CHAPTER II

IDEALS AND DUAL IDEALS

CHAPT ER II

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INTRODUCTION

Ideals and dual ideals play an important role in lattice, especially in distrubutive semilattice. As distributive semilattice is a generalization of a distributive lattice interestingly we shall study some properties of ideals & dual ideals in a distributive Λ semilattice.

In this chapter we have collected some properties of ideals and dual ideal in distrubutive Λ semilattice.

Throught out this chapter S stands for bounded distrubutive Λ semilattice.

At the out set we prove that an annihilator $(a)^*$ of an element a in a distributive semilattice S is an ideal.

Result : 2.1: Let S be a distributive semilattice with 0. Then $(a)^*$ is an ideal for any $a \varepsilon S$

Proof: We prove that $(a)^*$ is an ideal

i) As $a \Lambda 0 = 0$ we get $0 \varepsilon (a)^*$

Therefore $(a)^* \neq \Phi$

ii) Let $y \leq x$ and $x \varepsilon (a)^*$ we get $x \Lambda a = 0$.

If $y \leq x$ we get $y \Lambda a \leq x \Lambda a = 0$

Therefore $y \Lambda a = 0$

Therefore $y \in (a)^*$

iii) Let $u, v \in (a)^*$

As $u \in (a)^*$ we get $u \wedge a = 0$ and as $v \in (a)^*$ we get $v \wedge a = 0$

From $u \geq a \wedge v = 0$, we get two elements s and $t \in S$ such that $s \geq a$, $t \geq v$ and $s \wedge t = u$,

by distributivity of S.

Thus $u \leq t$ and $v \leq t$

Therefore the element t is an upper bound of $\{u,v\}$

Further $t \wedge a \quad = t \wedge (a \wedge s) \qquad$ (as $s \geq a$, $a \wedge s = a$)

$$= (t \wedge s) \wedge a$$

$$= u \wedge a$$

$$= 0$$

This shows that for any $u, v \in (a)^*$ there exist $t \in (a)^*$ such that $t \geq u$, $t \geq v$.

Thus from (i),(ii)and (iii) we get $(a)^*$ is an ideal.

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Now more generally we prove that an annihilator of a relative to b, i.e. $<a,b>$ is an ideal in a distributive semilattice S for all $a, b \in S$.

Result : 2.2 : In a distributive semilattice S, $<a,b>$ is an ideal in S. For all $a,b, \in S$.

Proof : Let $I = <a,b> = \{x \in S / x \wedge a \leq b\}$

we prove that I is an ideal

i) As $b \wedge a \leq b$ we get $b \in I$ .Therefore $I \neq \Phi$

ii) Let $y \leq x$ and $x \in I$. We get $x \wedge a \leq b$.

As $y \leq x$ we get $y \wedge a \leq x \wedge a$. Therefore $y \wedge a \leq b$

Thus $y \varepsilon$ I

iii) Let $x,y \varepsilon$ I. As $y \varepsilon$ I we get $y \wedge a \leq b$.

But by distributivity of S we get $b = a_1 \wedge y_1$ for some $a_1 \geq a$ and $y_1 \geq y$ in S.

Again as $x \varepsilon$ I we get $a \wedge x \leq b$

As $a \wedge x \leq b$ and $b \leq y_1$ we get $a \wedge x \leq y_1$

And hence by distributivity of S we get $y_1 = a_2 \wedge z$, for some $a_2 \geq a$ and $z \geq x$ in S.

Further $z \wedge (a_1 \wedge a_2) = (z \wedge a_2) \wedge a_1 = y_1 \wedge a_1 = b$.

Thus $z \wedge a \leq z \wedge (a_1 \wedge a_2)$ imply $z \wedge a \leq b$ and hence $z \varepsilon$ I.

Thus for $x,y \varepsilon$ I there exist $z \varepsilon$ I such that $z \geq x$ and $z \geq y$.

Therefore from (i),(ii) and (iii) we get I is an ideal in S.

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It is well known that every chain is a distributive lattice and hence a distributive semilattice. Interestingly we get,

Result :2.3 : A distributive semilattice S is a chain if and only if every ideal in S is prime

Proof : If part : Assume that I is a prime ideal in S.

