CHAPTER - SIX

FUZZY LATTICE HOMOMORPHISM

Chapter 6

FUZZY LATTICE HOMOMORPHISM.

§6.1 Fuzzy Isotone Map:

Definition: "Let P and Q be any two fuzzy partial order set. $f: P \to Q$ is a fuzzy isotone map if $R(x, y) > 0 \Rightarrow R(f(x), f(y)) > 0$."

§6.2 Fuzzy Lattice Homomorphism:

Definition: "Let $(X, \wedge, \vee)$ and $(X', \bar{\wedge}, \bar{\vee})$ be any two fuzzy lattices.

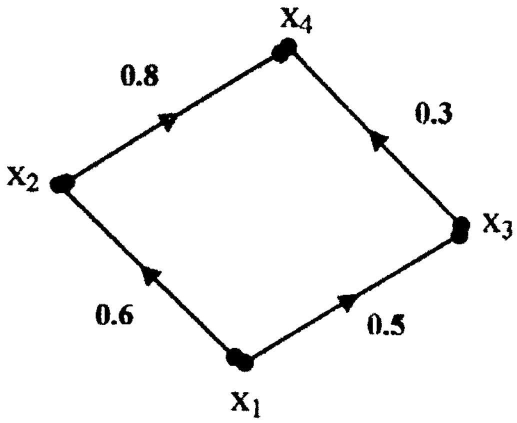
A mapping $f: X \to X'$ is a fuzzy lattice homomorphism if,

$\forall x, y \in X$, $f(x \wedge y) = f(x) \bar{\wedge} f(y)$ and

$f(x \vee y) = f(x) \bar{\vee} f(y)$."

Example: Consider, $X = \{ x_1, x_2, x_3, x_4\}$

The Hasse diagram is,



Here the ordered pair X is a fuzzy lattice.

Let X' = [a, a + n] where a, n ∈ IR and n > 0.

Let R = "**almost** less than or equal to" be a fuzzy relation defined on X as a function R: X x X → [0,1] defined by,

R (x, y)	=	1	if	x = y
	=	(y - x) / n	if	x < y
	=	0	else.	

Here (X', R) is a fuzzy lattice.

Define f: X → X' by $f(x_1) = f(x_3) = a$, $f(x_2) = f(x_4) = a + n$.

Then f is a fuzzy lattice homomorphism.

➢ **Theorem 6.1:**

Every fuzzy lattice homomorphism is a fuzzy isotone map.

Proof: Let f: $X_1 \rightarrow X_2$ be fuzzy lattice homomorphism.

Let R(x, y) > 0 where x, y ∈ X_1.

Now, R(x, y) > 0 ⇒ x Λ y = x (by theorem 2.5)

Now, $f(x) = f(x \wedge y) = f(x) \barwedge f(y)$

(since f is fuzzy lattice homomorphism).

Thus, $f(x) = f(x) \barwedge f(y)$

∴ R(f(x), f(y)) > 0 (by theorem 2.5)

∴ f is a fuzzy isotone map. □

§6.3 Kernel of fuzzy lattice homomorphism:

Definition: Let X_1 and X_2 be any two fuzzy lattices.

Let f: $X_1 \rightarrow X_2$ be fuzzy lattice homomorphism.

Define ker f = {x ∈ X_1 / f(x) = **0**, fuzzy zero of X_2}

- **Theorem 6.2:**

Let X_1 and X_2 be any two fuzzy lattices. Let $f: X_1 \to X_2$ be fuzzy lattice homomorphism. Let $\mathbf{0}$ be the fuzzy zero of X_1 and $\mathbf{0}'$ be the fuzzy zero of X_2. Then ker f is a fuzzy ideal.

Proof: Now, $f(\mathbf{0}) = \mathbf{0}'$ always. Hence $\mathbf{0} \in \ker f$. Hence $\ker f \neq \varnothing$.

