C H A P T E R - I

CHAPTER - I

INTRODUCTION

1.1 NOTATIONS AND DEFINITIONS :

Orthogonal functions :

Suppose we are given a function $w(x)$ which is non-negative in some closed interval [a,b] together with a sequence of functions {fn(x)}, n=0, 1, 2, ... defined in [a,b]. Suppose further that the following relations are satisfied

$$\int_a^b w(x)\, f_m(x) \,.\, f_n(x)dx = 0,\ m \neq n$$

and

$$\int_a^b w(x)\, f_n^2(x)dx = C_n$$

where C_n is, in general a non zero constant (dependent on n). Then the functions $f_n(x)$ are termed orthogonal functions relative to the positive weighting function $w(x)$.

Function of exponential order

Let $F(t)$ be a function of t specified for $t > 0$.
If real constants $M > 0$ and r exist such that for all $t > N$

$$|e^{-rt} F(t)| < M \text{ or } |F(t)| < Me^{rt}$$

we say that $F(t)$ is a function of exponential order r as $t \longrightarrow \infty$ or is of exponential order.

Gamma function

A function $\Gamma(z)$ defined by

$$\frac{1}{\Gamma(z)} = ze^{rz} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n} \right) \exp\left(- \frac{z}{n} \right) \right]$$

in which r is the Euler constant and the product is absolutely convergent for all finite z.

Some formulae

$\Gamma(1) = 1$

$\Gamma'(1) = -r$

$\Gamma(z+1) = Z\Gamma(z)$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(z) = (z-1)!$

Hypergeometric function

A function has expression

$$F(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n \, (b)_n \, z^n}{(C)_n \, n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n \, z^n}{(C)_n \, n!}$$

where c : neither zero nor a negative integer.

$(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + n - 1), \quad n \geq 1$

$(\alpha)_o = 1$

Then a function $F(a,b;c;z)$ is called the hypergeometric function.

Sectional or Piecewise Continuity

Let $F(t)$ be a function of t specified for $t > 0$, is called sectionally continuous or piecewise continuous in an interval $\alpha \le t \le \beta$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

Analytic Function

A function $f(z)$ of the complex variable z is analytic at a point z_o if its derivative $f'(z)$ exists not only at z_o but at every point z in some neighbourhood of z_o.

It is analytic in a domain of the z-plane if it is analytic at every point in that domain.

Laguerre Polynomial

The Laguerre polynomials $L_n(t)$ are defined by,

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n . e^t), \quad n = 0,1,2, \ldots$$

Lagrange's Formula

Lagrange's formula is given by,

$$P(x) = \sum_{i=o}^{n} L_i(x) y_i$$

where $L_i(x)$ is the Lagrange multiplier funcion

$$L_i(x) = \frac{(x-x_o)\ (x-x_1)\ \ldots\ (x-x_{i-1})\ (x-x_{i+1})\ \ldots\ (x-x_n)}{(x_i-x_o)(x_i-x_1)\ldots\ (x_i-x_{i-1})(x_i-x_{i+1})\ldots\ (x_i-x_n)}$$

having the properties,

$$L_i(x_k) = 0 \text{ for } k \neq i,\ L_i(x_i) = 1$$

Lagrange's formula does represent the collocation polynomial

i.e. $P(x_k) = y_k$ for $k = 0, 1, \ldots n$

The function

$$\pi(x) = (x-x_o)\ \ldots\ (x-x_n) = \prod_{i=0}^{n} (x-x_i)$$

1.2 INTEGRAL TRANSFORMS

Using partial differential equation, many of the phenomena of classical physics may be described several physical problem may be discussed with reference to laplace equation, poisson's equation, wave equation, diffusion equation and so on. The field variable is determined not only by the partial differential equation but by the initial values or boundary values assumed by the function.

e.g. The function

$$u(x,y) = e^{i\mu x - |\mu| y}, \quad \mu \varepsilon R$$

satisfies the laplace equation but it is the solution only of the boundary value problem

$$\begin{aligned} \triangle_2 u(x,y) &= 0, \quad -\infty < x < \infty, \; y \geq 0 \\ u(x,y) &= e^{i\mu x}, \quad -\infty < x < \infty \\ u(x,y) &\longrightarrow 0 \text{ as } \sqrt{x^2 + y^2} \longrightarrow \infty \end{aligned}$$

for other forms of u(x,0) we have to construct other solution. The forms of the function F(μ) the function.

$$u(x,y) = \int_{-\infty}^{\infty} F(\mu) e^{i\mu x - |\mu| y} \, d\mu$$

would also satisfy laplace equation in the half plane $y \geq 0$ and hence gives another solution. So there is a question arise that can we find the solution corresponding to the boundary condition $u(x,0) = f(x)$, $-\infty < x < \infty$? This problem can be generalized as follows.