We shall prove that S is a chain.

Let $a,b \varepsilon$ S.

Consider $I = (a \wedge b]$.

By data, I is prime.

Hence $a \wedge b \varepsilon$ I implies $a \varepsilon$ I or $b \varepsilon$ I.

Let $a \varepsilon$ I. Then $a \leq a \wedge b$.

Therefore $a = a \wedge b$.

i.e. $a \leq b$.

Hence S is a chain.

Only if part : Assume that S is a chain. We shall prove that I is a prime.

Let $a \wedge b \varepsilon S$

As S is a chain either $a \leq b$ or $b \leq a$

Let $a \leq b$. Then $a \wedge b = a$ and hence $a \varepsilon$ I

Hence I is prime.

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Any proper dual ideal in any semilattice with 0 is contained in a maximal dual ideal is proved in the following result.

Result : 2.4 : Any proper dual ideal of a semilattice S with 0 is contained in a maximal dual ideal.

Proof : Let D be a proper dual ideal of S with 0. As $D \neq S$, there exists $a \varepsilon S$, such that $a \notin D$. Define $\mathcal{K} = \{F / D \subseteq F$ and is proper dual ideal $\}$

Since $D \subseteq D$ and D is a proper dual ideal, we get $D \varepsilon \mathcal{K}$. Therefore $\mathcal{K} \neq \Phi$

Let ζ be any chain in $\mathcal{K}$.

We shall prove that $\bigcup_{C \in \zeta} C \in K$

Let $M = \bigcup_{C \in \zeta} C$

We shall prove that $M \in K$

i) Let $a \leq b$.

For $a \in M$ we get $a \in C$, for some $C \in \zeta$

Therefore $b \in C$ [As C is a dual ideal]

Thus $b \in M$

ii) For $a \wedge b \in M$ we get $a \wedge b \in C$, for some $C \in \zeta$

Therefore $a \in C$ and $b \in C$ [As C is dual ideal]

Hence $a \in M$ and $b \in M$.

Conversely,

For $a, b \in M$ we get $a \in X$ and $b \in Y$, for $X, Y \in \zeta$ As ζ is a chain we have $X \subseteq Y$ or $Y \subseteq X$.

Assume that $X \subseteq Y$.

Then $a, b \in Y$ and Y is a dual ideal, we get $a \wedge b \in Y$ and hence $a \wedge b \in M$.

Therefore from (i)&(ii) we get M is a dual ideal.

Now, since $0 \notin M$ we get M is proper.

Now as $\zeta \in K$, $D \subseteq C$, for each $C \in \zeta$, we get $D \subseteq \bigcup_{C \in \zeta} C$

Hence $D \subseteq M$ (as $M = \bigcup_{C \in \zeta} C$)

Hence we get $M \in K$.

Hence K contains a maximal element, say P .

Now we have to prove that P is maximal dual ideal.

Suppose if possible P is not maximal dual ideal in S. Then $P \subset Q \subset S$.

Now $F \subseteq P \subset Q \subset S$ (as $P \varepsilon K$)

Therefore we get $F \subseteq Q$ and hence $Q \varepsilon K$, which contradicts the maximality of P.

Therefore P is a maximal dual ideal in S.

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We characterise maximal dual ideal as,

Result : 2.5 : Let S be a distributive semilattice with 0. A proper dual ideal M in S is maximal if and only if for any element $a \notin M$ ($a \varepsilon S$) , there exists an element $b \varepsilon M$ such that $a \wedge b = 0$.

Proof: If Part : Assume that for any element $a \notin M$ ($a \varepsilon S$), there exist an element $b \varepsilon M$ such that $a \wedge b=0$.

We shall prove that M is maximal.

Let J be a dual ideal in S such that $M \subset J \subseteq S$.

As $M \subset J$ we get $a \varepsilon J$ such that $a \notin M$.

Now by data there exist $b \varepsilon M$ such that $a \wedge b=0$

As $b \varepsilon M$ we get $b \varepsilon J$.