Let $x, y \in \ker f$. To Prove: $x \vee y \in \ker f$

$x \in \ker f \Rightarrow f(x) = \mathbf{0}'$ (by definition of ker f)

$y \in \ker f \Rightarrow f(y) = \mathbf{0}'$ (by definition of ker f)

As f is fuzzy lattice homomorphism,

$\therefore f(x \vee y) = f(x) \triangledown f(y) = \mathbf{0}' \triangledown \mathbf{0}' = \mathbf{0}'$

Thus, $f(x \vee y) = \mathbf{0}'$

$\therefore x \vee y \in \ker f$ (by definition of ker f)

Thus $x, y \in \ker f \Rightarrow x \vee y \in \ker f$

Let $R(x, y) > 0$, $x \in X_1$, $y \in \ker f$ To Prove: $x \in \ker f$.

$y \in \ker f \Rightarrow f(y) = \mathbf{0}'$ (by definition of ker f)

As f is fuzzy lattice homomorphism,

$\therefore f$ is a fuzzy isotone map.

$\therefore R(x, y) > 0 \Rightarrow R(f(x), f(y)) > 0 \Rightarrow R(f(x), \mathbf{0}') > 0$

Also, $R(\mathbf{0}', f(x)) > 0$ (by definition of fuzzy zero)

Thus, $R(f(x), \mathbf{0}') > 0$ and $R(\mathbf{0}', f(x)) > 0$

$\Rightarrow f(x) = \mathbf{0}'$ (by theorem 2.7)

$\therefore x \in \ker f$ (by definition of ker f)

Thus, $R(x, y) > 0$, $x \in X_1$, $y \in \ker f \Rightarrow x \in \ker f$.

Thus, ker f is a fuzzy ideal. □

§6.4 Fuzzy Isomorphism:

Definition: "A fuzzy lattice homomorphism f: $X_1 \rightarrow X_2$ which is one-one and onto mapping is said to be fuzzy lattice isomorphism."

Note: "Two fuzzy lattices X_1 and X_2 are said to be fuzzy lattice isomorphic if there is an fuzzy lattice isomorphism f: $X_1 \rightarrow X_2$."

§6.5 Fuzzy Congruence Relation:

"Let (X, R) be a fuzzy lattice. A Similarity Relation 'R_θ' defined on a fuzzy lattice X is called a fuzzy congruence relation, if $x_1 \equiv y_1$ (R_θ), $x_2 \equiv y_2$ (R_θ)

$$\Rightarrow x_1 \wedge x_2 \equiv y_1 \wedge y_2 \ (R_\theta) \text{ and } x_1 \vee x_2 \equiv y_1 \vee y_2 \ (R_\theta)."$$

Example: Let (X, R) be fuzzy lattice. Define a fuzzy relation 'R_θ' as, R_θ: X x X $\rightarrow$ [0,1] defined by,

$$R_\theta(x, y) = 1 \quad \text{if} \quad x = y$$
$$= 0 \quad \text{else},$$

Then 'R_θ' is Fuzzy Congruence Relation.

Solution: We will first prove that 'R_θ' is a Similarity Relation.

1) **Fuzzy Reflexive Relation:**

Now, by definition of R_θ

$$R_\theta(x, x) = 1 \qquad \forall\, x \in X.$$

$\therefore$ R_θ is a fuzzy reflexive relation.

2) **Fuzzy Symmetric Relation**:

Let x, y $\in$ X

If $x = y$, $R_\theta(x, y) = 1$, $R_\theta(y, x) = 1 \Rightarrow R_\theta(x, y) = R_\theta(y, x)$

If $x \neq y$, $R_\theta(x, y) = 0$, $R_\theta(y, x) = 0 \Rightarrow R_\theta(x, y) = R_\theta(y, x)$

Thus, $\forall$ x, y $\in$ X, $R_\theta(x, y) = R_\theta(y, x)$

$\therefore$ R_θ is a fuzzy symmetric relation.