Suppose we have to find the solution of a homogenous partial differential operator of the form.

$$Lu(\bar{r}) = 0 \qquad \ldots. (1.2.1)$$

in a domain D. where $\bar{r}$ is the position vector with components $(x_1, x_2, \ldots x_n)$ of a field point in E_n, the euclidean space of n dimensions and L is a linear differential operator in the variables $x_1, x_2 \ldots x_n$.

If $L_n(\bar{r};\mu) = 0, \quad \bar{r}\varepsilon D, \ \mu\varepsilon\Omega$

then since the operator L is linear, it is possible that the function,

$$u(\bar{r}) = \int_\Omega F(\mu) n(\bar{r};\mu)\, d\mu, \ \bar{r}\varepsilon D$$

will also be a solution of (1.2.1) for any arbitrary function F of μ, if the integral converges uniformly for all $\bar{r}\varepsilon D$, where $n(\bar{r};\mu)$ is a simple solution of (1.2.1). The boundary conditions imposed in the problem are sufficient to determine the form of function $F(\mu)$.

To illustrate the situation which may arise we consider the solution of the form

$$u(x,y) = \int_\Omega F(\mu) n(x,y,\mu)\, d\mu, \ (x,y)\varepsilon D$$

If $u(x,0) = f(x)$, we have the relation

$$f(x) = \int_\Omega F(\mu) \, . \, k(x;\mu)\, d\mu, \ x\varepsilon D_1 \qquad \ldots. \ (1.2.2)$$

where D_1 is the domain of x and $k(x,\mu) = n(x,o;\mu)$.

When a function f is defined in terms of a function F as in (1.2.2) we say that f(x) is the integral transform of the function $F(\mu)$ by the kernel $k(x,\mu)$.

In boundary value problem discussion for partial differential equations the basic problem comes to be the determination of the function $F(\mu)$ when the function $f(x)$ is given. In many cases we can find a solution of an integral equation of this type in the form

$$F(\mu) = \int_{D_1} f(x) \cdot H(x;\mu)\, dx, \quad \mu\varepsilon\Omega$$

where D_1 is the domain of x. Such a result is called an inversion formula for the transform (1.2.2).

If we take D_1 to be the positive real line and kernel to be $e^{-x1\mu}$ we get the Laplace transform defined by the equation.

$$\bar{u}(s,y) = \int_0^\infty u(x,y)e^{-sx}\, dx$$

is called the Laplace transform with respect to x of the function u(x,y) and

$$\bar{u}(x,s) = \int_0^\infty u(x,y)e^{-sy}\, dy$$

its Laplace transform with respect to y.

Also if we take D_1 to be positive real line and kernel x_1^{s-1} we get the Mellin transform

$$u^*(s,y) = \int_0^\infty u(x,y)x^{s-1}\, dx$$

is the Mellin transform with respect to x of the function $u(x,y)$.

1.3 PROCEDURE

In briefly the process in solving a differential equation with boundary and initial conditions by the use of any integral transform as follows.

i. select the suitable transform.

ii. Multiply the differential equation and boundary conditions by the selected kernel and integrate between limits with respect to the selected variable for exclusion.

iii. While performing the integration in (1.2.2) make use of the suitable initial or boundary conditions.

iv. To obtain the transform of the required function, solve the resulting auxiliary equation.

v. Lastly invert to obtain the wanted function itself.

1.4 OCCURRENCE OF LAPLACE TRANSFORM

Laplace transforms arise in a natural fashion by considering the simple problem of determining the solution $u(x,t)$ of the one-dimensional diffusion equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t} \qquad \text{.... (1.4.1)}$$

in the region $t > 0$, $x > 0$ subject to the initial condition

$$u(x,0) = 0 \qquad \ldots. \ (1.4.2)$$

and the boundary conditions

$$u(0,t) = F(t), \qquad \ldots. \ (1.4.3)$$

$$u(x,t) \longrightarrow 0 \text{ as } x \longrightarrow \infty \qquad \ldots. \ (1.4.4)$$

Also assume that u(x,t) remains finite as t ⟶ ∞. We multiply both sides of (1.4.1) by $\bar{e}^{s+}$ and integrating with respect to t from 0 to ∞.

