CHAPTER - ONE

FUZZY SETS AND FUZZY RELATIONS

CHAPTER 1

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* **Introduction: [1]**

Throughout this work X stands for the universal set.

Let X be classical set of object called the universe, whose generic elements are denoted by x. Membership in a classical subset A of X is often viewed as a characteristic function $f : X \rightarrow \{0, 1\}$ defined by,

$$f(x) = 1, \text{ iff } x \in A$$
$$= 0, \text{ iff } x \notin A$$

Here {0,1} is called as a valuation set.

If the valuation set is allowed to be the real interval [0, 1] then the set A is called a fuzzy set. f(x) is the grade membership of x in A. Thus closer the value of f(x) is to 1, the more x belongs to A. Clearly, Set A is a subset of X that has no sharp boundary.

♦ Preliminary Definitions: -

§1.1 Fuzzy Set: -

Definition: "Let X be the universal set. A Fuzzy set 'A' in X is a function $A : X \rightarrow [0, 1]$."

Set A is completely characterized by the set of pairs $A=\{(x, A(x)) / x \in X\}$. Lofti A. Zadeh proposed a convenient notation. When X is a finite set {x1, x2, x3.... x_n}, a fuzzy set on X is expressed as,

$$A = \frac{A(x_1)}{x_1} + \frac{A(x_2)}{x_2} + \frac{A(x_3)}{x_3} + \dots + \frac{A(x_n)}{x_n}$$

$$A = \sum_{i=1}^{n} \frac{A(x_i)}{x_i}$$

Whereas X is not finite, we write $A = \int_X \frac{A(x)}{x}$

Note: The set of all fuzzy sets in X is denoted by F(X).

§1.2.1 Fuzzy Subsets: [1]

Definition: "Let A and B be the two fuzzy sets in X. A is said to be subset of B, denoted by $A \subseteq B$, if $A(x) \leq B(x) \;\; \forall x \in X$."

§1.2.2 Fuzzy Proper Subsets:

Definition: "Let A and B be the two fuzzy sets in X. A is said to be proper subset of B, denoted by $A \subset B$, if $A(x) < B(x) \;\; \forall x \in X$."

Remark: From definition of fuzzy subset it follows that,

Two fuzzy sets A and B are said to be equal iff

$A(x) = B(x) \qquad \forall x \in X$

§1.3 Union and Intersection of Fuzzy sets: - [1]

Let A and B be two fuzzy sets in X.

Then Fuzzy Union, denoted as A U B, is defined as,

$$A \cup B(x) = \max(A(x), B(x)) \quad \forall x \in X$$

Fuzzy Intersection, denoted as $A \cap B$, is defined as,

$$A \cap B(x) = \min(A(x), B(x)) \quad \forall x \in X$$

Note: Clearly, if A and B are two fuzzy sets in X then $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Also, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

§1.4 Complement of a fuzzy set: - [1]

Definition: "Let A be fuzzy set in X. The complement of A is a fuzzy set $\bar{A}$ defined as, $\bar{A}(x) = 1 - A(x) \quad \forall x \in X$."

§1.5 Fuzzy Relations: - [1]

Definition: "Let $X_1, X_2, \ldots X_n$ be n universes. An n-ary fuzzy relation R in $X_1 \times X_2 \times X_3 \times \ldots \times X_n$ is a fuzzy set R in $X_1 \times X_2 \times X_3 \times \ldots \times X_n$."

An ordinary crisp relation is a particular case of fuzzy relation.

Let us consider an example of fuzzy relation,

Let $X_1 = X_2 = \mathbb{R}^+ - \{0\}$ and

R = "much greater than" defined by a function $R: X_1 \times X_2 \rightarrow [0, 1]$

as,
$$R(x, y) = 0 \quad \text{iff } x \leq y$$
$$= \min(1, (x-y)/9y) \quad \text{iff } x \geq y$$
$$= 1 \quad \text{iff } x \geq 10y$$

Clearly R is a fuzzy relation on $X_1 \times X_2$.

§1.6 Binary Fuzzy Relations: -

Definition: "Let X and Y be two universes.

A function $R : X \times Y \rightarrow [0, 1]$ is called a fuzzy binary relation or fuzzy relation from X to Y."

Its domain is a fuzzy set in X, domain of R, defined as,

$$\text{dom } R(x) = \max_{y \in Y} R(x, y) \quad \forall x \in X.$$

Its range is a fuzzy set in Y, range of R, defined as,

$$\text{ran } R(y) = \max_{x \in X} R(x, y) \quad \forall y \in Y$$

The inverse of a fuzzy relation R(x, y) which is denoted by

$R^{-1}(y, x)$ is a fuzzy relation on Y x X defined by,

$R^{-1}(y, x) = R(x, y) \qquad \forall\, x \in X$ and $\forall\, y \in Y$

A membership matrix $R^{-1}=[r^{-1}_{yx}]$ representing $R^{-1}(Y, X)$ is the transpose of the matrix R for R(X, Y), which means that the rows of R^{-1} equal to the columns and the columns of R^{-1} equal to the rows of R. Clearly $(R^{-1})^{-1} = R$ for any Binary Fuzzy Relation R. Binary Fuzzy Relation on single set is denoted by R(X, X).

§1.7 Fuzzy Reflexive Relations: -

Definition: "A fuzzy relation R on X is said to be **Reflexive**

if $R(x, x) = 1 \qquad \forall\, x \in X$."

§1.8 Fuzzy Symmetric Relations: -

Definition: "A fuzzy relation R on X is said to be **Symmetric**

if $R(x, y) = R(y, x) \qquad \forall\, x \in X$ and $\forall\, y \in X$."

§1.9 Fuzzy Perfect Antisymmetric Relations: -[1]

Two definitions of antisymmetry can be found in the literature. They are,

Perfect Antisymmetry (Zadeh 1971):

"A fuzzy relation R in X is perfectly antisymmetric

if $x \neq y$ and $R(x, y) > 0 \Rightarrow R(y, x) = 0 \qquad \forall\, (x, y) \in X^2$

Antisymmetry (Kaufmann 1975):

"A fuzzy relation R in X is antisymmetric

if $x \neq y$ either $R(x, y) \neq R(y, x)$ or $R(x, y) = R(y, x) = 0$

Note: Perfect antisymmetric implies antisymmetry.

§1.10 Fuzzy Max-Min Transitivity Relations: -

Definition: "A fuzzy relation R on X is said to be **Max-Min Transitive**

$$\text{if } R(x, z) \geq \max_{y \in X} \min\{R(x, y), R(y, z)\} \quad \forall (x, z) \in X^2$$

§1.11 Similarity Relations / Fuzzy Equivalence Relations: - [2]

"A Binary fuzzy relation R which is reflexive, symmetric and max-min transitive is known as a Fuzzy Equivalence Relation or Similarity Relation."

e.g. X={A, B, C, D, E, F, G} and

R(X,X) is given by grade membership matrix as follows:

	A	B	C	D	E	F	G
A	1	0.8	0	0.4	0	0	0
B	0.8	1	0	0.4	0	0	0
C	0	0	1	0	1	0.9	0.5
D	0.4	0.4	0	1	0	0	0
E	0	0	1	. 0	1	0.9	0.5
F	0	0	0.9	0	0.9	1	0.5
G	0	0	0.5	0	0.5	0.5	1

Here R(X,X) is a Similarity Relation on X.

Note: The above grade membership matrix which is also known as triangular matrix can be read as, R(A, D) = 0.4 where A is a row element and D is a column element.