Chapter II

CHAPTER - II

Hankel Type transform of Distributions

2.1 Introduction:

The Hankel type transformation of a function $f(x)$ is defined by

$$F(y) = h_{\lambda} f = \int_0^{\infty} \left(\frac{y}{x}\right)^{\lambda/2} J_{\lambda}\left(2\sqrt{xy}\right) f(x) dx \text{ ------- } (2.1.1).$$

where $J_{\lambda}(x)$ is the Bessel function of first kind and of order λ.

The inversion formula for this is given by

$$f(x) = \int_0^{\infty} \left(\frac{x}{y}\right)^{\lambda/2} J_{\lambda}\left(2\sqrt{xy}\right) F(y) dy \text{ ------------ } (2.1.2)$$

In this chapter we have studied in details the extension of this Hankel type transform to a class of distributions and use it to solve a distributional integral equation.

For a real number λ and a positive integer a. we construct a testing function space $H_{a.\lambda}$, which contain the kernel $\left(\frac{y}{x}\right)^{\lambda/2} J_{\lambda}\left(2\sqrt{xy}\right)$ as a function on $0 < x < \infty$ for each fixed y.

The Hankel type transform $F(y)$ of a distribution f in the dual space $H'_{a.\lambda}$ is defined by

$$F(y) = h'_{\lambda} f = \left\langle f(x), \left(\frac{y}{x}\right)^{\lambda/2} J_{\lambda}\left(2\sqrt{xy}\right) \right\rangle \text{ for suitably restricted y.}$$

In section 2.2, the spaces $H_{a.\lambda}$ and $H'_{a.\lambda}$ are developed. In sec.2.3 and 2.4, we discuss the distributional Hankel type transformation and its inversion.

Sec.2.5 is devoted to an application to the solution of distributional integral equation.

$$f + kh'_{\lambda} f = g$$

where g is known distribution in certain space

For any real number a, let $L_{+,a}$ be the space of all smooth functions $\phi(t)$ on I such that $B_{a,k}(\phi) = \sup_{0<t<\infty}\left|e^{at}\frac{d^k}{dx^k}\phi(t)\right| < \infty$, k=0, 1, 2, 3-------

$L_{+,a}$ is a testing functions space such that $e^{-st} \in L_{+,a}$ if $\operatorname{Re} s > a$ [26, p.90].

$L'_{+,a}$ denotes the dual space of $L_{+,a}$.

If $f \in L'_{+,a}$, the Laplace transform of f is defined by

$$F(s) = < f(t), e^{-st} > \quad (\operatorname{Re} s > a\ )\text{-----------}(2.1.3)$$

If f is a locally integrable function on I and if, then f generates a regular generalized function in $L'_{+,a}$ through the definition,

$$< f, \phi > = \int_0^{\infty} f(t)\phi(t)dt\,,\ \phi \in L'_{+,a}.$$

2.2 The Testing Function Space $H_{a.\lambda}$ and its dual $H'_{a.\lambda}$:

For $a > 0$ and λ be any real number. Let $H_{a.\lambda}$ be the collection of all infinite differentiable functions $\phi(x)$ defined on I, such that for every nonnegative integer k,

$$T_k(\phi) = T_k^{\lambda,a}(\phi) = \sup_{x\in I}\left|e^{-ax}\Delta^k_{\lambda,x}\phi(x)\right| < \infty \quad \text{------------}(2.2.1)$$

$$\Delta^k_{\lambda,x} = \left[Dx^{-\lambda+1}Dx^{\lambda}\right]^k,\ \mathrm{D} = \frac{d}{dx}$$

$H_{a.\lambda}$ is a linear space over the field of complex numbers. We assign to it the topology generated by the separating countable collection $\{T_k\}_{k=0}^{\infty}$ of seminorms. Hence $H_{a.\lambda}$ is Hausdorff locally convex topological vector space that satisfies the first axiom of countability. The dual space $H'_{a.\lambda}$ consists of all continuous linear functionals on $H_{a.\lambda}$.

The dual is linear space to which weak topology generated by the multinorm $\{\xi_\phi\}_\phi$, where

$$\xi_\phi(f) = |< f, \phi >| \text{ and } \phi \text{ varies through } H_{a.\lambda}.$$

Lemma 2.2.1: $H_{a,\lambda}$ is complete and therefore a Frechet space

Proof: Let $\{\phi_m\}_{m=1}^{\infty}$ be a Cauchy sequences in $H_{a,\lambda}$. Then by equation (2.2.1) we have a uniform Cauchy sequence $\{\psi_m\}_{m=1}^{\infty}$ on I.

For each k,

$$\psi_m(x) = e^{-ax}\Delta_{\lambda,x}^{k}\phi_m(x) \quad \text{--------(2.2.2)}$$

By Cauchy write $\{\psi_m\}_{m=1}^{\infty}$ converges uniformly to $\{\psi\}_{m=1}^{\infty}$ on I for all k.

Hence by Theorem of mathematical analysis, there is a smooth function $\psi(x)$ on I such that $\psi_m(x) \to \psi(x)$ uniformly on I, where $\psi(x) = e^{-ax}\Delta_{\lambda,x}^{k}\phi(x)$.

since $\psi_m(x)$ is a uniformly Cauchy sequence then for each $\in > 0$, there is an integer N such that

$$\sup_{0<x<\infty} |\psi_m(x) - \psi_n(x)| < \in \quad \forall m, n > N.$$

Taking the limit as $n \to \infty$.