3) **Fuzzy Max-Min Transitive Relation:**

Let $(x, z) \in X^2$

To prove: $R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

If $x = z$ then $R_\theta(x, z) = 1$ (by definition of R_θ)

$\therefore R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

If $x \neq z$ then $R_\theta(x, z) = 0$

Now, for all $y \in X$ such that $y = x$ and $y \neq z$

$R_\theta(x, y) = 1$ and $R_\theta(y, z) = 0$

$\therefore \min(R_\theta(x, y), R_\theta(y, z)) = R_\theta(y, z) = 0$

Now, for all $y \in X$ such that $y = z$ and $y \neq x$

$R_\theta(x, y) = 0$ and $R_\theta(y, z) = 1$

$\therefore \min(R_\theta(x, y), R_\theta(y, z)) = R_\theta(x, y) = 0$

Now, for all $y \in X$ such that $y \neq x$ and $y \neq z$

$R_\theta(x, y) = 0$ and $R_\theta(y, z) = 0$

$\therefore \min(R_\theta(x, y), R_\theta(y, z)) = 0$

Thus, $\forall y \in X$, $\min(R_\theta(x, y), R_\theta(y, z)) = 0$

Thus, $R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

Thus, in all cases,

$R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

Thus, R_θ is a fuzzy max-min transitive relation.

Thus, R_θ is a Similarity Relation.

Let $x_1 \equiv y_1\ (R_\theta)$ and $x_2 \equiv y_2\ (R_\theta)$ where $x_1, x_2, y_1, y_2 \in X$.

$\therefore R_\theta(x_1, y_1) > 0$ and $R_\theta(x_2, y_2) > 0$

$\therefore R_\theta(x_1, y_1) = 1$ and $R_\theta(x_2, y_2) = 1$

$\therefore x_1 = y_1$ and $x_2 = y_2$

$\therefore x_1 \wedge x_2 = y_1 \wedge y_2$ and $x_1 \vee x_2 = y_1 \vee y_2$

$\therefore R_\theta(x_1 \wedge x_2, y_1 \wedge y_2) = 1$ and $R_\theta(x_1 \vee x_2, y_1 \vee y_2) = 1$

$\therefore x_1 \wedge x_2 = y_1 \wedge y_2\ (R_\theta)$ and $x_1 \vee x_2 = y_1 \vee y_2\ (R_\theta)$

Thus, R_θ is a Fuzzy Congruence Relation.

§6.6 Fuzzy Congruence Class:

Definition: "Let 'R_θ' be a fuzzy congruence relation defined on a fuzzy lattice (X, R). Define, $[a]^{R_\theta} = \{x \in X / x \equiv a\ (R_\theta)\}$, $a \in X$. Then $[a]^{R_\theta}$ is called a fuzzy congruence class containing a."

§6.7 Fuzzy Quotient Lattice:

"Let 'R_θ' be a fuzzy congruence relation defined on a fuzzy lattice (X, R). Define, $\dfrac{X}{R_\theta} = \{ [a]^{R_\theta} / a \in X \}$

Define, $\barwedge$ and $\veebar$ on X / R_θ by,

$[a]^{R_\theta} \barwedge [b]^{R_\theta} = [a \wedge b]^{R_\theta}$

$[a]^{R_\theta} \veebar [b]^{R_\theta} = [a \vee b]^{R_\theta}$

Then, $< \dfrac{X}{R_\theta}, \barwedge, \veebar >$ is a fuzzy lattice.

This fuzzy lattice is called as fuzzy quotient lattice of X by fuzzy congruence relation R_θ.

Fundamental Theorem of Fuzzy Lattice Homomorphism:

Every fuzzy homomorphic image of a fuzzy lattice X is fuzzy lattice isomorphic with its suitable fuzzy quotient lattice

OR

if f: $X \rightarrow X_1$ is an onto fuzzy homomorphism then

$X_1 \cong \frac{X}{R_\theta}$ for some fuzzy congruence relation R_0.

Proof: Let X and X_1 be two fuzzy lattices.

Let f: $X \rightarrow X_1$ is an onto fuzzy homomorphism.

Define a fuzzy relation 'R_θ' on X, denoted by $x \equiv y\ (R_\theta)$ for $x, y \in X$, by function as, R_θ: X x X $\rightarrow$ [0,1] defined by,

$$R_\theta(x, y) = \begin{cases} 1 & \text{if } f(x) = f(y) \\ 0 & \text{else} \end{cases}$$

We will first prove that 'R_θ' is a Similarity Relation.