Thus we have $a,b \varepsilon J$ and J is a dual ideal

Therefore , $a \wedge b \varepsilon J$. i.e. $0 \varepsilon J$

Therefore $J = S$.

This shows that M is a maximal.

Only if part : Assume that M is maximal.

For any element $a \notin M$ ($a \in S$) we have to prove that there exists an element $b \in M$ such that $a \wedge b = 0$.

As $a \notin M$ we get $M \vee [a) \subseteq S$.

i.e. $M \subset M \vee [a) \subseteq S$.

Now as M is maximal we get $M \vee [a) = S$.

Therefore $0 \in M \vee [a)$ gives $0 \geq m \wedge t$, for $m \in M, t \in [a)$. $\Rightarrow m \wedge t = 0 \Rightarrow m \wedge a = 0$

As S is distributive there exists $b \geq m$ & $a \geq t$ such that $a_1 \geq a$ s.t. $0 = b \wedge a_1$

$0 = a \wedge b \geq m \wedge t$, for $b \in M$ & $a \notin M$. $\Rightarrow b \wedge a = 0$

Therefore $a \wedge b = 0$, for $b \in M$.

Hence the result.

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As in a bounded distributive lattice we get

Result :2.6: Any maximal dual ideal of a bounded distributive semilattice S is prime.

Proof : Let M be a maximal dual ideal which is not prime . Then there exists two dual ideals D_1 and D_2 such that $D_1 \cap D_2 \subseteq M$, but neither $D_1 \subseteq M$ nor $D_2 \subseteq M$.

Therefore we get $x \in D_1$ such that $x \notin M$ and $y \in D_2$ such that $y \notin M$.

Since M is maximal there exists z and t in M such that $x \wedge z = 0$ and $y \wedge t = 0$.

[By **Result 2.5**]

Now since $z \wedge t \leq z$ and $z \wedge t \leq t$,then

$x \wedge (z \wedge t) \leq x \wedge z = 0$ and $y \wedge (z \wedge t) \leq y \wedge t = 0$

Thus we get x Λ(z Λt)=0 and y Λ(z Λt) =0

Therefore x,y ε {zΛt}* [By **Result 2.1**]

As x,y ∉ M, we take u≥ x,y such that uΛ(zΛt)=0. Since u,z,t belongs to M and u Λ(z Λt)=0,we get o ε M , which is a contradiction . Hence M is a maximal dual ideal of S which is prime.

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We first define D(x) as follows

Def.2.7 : Let S be a bounded distributive Λ- semilattice, for x ε S, define

D(x) = {y εS/ 1 is the only upper bond of x & y}.

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Using **Def.2.7** we prove the following result.

Result :2.8 : In a bounded distributive Λ- semilattice S, D(x) is a dual ideal for x ε S.

Proof : (i) Since 1 is the only upper bound of 1 and x. i.e. 1≥ 1 and 1 ≥ x.

Therefore 1 ε D(x) . Hence D(x) ≠ Φ .

ii) Let a ≤ b and a ε D(x) we get 1 ≥ a,x . i.e. 1 is the only upper bound of a and x. If a ≤ b then we get 1 ≥ a and 1 ≥ b. Thus 1 ≥ b and 1 ≥ x . This shows that 1 is the only upper bound of b and x.

Therefore b ε D (x).

iii) Let x, a ε D(x). a, b ∈ D(x)

Let k be any upper bound of y and x Λ a.

Since k ≥ x Λ a, there exists m,n such that

m ≥ x, n ≥ a and mΛ n = k

Let $k \geq x$, $k \geq a \wedge b$

$\Rightarrow k = m_1 \wedge n_1$, $a \leq m_1$, $b \leq n_1$

$\Rightarrow m_1$ is an u.b. for {x, a}

$\Rightarrow m_1 = 1$, Similarly $n_1 = 1$

$\Rightarrow k = 1$

$\Rightarrow a \wedge b \in D(x)$

As $m \geq y$ and $m \geq x$ we get $m = 1$

Similarly as $n \geq y$ and $n \geq a$ we get $n = 1$

Hence $k = 1$

i.e. $k = 1$ is the only upper bound of y and $x \wedge a$

Therefore $x \wedge a \in D(x)$.