We get,

$$\sup_{0<x<\infty} |\psi_m(x)-\psi(x)| \leq \in \quad \text{-------}(2.2.3)$$

Thus for each k, $m > N$,

$$T_k^{\lambda,a}(\phi_m - \phi) \to 0 \text{ as } m \to \infty.$$

Finally because of the uniformly of the convergence and the fact that each $\psi_m(x)$ is bounded on I, there exist a constant C not depending on m such that

$$|\psi_m(x)| < C \qquad \forall x$$

Then (2.2.3), we get

$$\sup_{0<x<\infty} |\psi(x)| \leq M \text{ where } M \text{ is constant.}$$

which shows that $\psi(x)$ is bounded on I.

Hence a function $\phi(x)$ which is the limit of a given sequence $\{\phi_m\}$ is a member of $H_{a,\lambda}$. Thus the sequence $\{\phi_n\}$ converges in $H_{a,\lambda}$ to the unique limit ϕ.

Hence $H_{a,\lambda}$ is a countably multinormed space which is complete.

Therefore $H_{a,\lambda}$ is a Frechet space.

Lemma 2.2.2: $H_{a,\lambda}$ is Testing function space.

Proof: Clearly, $H_{a,\lambda}$ satisfies the first two conditions of testing functions space.

We shall prove the third condition.

Let $\{\phi_m\}$ converges in $H_{a,\lambda}$ to zero.

In view of equation (2.2.2) and the seminorm defined by (2.2.1) it follows that by an induction on k, that for each k, $\{D_x^k \phi_m\}$ converges uniformly to zero function. Therefore $H_{a,\lambda}$ is Testing function space.

We now list some properties of $H_{a,\lambda}$ spaces:

i). $H_{a,\lambda}$ is sequentially complete space hence $H'_{a,\lambda}$ is also complete [26].

ii). Let $\lambda \geq -\frac{1}{2}$, for fixed complex number y belonging to the strip Ω

$$\Omega = \{ y : \left|\operatorname{Im}\sqrt{y}\right| < \frac{a}{2} , y \neq 0 \text{ or } \text{ a negative number}\}$$

then $\left(\frac{y}{x}\right)^{\lambda/2} J_\lambda\left(2\sqrt{xy}\right) \in H_{a,\lambda}$ --------------(2.2.4).

Indeed, by analyticity of $z^{-\lambda}J_\lambda(z), z \neq 0$ it follows that $\left(\frac{y}{x}\right)^{\lambda/2} J_\lambda\left(2\sqrt{xy}\right)$ is smooth on $o < x < \infty$. Also in view of the property

$$\Delta^k_{\lambda,x}\left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})\right] = (-1)^k y^k \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \text{ ----------(2.2.5)}$$

and the fact $\left|e^{-ax}(2\sqrt{xy})^{-\lambda} J_\lambda(2\sqrt{xy})\right|$ is bounded for $0 < x < \infty$, $y \in \Omega$ [9], the quantities

$$T_k^{\lambda,a}\left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})\right] \text{ are finite for all k=0, 1, 2, 3----}$$

Therefore $\left(\frac{y}{x}\right)^{\lambda/2} J_\lambda\left(2\sqrt{xy}\right) \in H_{a,\lambda}$

iii. Let $0 < b < a$, Then $H_{b,\lambda} \subset H_{a,\lambda}$ and the topology of $H_{b,\lambda}$ is stronger than the topology induced on it by $H_{a,\lambda}$. This follows from the equality

$T_k^{\lambda,a}(\phi) \leq T_k^{\lambda,b}(\phi)$ for $\phi \in H_{a,\lambda}$.

Let $0 < b < a$

Let $0 < e^{-ax} < e^{-bx}$ on I.

Then $\left|e^{-ax}\Delta_{\lambda,x}^{k}\phi(x)\right| \le \left|e^{-bx}\Delta_{\lambda,x}^{k}\phi(x)\right|$

So that $T_k^{\lambda,a}(\phi) \le T_k^{\lambda,b}(\phi)$ for $\phi \in H_{a,\lambda}$.

Hence the restriction of $f \in H'_{a,\lambda}$ to $H_{b,\lambda}$ is in $H'_{b,\lambda}$.

iv. $D(I) \subset H_{a,\lambda}$, and the topology of $D(I)$ is stronger than that induced on it by $H_{a,\lambda}$. Hence, the restriction of $f \in H'_{a,\lambda}$ to $D(I)$ is in $D'(I)$. Thus members of $H'_{a.\lambda}$ are distributions in Zemanian's sense [26].

v. Let $f(x)$ be locally integrable function on $0 < x < \infty$ and such that $\int_0^\infty e^{-ax}\left|f(x)\right|dx < \infty$, then f generates a regular generalized function in $H'_{a.\lambda}$ defined by

$$< f,\phi >= \int_0^\infty f(x)\phi(x)dx \text{ ---------(2.2.6)}$$

Let $< f,\phi >= \int_0^\infty f(x)\phi(x)dx$.

$$\left|< f,\phi >\right| = \left|\int_0^\infty \frac{f(x)}{e^{-ax}} e^{-ax}\phi(x)dx\right|$$

$$\le T_0^{\lambda,a}\phi(x)\int_0^\infty \left|\frac{f(x)}{e^{-ax}}\right|dx$$

which shows that (2.2.6) truly defines a functional f on $H_{a,\lambda}$.