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} \cdot \bar{e}^{st}\, dt = \frac{1}{k} \int_0^\infty \frac{\partial u}{\partial t} \cdot \bar{e}^{st}\, dt$$

$$= \frac{1}{k} \{ u(x,t)\ \bar{e}^{s+} \}_0^\infty + \frac{s}{k} \int_0^\infty u(x,t) \bar{e}^{s+} dt$$

$$= \frac{1}{k} \{ u(x,t)\ \bar{e}^{s+} \}_0^\infty + \frac{s}{k}\, \bar{u}(x,s)$$

where $\bar{u}(x,s) = \int_0^\infty u(x,t) e^{-st} dt \qquad \ldots. \ (1.4.5)$

using initial condition (1.4.2) and u remains finite as t⟶∞

we have

$$\frac{1}{k} \int_0^\infty \frac{\partial u}{\partial t} \cdot \bar{e}^{st}\, dt = \frac{s}{k}\, \bar{u}(x,s)$$

so that,

$$\frac{\partial^2 \bar{u}}{\partial x^2} = \frac{s}{k} \bar{u}(x,s) \qquad \text{.... (1.4.6)}$$

Also from equation (1.4.3)

$$\bar{u}(o,s) = \int_0^\infty u(o,t)e^{-s+}dt$$

$$= \int_0^\infty F(t)e^{-st}dt$$

$$\bar{u}(o,s) = \bar{f}(s)$$

we observed that $\bar{f}(s)$ is the Laplace transform of the prescribed function F(t).

1.5 OCCURRENCE OF MELLIN TRANSFORM

The occurrence of Mellin transforms can be demonstrated by considering the simple problem of determining the solutions $u(\varrho, \Phi)$ of the two dimensional Laplace equation appropriate to the infinite wedge $\varrho > 0$, $|\Phi| < \alpha$, where ϱ and Φ are plane polar co-ordinates. Using these co-ordinates, Laplace's equation can be written.

$$\frac{\partial^2 u}{\partial \Phi^2} + \frac{1}{\varrho}\frac{\partial u}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2 u}{\partial \Phi^2} = 0 \qquad \text{.... (1.5.1)}$$

By formula for integrating by parts, we have

$$\int_0^\infty \rho^{s+1} \left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}\right) d\rho = \int_0^\infty \rho^s \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) d\rho$$

$$= \left[\rho^{s+1} \frac{\partial u}{\partial \rho}\right]_0^\infty - s \int_0^\infty \rho^{s-1} \cdot \rho \frac{\partial u}{\partial \rho} d\rho$$

$$= \left[\rho^{s+1} \frac{\partial u}{\partial \rho}\right]_0^\infty - s \int_0^\infty \rho^{s} \cdot \frac{\partial u}{\partial \rho} d\rho$$

$$= \left[\rho^{s+1} \frac{\partial u}{\partial \rho}\right]_0^\infty [s\ \rho^s u(\rho,\Phi)]_0^\infty + s^2 \int_0^\infty u(\rho,\Phi)\ \rho^{s-1} d\rho$$

If the functions

$$\rho^{s+1} \frac{\partial u}{\partial \rho}, \quad \rho^s u(\rho, \Phi)$$

tends to zero as $\rho \longrightarrow 0$ and as $\rho \longrightarrow \infty$ we find that,

$$\int_0^\infty \rho^{s+1}\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}\right) d\rho = s^2 \int_0^\infty \rho^{s-1} u(\rho, \Phi) d\rho \quad \ldots. (1.5.2)$$

$$= s^2 \ u^*(s, \Phi)$$

where $u^*(s, \Phi) = \int_0^\infty \rho^{s-1} \cdot u(\rho, \Phi) d\rho \quad \ldots. (1.5.3)$

we see that $u^*(s, \Phi)$ is the Mellin transform of the function $u(\rho, \Phi)$ with respect to the variable ρ.

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