Conversely, let $x \wedge a \in D(x)$.

Then as $x \wedge a \leq x$ we get $x \in D(x)$.

Similarly as $x \wedge a \leq a$ we get $a \in D(x)$.

Thus for $x \wedge a \in D(x)$, we get $x \in D(x)$ and $a \in D(x)$

Therefore from (i), (ii), and (iii) we get $D(x)$ is a dual ideal.

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We define $W(M)$ as follows.

Def.2.9 : Let S be a bounded semilattice and let M be a maximal dual ideal in S we define $W(M) = \{x \in S / [x) \cap [y) = \{1\}$ for some $y \notin M\}$.

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Using the **Def. 2.9** we prove the following

Result : 2.10: In a bounded semilattice S, the set W(M) is a dual ideal contained in M.

Proof : We shall prove that W(M) is dual ideal in S contained in M.

i) Since $[1) \cap [y) = \{1\}$ for each $y \notin M$.

Therefore $1 \in W(M)$ and hence $W(M) \neq \Phi$

ii) Let $a \leq b$ and $a \in W(M)$, we get

$[a) \cap [y) = \{1\}$, for each $y \notin M$

If $a \leq b$ then $[b) \subseteq [a)$

Therefore $[b) \cap [y) \subseteq [a) \cap [y) = \{1\}$

Therefore $[b) \cap [y) = \{1\}$

This shows that $b \varepsilon W(M)$.

iii) Let $x_1, x_2 \varepsilon W(M)$

We get $[x_1) \cap [y_1) = \{1\}$ for $y_1 \notin M$

and $[x_2) \cap [y_2) = \{1\}$ for $y_2 \notin M$

As $y_1 \notin M$ and M is maximal dual ideal we get $y_1 \wedge m_1 = 0$ for some $m_1 \varepsilon M$.

Similarly, as $y_2 \notin M$ and M is maximal dual ideal we get $y_2 \wedge m_2 = 0$, for some $m_2 \varepsilon M$.

[By Result 2.5]

Therefore $y_1 \wedge (m_1 \wedge m_2) = 0$ and $y_2 \wedge (m_1 \wedge m_2) = 0$

Therefore $y_1, y_2 \varepsilon \{m_1 \wedge m_2\}^*$ **[By Result 2.1]**

As $\{m_1 \wedge m_2\}^*$ is an ideal, there exists $t \geq y_1$, $t \geq y_2$ such that $t \varepsilon \{m_1 \wedge m_2\}^*$ **[By Def. 1.16]**

i.e $t \wedge (m_1 \wedge m_2) = 0$. Now $(m_1 \wedge m_2) \varepsilon M$.

If $t \varepsilon M$ then $t \wedge (m_1 \wedge m_2) \varepsilon M$.

But $t \wedge (m_1 \wedge m_2) = 0 \varepsilon M$, contradicts the maximality of M.

Therefore $t \notin M$.

Thus $t \geq y_1$, $t \geq y_2$ and $t \notin M$

We prove that $[x_1 \wedge x_2) \cap [t) = \{1\}$

Let k be an upper bound of t and $x_1 \wedge x_2$

Therefore $k \geq t$ and $k \geq x_1 \wedge x_2$

As S is distributive, there exists m, n such that $m \geq x_1$, $n \geq x_2$ and $m \wedge n = k$.

As $m \geq k$ and $k \geq t$ we get $m \geq t$ (as $k = m \wedge n$)

Thus as $m \geq x_1$ and $m \geq t$ we get $m=1$ (since $[x_1) \cap [t) = \{1\}$)

Similarly as $n \geq x_2$ and $m \geq t$ we get $n=1$ (since $[x_2) \cap [t) = \{1\}$)

Therefore $k = m \wedge n = 1 \wedge 1 = 1$

Therefore 1 is the only upper bound of $x_1 \wedge x_2$ and t

Therefore $[x_1 \wedge x_2) \cap [t) = \{1\}$ for $t \notin M$

Therefore $x_1 \wedge x_2 \in W(M)$

Conversely, let $x_1 \wedge x_2 \in W(M)$

Then as $x_1 \wedge x_2 \leq x_1$ we get $x_1 \in W(M)$

Similarly as $x_1 \wedge x_2 \leq x_2$ we get $x_2 \in W(M)$

Therefore from (i), (ii) and (iii) we get W (M) is a dual ideal

Now to prove that $W(M) \subseteq M$

Let $x \in W(M)$. We get $[x) \cap [y) = 1$ for $y \notin M$.