This functional is clearly a linear one.

Moreover, if $\{\phi_m\}_{m=1}^{\infty}$ converges in $H_{a,\lambda}$ to zero, then $T_o^{\lambda,a}(\phi_m) \to 0$ so that $\left|< f,\phi_m >\right| \to 0$.

Thus f is also continuous on $H_{a,\lambda}$.

Hence f generates a regular generalized function in $H'_{a,\lambda}$.

2.3 The distributional Hankel Type Transformation:

Let $-\frac{1}{2} \leq \lambda < \infty$, $a > 0$. In view of note iii (sec. 2.2), for every $f \in H'_{a,\lambda}$ there exists a unique real $\sigma_f > 0$ (possibly $\sigma_f = \infty$) such that $f \in H'_{b,\lambda}$ if $b < \sigma_f$ and $f \notin H'_{b,\lambda}$ if $b > \sigma_f$.

For $f \in H'_{a,\lambda}$ and $\lambda \geq -\frac{1}{2}$, We define the λ^{th} order Hankel type transform $h'_\lambda f$ of f as the application of f to the kernel $\left(\frac{y}{x}\right)^{\lambda/2} J_\lambda\left(2\sqrt{xy}\right)$.

i.e. $F(y) = h'_\lambda f = < f(x), \left(\frac{y}{x}\right)^{\lambda/2} J_\lambda\left(2\sqrt{xy}\right) >$--------- (2.3.1)

Where $y \in \Omega_f = \{y : \left|\mathrm{Im}\sqrt{y}\right| < \frac{\sigma_f}{2}$, $y \neq 0$ or a negative number$\}$.

The right hand side (2.3.1) has a sense because, by note ii, $\left(\frac{y}{x}\right)^{\lambda/2} J_\lambda\left(2\sqrt{xy}\right) \in H_{b,\lambda}$ for every $b < \sigma_f$ and $y \in \Omega_f$.

If $f(x)$ satisfies the conditions of note v. sec (2.2) for every $a < \sigma_f$, then we may write

$$F(y) = (h_\lambda f)(y) = \int_0^\infty \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) f(x) dx, \; y \in \Omega_f .$$

Theorem 2.3.1: For $f \in H'_{a,\lambda}$ and $\lambda \geq -\frac{1}{2}$, Let $F(y)$ be defined by

$$F(y) = h'_\lambda f(y) = < f(x), \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda\left(2\sqrt{xy}\right) >$$

Then $F(y)$ is analytic function of y on Ω_f and

$$DF(y) = < f(x), D[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})] > \text{ where } D = \frac{d}{dy}.$$

Proof: Let y be an arbitrary but fixed point in Ω_f.

$$\Omega_f = \left\{ y : \left|\operatorname{Im}\sqrt{y}\right| < \frac{\sigma_f}{2}, y \neq 0 \text{ Or a negative number } \right\}$$

Let C denote the circle whose centre is at y and whose radius is r_1. Restrict r_1 still further by requiring that C lie entirely with in Ω_f.

Finally, let Δy be a nonzero complex increment such that

$|\Delta y| < r$ and $\left|\operatorname{Im}\sqrt{y+\Delta y}\right| < \frac{\sigma_f}{2}$.

Consider $\frac{F(y+\Delta y) - F(y)}{\Delta y} - < f(x), \frac{d}{dy}\left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})\right] > = < f(x), \psi_{\Delta y}(x) >$

---------(2.3.2)

$$\psi_{\Delta y} = \frac{1}{\Delta y}\left\{\left(\frac{y+\Delta y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{x(y+\Delta y)}) - \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})\right\} - \frac{d}{dy}\left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})\right]$$

Our theorem will be proven when we show that (2.3.2) converges to zero as $|\Delta y| \to 0$.

This can be done by showing that $\psi_{\Delta y}(x)$ converges in $H_{a,\lambda}$ to zero as $|\Delta y| \to 0$,

Using the fact that

$$\Delta_{\lambda,x}^{k}\left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy})\right] = (-1)^{k} y^{k}\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy}) \text{ -----(2.3.3)}$$

and

$\Delta_{\lambda,x}^{k}\psi_{\Delta y}(x)$ can be written as a closed interval on C by using Cauchy's integral formulas.