1) **Fuzzy Reflexive Relation:**

Now, $f(x) = f(x) \quad \forall x \in X$. (Since f is a function)

Now, by definition of R_θ,

$\therefore R_\theta(x, x) = 1 \quad \forall x \in X$.

$\therefore R_\theta$ is a fuzzy reflexive relation.

2) **Fuzzy Symmetric Relation**:

Let $x, y \in X$

If $f(x) = f(y) \Rightarrow f(y) = f(x)$

$\therefore R_\theta(x, y) = 1, R_\theta(y, x) = 1 \Rightarrow R_\theta(x, y) = R_\theta(y, x)$

If $f(x) \neq f(y) \Rightarrow f(y) \neq f(x)$

$\therefore R_\theta(x, y) = 0, R_\theta(y, x) = 0 \Rightarrow R_\theta(x, y) = R_\theta(y, x)$

Thus, $\forall x, y \in X, R_\theta(x, y) = R_\theta(y, x)$

$\therefore R_\theta$ is a fuzzy symmetric relation.

3) **Fuzzy Max-Min Transitive Relation:**

Let $(x, z) \in X^2$

To prove: $R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

If $f(x) = f(z)$ then $R_\theta(x, z) = 1$ (by definition of R_θ)

$\therefore R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

If $f(x) \neq f(z)$ then $R_\theta(x, z) = 0$

for all $y \in X$ such that $f(y) = f(x)$ and $f(y) \neq f(z)$

$R_\theta(x, y) = 1$ and $R_\theta(y, z) = 0$

$\therefore \min(R_\theta(x, y), R_\theta(y, z)) = R_\theta(y, z) = 0$

for all $y \in X$ such that $f(y) = f(z)$ and $f(y) \neq f(x)$

$R_\theta(x, y) = 0$ and $R_\theta(y, z) = 1$

$\therefore \min(R_\theta(x, y), R_\theta(y, z)) = R_\theta(x, y) = 0$

for all $y \in X$ such that $f(y) \neq f(x)$ and $f(y) \neq f(z)$

$R_\theta(x, y) = 0$ and $R_\theta(y, z) = 0$

$\therefore \min(R_\theta(x, y), R_\theta(y, z)) = 0$

Thus, $\forall\, y \in X, \min(R_\theta(x, y), R_\theta(y, z)) = 0$

Thus, $R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

Thus, in all cases,

$R_\theta(x, z) \geq \max_{y \in X} \min(R_\theta(x, y), R_\theta(y, z))$

Thus, R_θ is a fuzzy max-min transitive relation.

Thus, R_θ is a Similarity Relation.

Let $x_1 \equiv y_1\ (R_\theta)$ and $x_2 \equiv y_2\ (R_\theta)$ where $x_1, x_2, y_1, y_2 \in X$.

$\therefore\ R_\theta(x_1, y_1) > 0$ and $R_\theta(x_2, y_2) > 0$

$\therefore\ R_\theta(x_1, y_1) = 1$ and $R_\theta(x_2, y_2) = 1$

$\therefore\ f(x_1) = f(y_1)$ and $f(x_2) = f(y_2)$

Now, f is a fuzzy lattice homomorphism,

$\therefore\ f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2) = f(y_1) \wedge f(y_2) = f(y_1 \wedge y_2)$

Thus, $f(x_1 \wedge x_2) = f(y_1 \wedge y_2)$

$\therefore\ R_\theta(x_1 \wedge x_2, y_1 \wedge y_2) = 1$

$\therefore\ x_1 \wedge x_2 \equiv y_1 \wedge y_2\ (R_\theta)$

Also, $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = f(y_1) \vee f(y_2) = f(y_1 \vee y_2)$

Thus, $f(x_1 \vee x_2) = f(y_1 \vee y_2)$

$\therefore\ R_\theta(x_1 \vee x_2, y_1 \vee y_2) = 1$

$\therefore\ x_1 \vee x_2 \equiv y_1 \vee y_2\ (R_\theta)$

Thus, R_θ is a Fuzzy Congruence Relation.