As $\{1\} \subseteq M$ we get $[x) \cap [y) \subseteq M$ and M is prime.

Therefore $[x) \subseteq M$. Thus we get $x \in M$.

Thus if $x \in W(M)$ then $x \in M$.

This shows that $W(M) \subseteq M$.

Hence the result.

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The nature of elements of I V J for any two dual ideals I & J in any semilattice is given in the following.

Result : 2.11 : Let I and J be dual ideals of a semilattice S Then.

$IVJ = [I \cup J) = \{t/t \geq i \wedge j, i \varepsilon I, j \varepsilon J\}$

Proof : Let $T = IVJ = [IUJ) = \{t / t \geq i \wedge j, i \varepsilon I, j \varepsilon J\}$

We have to prove that T is

I) a dual ideal

II) $I \subseteq T$, $J \subseteq T$

and (III) is there exists a dual ideal D such that $I \subseteq D$ and J $\subseteq D$ then $T \subseteq D$.

$I \subseteq D$ and $I \subseteq D$ then $T \subseteq D$.

(I) (a) since $1 \geq i \wedge j$, $i \varepsilon I$, $j \varepsilon J$

We get $1 \varepsilon T$. Therefore $T \neq \Phi$

b) Let $x \leq y$. As $x \varepsilon T$, we get $x \geq i \wedge j$, $i \varepsilon I$, $j \varepsilon J$

As $x \leq y$ we get $y \geq i \wedge j$, $i \varepsilon I$, $j \varepsilon J$. Therefore $y \varepsilon T$

C) Let $x, y \varepsilon T$.

As $x \varepsilon T$ we get $x \geq i_1 \wedge j_1$, $i_1 \varepsilon I$, $j_1 \varepsilon J$.

As $y \varepsilon T$ we get $y \geq i_2 \wedge j_2$, $i_2 \varepsilon I$, $j_2 \varepsilon J$

Then $x \wedge y \geq (i_1 \wedge j_1) \wedge (i_2 \wedge j_2)$

i.e. $x \wedge y \geq (i_1 \wedge i_2) \wedge (j_1 \wedge j_2)$

i.e. . $x \wedge y \geq i \wedge j$ as $i_1 \wedge i_2 = i \varepsilon I$

& $j_1 \wedge j_2 = j \varepsilon J$

Therefore $x \wedge y \in T$.

Conversely, let $x \wedge y \in T$.

We get $x \wedge y \geq i \wedge j$, $i \in I, j \in J$.

Then $x \geq x \wedge y \geq i \wedge j$, $i \in I, j \in J$.

Thus $x \geq i \wedge j$, $i \in I, j \in J$. Therefore $x \in T$.

Similarlly, as $x \wedge y \in T$, we get $y \geq x \wedge y \geq i \wedge j$, $i \in I, j \in J$.

Therefore $y \in T$.

II) Let $i \in I$, we get $i \geq i \wedge j$, $j \in J$. Therefore $i \in T$

This shows that $I \subseteq T$

similarlly, let $j \in J$, we get $j \geq i \wedge j$, $i \in I$. Therefore $j \in T$.

This shows that $J \subseteq T$.

Thus we get $I \subseteq T$ and $J \subseteq T$

III) We have to prove that T is the smallest dual ideal containing I and J

As $I \subseteq T$ and $J \subseteq T$, assume D is a dual ideal such that $I \subseteq D$ and $J \subseteq D$

Let $t \in T$ then $t \geq i \wedge j$, $i \in I, j \in J$.

As $I \subseteq D$, $J \subseteq D$, we get $i, j \in D$. But D being dual ideal, we get $i \wedge j \in D$.

As $t \geq i \wedge j \in D$ and D is dual ideal we get $t \in D$.