This gives

$$\Delta_{\lambda,x}^{k}\psi_{\Delta y}(x) = \frac{1}{2\pi i}\int_{C}(-1)^{k}\xi^{k}\left(\frac{\xi}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy})\left[\frac{1}{\Delta y}\left(\frac{1}{\xi-y-\Delta y}-\frac{1}{\xi-y}\right)-\frac{1}{(\xi-y)^{2}}\right]d\xi$$

$$= \frac{\Delta y}{2\pi i}\int_{C}\frac{(-1)\xi^{k}\left(\frac{\xi}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{\xi x})}{(\xi-y)^{2}(\xi-y-\Delta y)}d\xi$$

Next for all $\xi \in C$ and $0 < x < \infty$,

We may write $$\left|e^{-ax}\Delta_{\lambda,x}^{k}\psi_{\Delta y}(x)\right| = \left|e^{-ax}\frac{\Delta y}{2\pi i}\int_{C}\frac{(-1)^{k}\xi^{k}\left(\frac{\xi}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy})}{(\xi-y)^{2}(\xi-y-\Delta y)}d\xi\right|$$

$$\leq \frac{|\Delta y|}{2\pi}\int_{C}\left|\frac{\xi^{k}\left(\frac{\xi}{x}\right)^{\frac{\lambda}{2}} e^{-ax}J_{\lambda}(2\sqrt{\xi x})}{(\xi-y)^{2}(\xi-y-\Delta y)}\right|d\xi$$

$$\leq \frac{|\Delta y|}{2\pi}\int_{C}\left|\frac{\xi^{k+\lambda}2^{\lambda}e^{-ax}(2\sqrt{\xi x})^{-\lambda}J_{\lambda}(2\sqrt{\xi x})}{(\xi-y)^{2}(\xi-y-\Delta y)}\right|d\xi$$

$$< \frac{|\Delta y| A_\lambda}{r_1^2 (r_1 - r)} \sup_{\xi \in C} |\xi^{k+\lambda}|$$

Here A_λ be a constant bounded on $e^{-ax}(2\sqrt{\xi x})^{-1} J_\lambda(2\sqrt{\xi x})$ for $0 < x < \infty$ and

Moreover, $|\xi - y| = r_1$ and $|\xi - y - \Delta y| > r_1 - r > 0$.

Thus $e^{-ax} \Delta^k_{\lambda,x} \psi_{\Delta y}(x)| \to 0$ as $|\Delta y| \to 0$.

This proves that $\psi_{\Delta y}(x)$ converges in $H_{a,\lambda}$ to zero as $|\Delta y| \to 0$.

Consequently, $< f(x), \psi_{\Delta y}(x) > \to 0$ as $|\Delta y| \to 0$.

Using equation (2.3.2), we say that $F(y)$ is analytic.

Theorem2.3.2 Let $F(y)$ be the distributional Hankel type transform of $f \in H'_{a,\lambda}$ as defined by (2.3.1) then, $F(y)$ satisfies the inequality

$$|F(y)| \le \begin{cases} ky^\lambda & 0 < y < 1 \\ ky^p & 1 < y < \infty \end{cases}$$

where p is a sufficiently large real number and k is chosen appropriately.

Proof: In view of a general result [26, Theorem 1.8.1], there exists a constant $c > 0$ and non negative integer r such that

$$|F(y)| \le C \max_{0 \le k \le r} \sup_{x \in I} \left| e^{-ax} \Delta^k_{\lambda,x} \left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \right] \right|$$

By (2.2.3), the right hand side is equal to

$$= \max_{0 \le k \le r} \sup_{x \in I} \left| e^{-ax} y^k \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \right|$$

$$= \max_{0\le k\le r} \ Sup_{x\in I} \left| e^{-ax} 2^{\lambda} y^{k+\lambda} \frac{J_\lambda(2\sqrt{xy})}{\left(2\sqrt{xy}\right)^{\lambda}} \right|$$

Since for all $y > 0$,

$$\left| e^{-ax} \frac{J_\lambda(2\sqrt{xy})}{(2\sqrt{xy})^{\lambda}} \right| < A_\lambda \quad [9]$$

where A_λ is constant with respect to x and y, the theorem follows.

Inversion Theorem for the Distributional Hankel type transform

This section is devoted to prove an inversion formula for distributional Hankel type transformation. This inversion formula determines the restrictions to $D(I)$ of any h_λ transformable generalized function from its Hankel type transform. Form this we will obtain an incomplete version of a uniqueness theorem, which states that two h_λ transformable generalized functions having the same transform must have the same restriction to $D(I)$.

Theorem 2.4 Let $f \in H'_{a,\lambda}$ and Let $F(y)$ be the distributional Hankel type transform of f defined by

$$F(y) = (h'_\lambda f)(y) = < f(x), \left(\frac{y}{x}\right)^{\lambda/2} J_\lambda\left(2\sqrt{xy}\right) >.$$

Let $\lambda \ge -\frac{1}{2}$.

Then for each $\phi \in D(I)$,

$$< \int_0^R F(y) \left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) dy, \phi(x) > \to < f, \phi > \text{ as } R \to \infty.$$

i.e. $f(x)=\lim_{R\to\infty}\int_0^R F(y)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dy$ --------------(2.4.1) .

Proof: Let $\phi\in D(I)$.

Choose a real number a such that $0<a<\sigma_f$.

Since the integral in (2.4.1) is a continuous function of x, it generates a regular distribution in $D(I)$. Hence, we have

$$<\int_0^R F(y)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dy,\phi(x)>=\int_0^\infty \phi(x)\int_0^R F(y)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dydx$$

Since from smoothness of $F(y)$ and ϕ is of bounded support and the integrand on the right hand side is a continuous function of x and y, we can change the order of integration and obtain

$$<\int_0^R F(y)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dy,\phi(x)>=\int_0^R F(y)\int_0^\infty \phi(x)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dxdy$$

$$=\int_0^R < f(t),\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})>\int_0^\infty \phi(x)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dxdy \text{ ---(2.4.2)}$$

Now, Let $\Phi(y)=\int_0^\infty \phi(x)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dx$ and $M_R(t)=\int_0^R \Phi(y)\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})dy$.