Now, $\dfrac{X}{R_\theta} = \{[a]^{R_\theta} / a \in X\}$

and $< \dfrac{X}{R_\theta}, \barwedge, \veebar >$ is a fuzzy quotient lattice by fuzzy congruence relation R_θ where $\barwedge$ and $\veebar$ are defined as

$\forall$ a, b $\in$ X,

$[a]^{R_\theta} \barwedge [b]^{R_\theta} = [a \wedge b]^{R_\theta}$

$[a]^{R_\theta} \veebar [b]^{R_\theta} = [a \vee b]^{R_\theta}$

Define, $g: \dfrac{X}{R_\theta} \longrightarrow X_1$ by $g([a]^{R_\theta}) = f(a) \quad \forall a \in X$.

Now, $[a]^{R_\theta} = [b]^{R_\theta}$ where $a, b \in X$

Consider, $a \in [a]^{R_\theta}$ $\Rightarrow a \in [b]^{R_\theta}$

$\Rightarrow a \equiv b(R_\theta)$

$\Rightarrow R_\theta(a, b) = 1$

$\Rightarrow f(a) = f(b)$

$\Rightarrow g([a]^{R_\theta}) = g([b]^{R_\theta})$

Thus, g is well-defined.

To Prove: g is fuzzy lattice homomorphism.

Consider, $g([a]^{R_\theta} \barwedge [b]^{R_\theta}) = g([a \wedge b]^{R_\theta})$ (By definition of $\barwedge$)

$= f(a \wedge b)$ (By definition of g)

$= f(a) \wedge f(b)$

(since f is fuzzy lattice homomorphism.)

$= g([a]^{R_\theta}) \wedge g([b]^{R_\theta})$

Thus, $g([a]^{R_\theta} \barwedge [b]^{R_\theta}) = g([a]^{R_\theta}) \wedge g([b]^{R_\theta})$ _____ (1)

Consider, $g([a]^{R_\theta} \veebar [b]^{R_\theta}) = g([a \vee b]^{R_\theta})$ (By definition of $\veebar$)

$= f(a \vee b)$ (By definition of g)

$= f(a) \vee f(b)$

(since f is fuzzy lattice homomorphism.)

$= g([a]^{R_\theta}) \vee g([b]^{R_\theta})$

Thus, $g([a]^{R_\theta} \veebar [b]^{R_\theta}) = g([a]^{R_\theta}) \vee g([b]^{R_\theta})$ _____ (2)

From (1) and (2) we get, g is a fuzzy lattice homomorphism.

To prove: g is onto.

Let $z \in X_1$

Now, $f : X \rightarrow X_1$ is an onto fuzzy homomorphism.

$\therefore$ There exists $p \in X$ such that $f(p) = z$.

For this p, $g([p]^{R_\theta}) = f(p) = z$. (By definition of g)

Thus there exists $[p]^{R_\theta} \in X / R_\theta$ such that $g([p]^{R_\theta}) = z$.

Thus, g is onto.

To Prove: g is one-one.

Let $g([x]^{R_\theta}) = g([y]^{R_\theta})$ where $x, y \in X$

$\therefore f(x) = f(y)$

$\therefore R_\theta(x, y) = 1$

$\therefore x \equiv y\ (R_\theta)$

$\therefore [x]^{R_\theta} = [y]^{R_\theta}$

Thus, $g([x]^{R_\theta}) = g([y]^{R_\theta}) \Rightarrow [x]^{R_\theta} = [y]^{R_\theta}$ $\forall x, y \in X$.

Thus, g is one-one.

Thus, g is fuzzy lattice homomorphism which is one-one and onto.

Thus g is fuzzy isomorphism.

Hence $X_1 \cong \dfrac{X}{R_\theta}$ for some fuzzy congruence relation R_θ.

Thus, Every fuzzy homomorphic image of a fuzzy lattice X is fuzzy lattice isomorphic with its suitable fuzzy quotient lattice. □

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