This shows that $T \subseteq D$.

Hence the Result.

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For distributive semilattice S we get.

Result :- 2.12 : Let I and J be dual ideals of a distributive semilattice S then

$I V J = [I U J) = \{ t / t = i \wedge j , i \varepsilon I , j \varepsilon J \}$

Proof :- Let $T = \{t / t = i \wedge j, i \varepsilon I , j \varepsilon J\}$

By **Result 2.11**, we have

$I V J = [I U J) = \{t / t \geq i \wedge j, i \varepsilon I, j \varepsilon J\}$

Let $t \varepsilon I V J$. Then $t \geq i \wedge j$, $i \varepsilon I , j \varepsilon J$

As S is distributive, there exists $i_1 , j_1 \varepsilon S$ such that $t = i_1 \wedge j_1$

As I is a dual ideal and $i \varepsilon I$, we get $i_1 \varepsilon I$ (since $i_1 \geq i$)

Simillarly J is a dual ideal and $j \varepsilon J$, we get $j_1 \varepsilon J$ (since $j_1 \geq j$)

Hence $t = i_1 \wedge j_1$, $i_1 \varepsilon I, j_1 \varepsilon J$

Therefore $t \varepsilon T$

Hence $I V J \subseteq T$.. (i)

Now obviously $T \subseteq I V J$.... (ii)

From (i) & (ii) we get

$IVJ = T$

Thus $IVJ = \{t / t = i \wedge j, i \varepsilon I , j \varepsilon J\}$

Hence the result.

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We use the following definition to proceed further.

De 2.13 : Let S be a distributive $\wedge$ - semilattice, $a,b \varepsilon S$. By $< a,b > \underline{V} < b,a >$ we mean ideal generated by $< a,b > U < b,a >$.

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In a bounded distributive Λ - semilattice, every prime dual idal is contained in a unique maximal dual idal then $< a,b > \underline{V} < b,a > = S$ identically for a,b belong to us with a Λ b=0 is proved in the following.

Result : 2.14 : Let S be bounded distributive Λ - semilattice. If every prime dual ideal in S is contained in a unique maximal dual ideal then $< a,b> \underline{V} <b,a> =S$ indentically for a, b ε S with $a \Lambda b = 0$.

Proof : Let a, b $\in$ S such that $a \; \Lambda \; b = 0$ and let $<a,b> \underline{V} <b,a> = I \; (\neq S)$. Then there exists a prime dual ideal P disjoint with I. Consider the dual ideal PV [a).If b$\in$ PV[a) then $b \geq t \; \Lambda \; a$ for some $t \in P$. Therefore, t ε $<a,b>$ [By **Result 2.2**]

and hence $t \in I \cap P = \Phi$, a contradiction. Hence $b \notin PV[\,a)$

This shows that PV [a) is a proper dual ideal.

Hence $PV[a) \subseteq M_1$, for some maximal dual ideal M_1. [By **Result 2.4**]

Hence $P \subseteq M_1$

Similarlly, $a \notin PV[b)$ imply PV[b) is a proper dual ideal and hence there exists a maximal dual ideal M_2 such that $Pv[b) \subseteq M_2$

Hence $P \subseteq M_2$

Thus as $a \in M_1$ and $a \Lambda b = 0$, we get $b \notin M_1$ [By **Result 2.5**]

Similarlly, as $b \in M_2$ and $a \wedge b = 0$, we get $a \notin M_2$

Thus $M_1 \neq M_2$. But this shows that $P \subseteq M_1$ and $P \subseteq M_2$ with $M_1 \neq M_2$, a contradiction; and hence $<a,b> \underline{V} <b,a> = S$, identically, for $a,b \in S$ with $a \Lambda b = 0$

Hence the result

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In a bounded distributive semilattice, $<a,b> \underline{V} <b,a> = S$ identically for $a,b \in S$ with $a \wedge b=0$ then for any prime dual ideal P of S, there exists $x \in P$ such that $a \wedge x$ and $b \wedge x$ are comparable is proved in the following.