Since $\left|e^{-at}\Delta_{\lambda,t}^k M_R(t)\right|=\left|e^{-at}\Delta_{\lambda,t}^k\int_0^R \Phi(y)\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})dy\right|$

$$=\left|\int_0^R y^k e^{-at}\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})\Phi(y)dy\right|$$

$$\le B\int_0^R |\Phi(y)|y^k dy$$

For some suitable constant, $M_R(t) \in H_{a,\lambda}$ for each $R > 0$.

Using the Riemann sum technique, (2.4.2) can be rewritten as

$$< \int_0^R F(y)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dy, \phi(x) >= \int_0^R < f(t), \left(\frac{y}{t}\right)^{\frac{\lambda}{t}} J_\lambda(2\sqrt{yt}) > \Phi(y)dy$$

$$= \int_0^R\int_0^\infty f(t)\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})dt\Phi(y)dy$$

By change the order

$$= \int_0^\infty f(t)\int_0^R\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})\Phi(y)dydt$$

$$= \int_0^\infty f(t)M_R(t)dt$$

$$=< f(t), M_R(t) > \ , \ R > 0.$$

This has sense because $M_R(t) \in H_{a,\lambda}$.

Hence now proof of the theorem will be complete, if we show that $M_R(t) \to \phi(t)$ in $H_{a,\lambda}$ as $R \to \infty$.

Since $\Phi(y)\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})$ is smooth and $\phi \in D(I)$, we may repeatedly differentiate under the integral sign and use the equation (2.2.3) to write

$$\Delta_{\lambda,t}^k M_R(t) = \int_0^R \Phi(y)\Delta_{\lambda,t}^k[\left(\frac{y}{t}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{yt})]dy$$

$$= \int_0^R (-1)^k y^k \left(\frac{y}{t}\right)^{\frac{\lambda}{t}} J_\lambda(2\sqrt{yt})\int_0^\infty \phi(x)\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})dxdy$$

$$= \int_0^R (-1)^k y^k \left(\frac{y}{t}\right)^{\frac{\lambda}{t}} J_\lambda(2\sqrt{yt}) \int_0^\infty \phi(x) \left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) dx dy$$

$$= \int_0^R y^{-\frac{\lambda}{2}} t^{-\frac{\lambda}{2}} J_\lambda(2\sqrt{yt}) \int_0^\infty x^\lambda \phi(x) (-1)^k y^k \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) dx dy$$

$$= \int_0^R y^{-\frac{\lambda}{2}} t^{-\frac{\lambda}{2}} J_\lambda(2\sqrt{yt}) \int_0^\infty x^\lambda \phi(x) \Delta^k_{\lambda,x} \left[\left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \right] dx dy$$

$$= \int_0^R t^{-\frac{\lambda}{2}} J_\lambda(2\sqrt{yt}) \int_0^\infty x^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \Delta^k_{\lambda,x}[\phi(x)] dx dy$$

The last equality is obtained by integrating by parts the inner integral $2k$ times and nothing that the limit terms are always equal to zero.

Now, reversing the order of integration and using the formula

$$L_R(x,t) = x^{\frac{\lambda}{2}} t^{-\frac{\lambda}{2}} \int_0^R J_\lambda(2\sqrt{xy}) J_\lambda(2\sqrt{yt}) dy \quad \text{--------------(2.4.3)}$$

$$= \frac{\sqrt{R} x^{\frac{\lambda}{2}} t^{-\frac{\lambda}{2}}}{\sqrt{x} - \sqrt{t}} \left\{ \sqrt{x} J_{\lambda+1}(2\sqrt{xr}) J_\lambda(2\sqrt{tr}) - \sqrt{t} J_\lambda(2\sqrt{xr}) J_{\lambda+1}(2\sqrt{tr}) \right\}$$

We obtain,

$$\Delta^k_{\lambda,t} M_R(t) = \int_0^\infty L_R(x,t) \Delta^k_{\lambda,x}[\phi(x)] dx \quad \text{--------------(2.4.4)}$$

Denote $\Delta^k_{\lambda,x}[\phi(x)]$ by $\phi_k(x)$.

Now suppose that the support of $\phi(x)$ is contained in $[A,B]$, where $0 < A < B < \infty$. Let $0 < \delta < A$. Let us break the integral in (2.4.4) into

$$\Delta^k_{\lambda,t} M_R(t) = V_1(t) + V_2(t) + V_3(t)$$

where $V_1(t) = \int_0^{t-\delta} L_R(x,t) \Delta^k_{\lambda,x}[\phi(x)] dx$

$$V_3(t) = \int_{t+\delta}^{\infty} L_R(x,t)\Delta_{\lambda,x}^{k}[\phi(x)]dx$$

We shall first show that $N_R(t) = e^{-at}[V_2(t) - \Delta_{\lambda,x}^{k}\phi(t)]$ converges uniformly to zero on $0 < t < \infty$ as $R \to \infty$. If $0 < t + \delta < A$ or $t - \delta > B$, then $V_2(t) = 0$ and $\Delta_{\lambda,t}^{k}\phi(t) = 0$.

