x^{2k};\ell)-Y_n^{-\beta\ell}(x^{2k};\ell)\}$$

$$-4\frac{(3+2n)(n\ell-\beta\ell+1-x^{2k})}{2(1-\beta+n)(1-\beta)_n}(3/2)_n(-1)^n 2^{2n+1}n!\,xY_n^{-\beta\ell}(x^{2k};\ell)$$

$$=\frac{-x(3+2n)}{k(1-\beta+n)}DT_{2n+1}(x;k,\ell)+\frac{(3+2n)}{k(1-\beta+n)}T_{2n+1}(x;k,\ell)$$

$$\frac{-2(3+2n)(n\ell-\beta\ell-x^{2k})}{(1-\beta+n)}T_{2n+1}(x;k,\ell)$$

$$=\frac{-(3+2n)}{k(1-\beta+n)}xDT_{2n+1}(x;k,\ell)$$

$$-\frac{2(3+2n)}{(1-\beta+n)}\left\{n\ell-\beta\ell+1-x^{2k}-\frac{1}{2k}\right\}T_{2n+1}(x;k,\ell)---(2.5.7)$$

Combine (2.5.6) and (2.5.7), we get,

$$\ell T_{n+1}(x;k,\ell)=\left(\frac{-1}{k}\right)\frac{(n+1+\in)}{\left(1+\left[\frac{n}{2}\right]-\beta\right)}xD.T_n(x;k,\ell)-$$

$$-\frac{2(n+1+\in)}{\left[1+\left(\left[\frac{n}{2}\right]-\beta\right)\in\right]}\left\{\ell\left[\frac{n}{2}\right]+1-x^{2k}+\beta-(\beta\ell+\frac{1}{2k}+\beta)\in\right\}.T_n(x;k,\ell).--(2.5.8)$$

(III) Pure Recurrence Relation:-

$$k(n+1)Y_{n+1}^{\alpha}(x;k)=(\alpha+1+kn)Y_n^{\alpha}(x;k)-x\,Y_n^{\alpha+1}(x;k)...(2.5.9)$$

Re placing.α.byβ, x.by.$x^{2k}$and.k.by.ℓ

$$\ell(n+1)Y_{n+1}^{\beta}(x^{2k};\ell)=(\beta+1+n\ell)Y_n^{\beta}(x^{2k};\ell)-xY_n^{\beta+1}(x^{2k};\ell)$$

Multiplying by $(-1)^{n+1}$ n! $(1+n+1)_{n+1}$ on both the sides we get,

$$\ell(-1)^{n+1}(n+1)!(1+n+1)Y_{n+1}^{\beta}(x^{2k};\ell)$$

$$=(-1)^{(n+1)}n!(1+n+1)_{n+1}(\beta+1+n\ell)Y_n^{\beta}(x^{2k};\ell)-x(-1)^{n+1}\cdot n!(1+n+n)_{n+1}$$

$$\cdot Y_n^{\beta+1}(x^{2k};\ell)$$

$$\ell T_{2n+2}(x;k,\ell)=-(\beta+1+n\ell)(-1)^{(n)}n!\cdot 2(2n+1)(1+n)_n\,Y_n^{\beta}(x^{2k};\ell)$$

$$+x2(2n+1)(-1)^n n!(1+n)_n\,Y_n^{\beta+1}(x^{2k};\ell)$$

$$=-2(2n+1)(\beta+1+n\ell)(-1)^{(n)}n!(1+n)_n\,Y_n^{\beta}(x^{2k};\ell)$$

$$+2(2n+1)x(-1)^{(n)}n!(1+n)_n\,Y_n^{\beta+1}(x^{2k};\ell)$$

$$=-2(2n+1)(\beta+1+n\ell)T_{2n}(x;k,\ell)+2(2n+1)xT_{2n}(x;k,\ell)\qquad ---(2.5.10)$$

Replace x by x^{2k}, k by ℓ and α by $-\beta\ell$ in (2.5.9)pure recurrence relation

$$\ell(n+1)Y_{n+1}^{-\beta\ell}\left(x^{2k};\ell\right)=\left(-\beta\ell+1+n\ell\right)Y_{n+1}^{-\beta\ell}\left(x^{2k};\ell\right)-xY_{n}^{-\beta\ell+1}\left(x^{2k};\ell\right)---(2.5.11)$$

$$\text{By..defnition.of.}T_{2n+1}(x;k,\ell)=\frac{(-1)^{n}n!2^{2n+1}\left(\tfrac{3}{2}\right)_{n}x}{(1-\beta)_{n}}Y_{n}^{-\beta\ell}\left(x^{2k};\ell\right)$$

Replace n by n+1

$$T_{2n+3}(x;k,\ell)=\frac{(-1)^{n+1}(n+1)!\cdot 2^{2n+3}\left(\tfrac{3}{2}\right)_{n+1}x}{(1-\beta)_{n+1}}Y_{n+1}^{-\beta\ell}\left(x^{2k};\ell\right)$$

Multiplying above equation (2.5.11) by

$$\frac{(-1)^{n+1}n!2^{2n+3}\left(\tfrac{3}{2}\right)_{n+1}x}{(1-\beta)_{n+1}}\quad\text{we get,}$$

$$\ell\frac{(-1)^{n+1}(n+1)!2^{2n+3}\left(\tfrac{3}{2}\right)_{n+1}x}{(1-\beta)_{n+1}}Y_{n+1}^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$=\frac{-4x\left(-\beta\ell+1+n\ell\right)(2n+3)}{(1-\beta+n)\cdot 2}\frac{(-1)^{n}n!2^{2n+1}\left(\tfrac{3}{2}\right)_{n}}{(1-\beta)_{n}}xY_{n}^{-\beta\ell}\left(x^{2k};\ell\right)$$

$$+\frac{4x(2n+3)}{(1-\beta+n)\cdot 2}\frac{(-1)^{n}n!2^{2n+1}\left(\tfrac{3}{2}\right)_{n}}{(1-\beta)_{n}}xY_{n}^{1-\beta\ell}\left(x^{2k};\ell\right)$$

Using definition of $T_n(x;k,\ell)$

$$\ell T_{2n+3}(x;k,\ell) = -2\frac{(-\beta\ell + n\ell + 1)(2n+3)}{(1-\beta+n)} T_{2n+1}(x;k,\ell) + \frac{2x(2n+3)}{(1-\beta+n)} T_{2n+1}(x;k,\ell)$$

We combine above even and odd cases,

$$\ell T_{n+2}(x;k,\ell) = \frac{-2(n+1+\in)\left[1+\left[\frac{n}{2}\right]\cdot\ell+\beta-(\beta\ell+\beta)\in\right]}{1+\left(\left[\frac{n}{2}\right]-\beta\right)\in} T_n(x;k,\ell).$$

$$+\frac{2x(n+1+\in)}{1+\left(\left[\frac{n}{2}\right]-\beta\right)\in} T_n(x;k,\ell)$$