Result: 2.15 : Let S be a bounded distributive semilattice. If $<a,b> \underline{V} <b,a> = S$ identically for $a,b \in S$ with $a \wedge b = 0$ then for any prime dual ideal P of S, there exist x in P such that $a \wedge x$ and $b \wedge x$ are comparable.

Proof :By data $<a,b> \underline{V} <b,a> = S$ identically for $a,b \in S$.

Let $t \in P$ we get $t \in S$

i.e. $t \in <a,b> \underline{V} <b,a>$

i.e. $t \in (<a,b> \cup <b,a>]$

Hence there exists $x \in <a,b>$ and $y \in <b,a>$ such that $t \leq x \wedge y$.

Are you using Lemma 1 (ii) p. 21 Grätzer [S]? Is it valid in a $\wedge$-semilattice?

Therefore $[t) \supseteq [x) \wedge [y)$

Thus $t \in P$ we get $[t) \subseteq P$.

Hence $[x) \cap [y) \subseteq P$

Since P is prime dual ideal of S, we get $[x) \subseteq P$ or $[y) \subseteq P$ **[By Def. 1.23]**

Let $[x) \subseteq P$. Then we get $x \in P$

Thus as $x \in <a,b>$ we get $a \wedge x \leq b$.

Therefore $a \wedge x \leq b \wedge x$.

Thus for given $a,b \in S$ with $a \wedge b = 0$ there exists $x \in P$ such that $a \wedge x \leq b \wedge x$. i.e. $a \wedge x$ and $b \wedge x$ are comparable.

Hence the result.

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For any prime dual ideal P of bounded distibutive semilattice there exists x in P such that a $\wedge$ x and b $\wedge$ x are comparable then P is contained in a unique maximal dual ideal is proved in the following.

Result : 2.16: Let S be a bounded distributive Λ - semilattice and let P be a prime dual ideal in S, for a,b ε S with a ,b ε S with aΛ b=0 there exists x in P such that a Λ x and b Λ x are comparable then P is contained in a unique maximal dual ideal.

Proof : Let $P \subseteq M_1$ and $P \subseteq M_2$ where M_1 and M_2 are distinct maximal dual ideals in S.

As $M_1 \neq M_2$ there exists $a_1 \in M_1$ such that $a_1 \notin M_2$.

But then there exists $a_2 \in M_2$ such that $a_1 \Lambda a_2 = 0$ [By **Result 2.5**]

By data, there exists x in P such that $a_1 \Lambda x$ and $a_2 \Lambda x$ are comparable.

Assume without loss of generality $a_1 \Lambda x \leq a_2 \Lambda x$

As $x \in P$ and $P \leq M_1$ imply $x \in M_1$

As $a_1 \in M_1$ and $x \in M_1$, we get $a_1 \Lambda x \in M_1$ [By **Def 1.21**]

Thus as $a_1 \Lambda x \leq a_2 \Lambda x$ and $a_1 \Lambda x \in M_1$ we get $a_2 \Lambda x \in M_1$ [By **Def. 1.21**]

Thus we get $a_2 \in M_1$

As $a_1 \in M_1$ and $a_2 \in M_1$ we get $a_1 \wedge a_2 \in M_1$ [By **Def. 1.21**]

i.e. $o \in M_1$, contradicting the maximality of M_1

Therefore $M_1 = M_2$

Hence the prime dual ideal P must be contained in a unique maximal dual ideal.

Hence the result.

***

Combining the **Results 2.14, 2.15 and 2.16** and putting them in the following elegant form we get.

Result : 2.17 : Let S be a bounded distributive $\wedge$ - semilattice. Then following are equvalent.

1. Every prime dual ideal in S is contained in a unique maximal dual ideal.
2. $\langle a,b\rangle \underline{\vee} \langle b,a\rangle = S$ identically for $a,b \in S$ with $a \wedge b = 0$
3. For any prime dual ideal P of S, there exists x in P such that a $\wedge$ x and b $\wedge$ x comparable.

***

The set of all dense elements in a bounded semilattice is a dual ideal is proved in the following result.

Result : 2.18: In a bounded semilattice S, the set of all dense elements , D(S) is a dual ideal.