Therefore, we need merely consider the interval $A - \delta < t < B + \delta$. Moreover, since the support of ϕ is an $[A,B]$, we can take the integral in (2.4.4) on (A,B). Using the asymptotic expansion of $J_\lambda(z)$ [8], we have for large R,

$$N_R(t) = \frac{1}{\pi}e^{-at}\int_{t-\delta}^{t+\delta} \phi_k(x)x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}t^{\left(-\frac{\lambda}{2}-\frac{1}{4}\right)}\frac{\sin(2\sqrt{xR}-2\sqrt{tR})}{\sqrt{x}-\sqrt{t}}dx$$

$$-\frac{1}{\pi}e^{-at}\int_{t-\delta}^{t+\delta} \phi_k(x)x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}t^{\left(-\frac{\lambda}{2}-\frac{1}{4}\right)}\frac{\cos(2\sqrt{xR}+2\sqrt{tR}-\lambda\pi)}{\sqrt{x}+\sqrt{t}}dx$$

$$-e^{-at}\Delta_{\lambda,t}^{k}\phi(t) \quad \text{---------(2.4.5)}$$

First consider the middle term. For all $R > 1$, the integrand is bounded on

$$\{(x,t)|\ A < x < B\ ,\ A - \delta < t < B - \delta\ \}$$

by a constant independent of R. Therefore, given $\in > 0$, we can choose δ so small, say $\delta = \delta_1$, that the magnitude of the middle term can be made less then $\frac{\in}{2}$ for all $R > 1$.

Now consider the sum of the first and last term in (2.4.5). This sum can be written as

$$\frac{1}{\pi}\int_{\sqrt{t-\delta}-\sqrt{t}}^{\sqrt{t+\delta}-\sqrt{t}} H(T,t)\sin(2\sqrt{R}T)dT + e^{-at}\Delta_{\lambda,t}^{k}\phi(t)\left[\frac{1}{\pi}\int_{-2\sqrt{R}(\sqrt{t}-\sqrt{t-\delta})}^{2\sqrt{R}(\sqrt{t+\delta}-\sqrt{t}}\frac{\sin y}{y}dy - 1\right] \text{-------(2.4.6)}$$

where $H(T,t) = \dfrac{e^{-at}2\phi_k[(T+\sqrt{t})^2(T+\sqrt{t})^{\lambda+\frac{1}{2}}t^{-\frac{\lambda}{2}-\frac{1}{4}} - \Delta_{\lambda,t}^{k}\phi(t)}{T}$

Since $H(T,t)$ is a continuous of (T,t) and supp $\phi(x)\subset[A,B]$, $H(T,t)$ is a bounded function of T on $\sqrt{t-\delta}-\sqrt{t} < T < \sqrt{t+\delta}-\sqrt{t}$ for all $0<t<\infty$. Hence choosing δ very small, say $\delta=\delta_2$, the first term in (2.4.6) can be made less then $\frac{\in}{2}$ for all $R>1$. Now, fix $\delta=\min(\delta_1,\delta_2)$.

The second term in (2.4.6) converges uniformly to zero on $0<t<\infty$ as $R\to\infty$.

Thus $|N_{f_1}(t)|\le\in$ on $0<t<\infty$.

Since $\in>0$ is arbitrary, $N_R(t)\to 0$ uniformly on $0<t<\infty$ as $R\to\infty$.

Now consider $P_R(t) = e^{-at}V_1(t) = e^{-at}\int_0^{t-\delta} L_R(x,t)\phi_k(x)dx$

For $t-\delta\le A$, $P_R(t)\equiv 0$

Now consider the range $t-\delta>A$ and using the asymptotic expressions as $R\to\infty$.

We obtain

$$P_R(t) = \frac{1}{\pi}e^{-at}\int_A^{\min(B,t-\delta)} \phi_k(x)x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}t^{\left(-\frac{\lambda}{2}-\frac{1}{4}\right)}\frac{\sin(2\sqrt{xR}-2\sqrt{tR})}{\sqrt{x}-\sqrt{t}}dx$$

$$-\frac{1}{\pi}e^{-at}\int_A^{\min(B,t-\delta)} \phi_k(x)x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}t^{\left(-\frac{\lambda}{2}-\frac{1}{4}\right)}\frac{\cos(2\sqrt{xR}+2\sqrt{tR}-\lambda\pi)}{\sqrt{x}+\sqrt{t}}dx \text{----(2.4.7)}$$

First note that $a > 0, e^{-at}$ is abounded function for $t-\delta > A$.

Similarly, the quantity

$$\phi_k(x)x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}t^{\left(-\frac{\lambda}{2}-\frac{1}{4}\right)}\frac{\sin(2\sqrt{xR}-2\sqrt{tR})}{\sqrt{x}-\sqrt{t}} - \phi_K x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}t^{\left(-\frac{\lambda}{2}-\frac{1}{4}\right)}\frac{\cos(2\sqrt{xR}+2\sqrt{tR}-\lambda\pi)}{\sqrt{x}+\sqrt{t}}$$

is bounded on the domain.

$$T = \{(x,t) : A < t-\delta,\ A < t < \min(B, t-\delta)\ \}$$

we integrate by parts the first term in the (2.4.7)

$$-\phi_K x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}t^{\left(-\frac{\lambda}{2}-\frac{1}{4}\right)}\frac{\cos(2\sqrt{xR}+2\sqrt{tR}-\lambda\pi)}{\sqrt{x}+\sqrt{t}}$$

$$+\frac{1}{R}\int_A^{\min(B,t-\delta)} D_x[\ \phi_K(x)\frac{x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}}{\sqrt{x}+\sqrt{t}}\ ]Sin(2\sqrt{xR}+2\sqrt{tR}-\lambda\pi)dx \text{----(2.4.8)}$$

The lower limit term is zero. So is the upper limit term if $B \le t-\delta$.

On the other hand, if $t-\delta < B$, the upper limit term is bounded by

$$\frac{1}{R\delta}\sup_{A<x<B}\left|x^{\frac{\lambda}{2}-\frac{1}{4}}\phi_k(x)\right|.$$

Consequently, this upper limit term converges uniformly to zero for $A < t-\delta$ as $R \to \infty$.

Moreover $D_x\left[\frac{x^{\left(\frac{\lambda}{2}-\frac{1}{4}\right)}\phi_k(x)}{\sqrt{x}+\sqrt{t}}\right]$ is also bounded on the domain K which implies that the second term in (2.4.8) also converges uniformly to zero for $A < t-\delta$ as.

A similar procedure of integration by parts may be applied to the second term in (2.4.7).

This proves that, as $R \to \infty$, $P_R(t)$ converges uniformly to zero on $0 < t < \infty$.

Consider $Q_R(t) = e^{-at} V_3(t) = e^{-at} \int\limits_{t+\delta}^{\infty} L_R(x,t)\phi_k(x)dx$

By the similar arguments as in [26] it can be show that $Q_R(t) \to 0$ uniformly on $0 < t < \infty$ as $R \to \infty$.

Thus $e^{-at} \Delta_{\lambda,t}^{k} [M_R(t) - \phi(t)] \to 0$ uniformly on $0 < t < \infty$ as $R \to \infty$ which implies that

$M_R(t) \to \phi(t)$ in $H_{a,\lambda}$ as $R \to \infty$, and the theorem is proved.

Theorem 2.4.2 Let $F(y) = h_\lambda f$ for $y \in \Omega_f$ and Let $G(y) = h_\lambda g$ for $y \in \Omega_g$ .If $F(y) = G(y)$ on $\Omega_f \cap \Omega_g$, then $f = g$ in the sense of equality in $D'(I)$.

Proof: By Theorem (2.4.1), in the sense of convergence in $D'(I)$,

We have $f - g = \lim\limits_{R \to \infty} \int\limits_{0}^{R} [F(y) - G(y)] \left(\frac{x}{y} \right)^{\frac{\lambda}{2}} J_\lambda (2\sqrt{xy}) dy = 0$.

Therefore $f = g$ in the sense of equality in $D'(I)$.

1.5 Application:

The classical problem is that of solving the integral equation

$$f(x)+k\int_{0}^{\infty}\left(\frac{x}{y}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy})dy = g(x) \text{ ------(2.5.1)}$$

where g is a prescribed function and λ is real number.

(2.5.1) is equivalent to the operation equation

$f(x)+kh_{\lambda}[f(y)(x)] = g(x)$

i.e. $f+kh_{\lambda}f = g$.

In general problem, g is a prescribed distribution from certain spaces and we seek a distribution f such that

$f+kh_{\lambda}^{'}f = g$ ---------(2.5.3)

in the sense of equality in certain space.

Lemma 2.5.1 Let $0<b<1$, $\lambda>-\frac{1}{2}$ and let $f\in H_{a,\lambda}^{'}\cap L_{+,b}^{'}$. If $h_{\lambda}f$ is a distributional Hankel type transform of f defined by (2.3.1).Then $h_{\lambda}f$ generates a regular generalized function in $L_{+,b}^{'}$ and the distributional Laplace transform of $h_{\lambda}f$ is given by

$$L[h_{\lambda}f](p) = p^{-\lambda-1}F\left(\frac{1}{p}\right) \text{ --------(2.5.4)}$$

where $F(P) = \langle f(x), e^{-px} \rangle$ for $b<\operatorname{Re} p<\frac{1}{b}$ is the distributional Laplace transform of f.

Proof: In the view of theorem (2.3.2)

$$|h(f)(y)| \leq \begin{cases} ky^{\lambda} & 0 < y < 1 \\ ky^{p} & 1 < y < \infty \end{cases}$$

where k and p are suitably chosen real numbers.

Hence, when $b > 0$, $\int_0^{\infty} e^{-by} |h_{\lambda}(f)(y)| dy \leq \infty$ which shows that $h_{\lambda}(f)$ generates a regular generalized function in $L'_{+,b}$ and therefore, if $\operatorname{Re} p > b$, the Laplace transform of $h_{\lambda}(f)$ is given by

$$L[h_{\lambda}(f)(y)](p) = \int_0^{\infty} (h_{\lambda} f)(y) e^{-py} dy$$

$$= \int_0^{\infty} < f(x), \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy}) > e^{-py} dy$$

$$=< f(x), \int_0^{\infty} \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy}) e^{-py} dy > \quad \text{------(2.5.5)}$$

$$=< f(x), p^{-\lambda-1} e^{-\frac{x}{p}} > \quad [22]$$

$$= p^{-\lambda-1} < f(x), e^{-\frac{x}{p}} >$$

$$= p^{-\lambda-1} F\left(\frac{1}{p}\right) \quad \text{if } b < \operatorname{Re} p < \frac{1}{b}$$

Step (2.5.5) can be justified as follows:

For any $R > 0$, $\lambda. > -\frac{1}{2}$

$$\text{Let } \phi_R(x) = \int_0^{R} \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_{\lambda}(2\sqrt{xy}) e^{-py} dy \quad \text{---------(2.5.6)}$$

By the smoothness of the integrand, we may carry operator $\Delta_{\lambda,x}^{k}$ under the integral sign in (2.5.6) to obtain

$$\left|e^{-at}\Delta_{\lambda,x}^{k}\phi_R(x)\right| = \left|2^{\lambda}\int_0^R e^{-ax}(2\sqrt{xy})^{-\lambda}J_\lambda(2\sqrt{xy})y^{k+\lambda}e^{-py}dy\right|$$

$$\leq 2^{\lambda}A_\lambda\int_0^R y^{k+\lambda}e^{-py}dy$$

$$< \infty$$

where A_λ is a constant bound on $e^{-ax}(2\sqrt{xy})^{-\lambda}J_\lambda(2\sqrt{xy})$ for $0 < x < \infty$, $0 < y < R$.

This shows that $\phi_R(x) \in H_{a,\lambda}$.

Also for each p, $b < \operatorname{Re} p < \frac{1}{b}$,

$$\phi(x) = \int_0^\infty \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})e^{-py}dy$$

$$= p^{-\lambda-1}e^{-\frac{x}{p}} \qquad [22]$$

is in $H_{a,\lambda}$. In view of Theorem (2.3.1) and Theorem (2.3.2), and the fact that $e^{-py} \in H_{a,\lambda}$, the left hand side of (2.5.5) can be written as

$$\int_0^R \left< f(x), \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \right> e^{-py}dy + \int_R^\infty \left< f(x), \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) \right> e^{-py}dy \text{----}(2.5.7)$$

Moreover, given any $\in > 0$, there exist R_1 such that, for all $R > R_1$, the second term in (2.5.7) is bounded in magnitude by $\frac{\in}{2}$. The right hand side of (2.5.7) is equal to

$$< f(x), \int_0^R \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})e^{-py}dy > + < f(x), \int_R^\infty \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})e^{-py}dy > \text{------}(2.5.8)$$

Using the Riemann sum technique, we can show that

$$\int_0^R < f(x), \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy}) > e^{-py}dy = < f(x), \int_0^R \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})e^{-py}dy > \text{----}(2.5.9)$$

For each R, $\int_R^\infty \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})e^{-py} \in H_{a,\lambda}$ and

$$e^{-ax}\Delta_{\lambda,x}^k \int_R^\infty \left(\frac{y}{x}\right)^{\frac{\lambda}{2}} J_\lambda(2\sqrt{xy})e^{-py}dy = (-1)^k 2^\lambda \int_R^\infty e^{-ax}(2\sqrt{xy})^{-\lambda} J_\lambda(2\sqrt{xy})y^{k+\lambda}e^{-py}dy$$

which tend to zero uniformly on $0 < x < \infty$. Hence, there is R_2 such that the second term in (2.5.8) is bounded by $\frac{\in}{2}$. Thus, by (2.5.9) the difference between the two side of (2.5.5) is less than $\in$. Since $\in > 0$ is arbitrary, equality in (2.5.5) follows.

Now we use these facts to solve the distributional integral equation

$$f + kh_\lambda f = g \quad \text{-------}(2.5.10)$$

in the space $H'_{a,\lambda} \cap L'_{+,b}$ where $0 < b < 1$, $a > 0$ and g is a known distribution in $H'_{a,\lambda} \cap L'_{+,b}$ and $k \neq -1$.

By applying the distributional Laplace transformation L (2.5.10) can be rewritten as

$$L(f) + kL[h_\lambda f] = L(g)$$

Using (2.5.4), we get

$$L(f)(p) + kp^{-\lambda-1}L(f)\left(\frac{1}{p}\right) = L(g)(p) \text{ if } b < \operatorname{Re} p < \frac{1}{b} \text{ ----(2.5.11)}$$

Replacing p by $\frac{1}{p}$ in (2.5.11), we have

$$L(f)\left(\frac{1}{p}\right) + kp^{+\lambda+1}L(f)(p) = L(g)\left(\frac{1}{p}\right), b < \operatorname{Re} p < \frac{1}{b} \text{ since } o < b < 1 \text{ ----(2.5.12)}$$

Therefore, eliminating $L(f)\left(\frac{1}{p}\right)$ from (2.5.11) and (2.5.12),

We get $L(f)(p) = \frac{1}{1-k^2}\left[L(g)(p) - kp^{-\lambda-1}L(g)\left(\frac{1}{p}\right)\right]$

$$= \frac{1}{1-k^2}\left[L(g)(p) - kL(h_\lambda g)(p)\right] \text{ using (2.5.4).}$$

i.e. $L(f) = \frac{1}{(1-k^2)}L\left[g - kh_\lambda f\right]$

which implies that $f = \frac{1}{1-k^2}(g - kh_\lambda(g))$.