Proof : We have, $D(S) = \{ x \in S / x \text{ is a dense element}\} = \{x \in S / \{x\}^* = \{0\} \}$

We have to prove that D(S) is a dual ideal in S

i. Since $\{1\}^* = \{0\}$, we get $1 \in D(S)$. Therefore $D(S) \neq \phi$

ii. Let $x \leq y$ and $x \in D(S)$. We get $\{x\}^* = \{o\}$

As $x \leq y$ we get $\{x\}^* \supseteq \{y\}^*$. i.e. $\{o\} \supseteq \{y\}^*$

i.e. $\{y\}^* = \{o\}$ Therefore $y \in D(S)$

iii. Let $x,y \in D(S)$

As $x \in D(S)$ we get $\{x\}^* = \{o\}$. i.e. $\{x\}^{**} = \{o\}^* = \{1\}$

As $y \in D(S)$, we get $\{y\}^* = \{o\}$. i.e. $\{y\}^{**} = \{o\}^* = \{1\}$

Now we know $(x \wedge y)^{**} = x^{**} \wedge y^{**}$

Therefore $(x \wedge y)^{**} = \{1\} \wedge \{1\} = \{1\}$

Therefore $(x \wedge y)^{***} = \{1\}^* = \{o\}$

i.e. $(x \wedge y)^* = \{o\}$ (As $a^{***} = a^*$)

Therefore $(x \wedge y) \in D(S)$

Conversely let $x \wedge y \in D(S)$

As $x \wedge y \leq x$. Therefore $\{ x \wedge y \}^* \supseteq \{x\}^*$

i.e. $\{x\}^* \subseteq \{x \; y\}^* = \{0\}$. i.e. $\{x\}^* = \{0\}$,

Therefore $x \in D(S)$

Similarlly, as $x \wedge y \leq y$, then $\{x \wedge y\}^* \supseteq \{y\}^* = \{0\}$

Therefore $\{y\}^* = \{0\}$. Therefore $y \in D(S)$

Hence from i,ii & iii, we get D(S) is a dual ideal in S.

Hence the result.

***

The relation between set of all dense elements and the set of all maximal dual ideal is exhibited in the following.

Result : 2.19: In a bounded distributive semilattice S, the set of all dense elements D(S) is the intersection of all maximal dual ideals in S.

Proof : We have to prove that $D(S) = \cap \mathcal{M}$, where $\mathcal{M}$ is the set of maximal dual ideals in S.

Let $x \in D(S)$

We have to prove that $x \in M$, for all $M \in \Pi$

Suppose $x \notin M$, then there exists $m \in M$ such that $x \wedge m = 0$ [By result 2.5]

But then $m = 0$, a contradiction

Therefore $x \in M$ for all $M \in \Pi$

Hence $D(S) \subseteq \cap \Pi$ ....(I)

Now, let $y \in \cap \Pi$ Then $y \in M$ for all $M \in m$

Suppose $y \notin D(S)$; then there exist $z \in S$, such that $y \wedge z = 0$ and $z \neq 0$

Now, since $z \neq 0$, then there exist a maximal dual ideal containing z. Suppose it is M.

i.e. [z) is a proper dual ideal and contained in M. [By Result 2.4]

Therefore $z \in M$

Now as $z \in M$ and $y \in M$, we get $y \wedge z \in M$

i.e. $0 \in M$, contradicting to the maximality of M. Therefore $y \in D(S)$

Thus $\cap \Pi \subseteq D(S)$ ....(II)

Hence from (I) & (II) we get

$D(S) = \cap \Pi$

Hence the result.

***

Further we have

Result : 2.20: In a bounded distributive semilattice $\Pi \subseteq \wp$, where Π is the set of all maximal dual ideals & $\wp$ is the set of all prime dual ideals containing all dense elements.

Proof : We have to prove that $\Pi \subseteq \wp$

As $\cap \mathcal{M} \subseteq M$, for all $M \in \mathcal{M}$

Then we get $D(S) \subseteq M$, for all $M \in \mathcal{M}$ (As $D(S) = \cap \mathcal{M}$)

Now as $M \in \wp$, for all $M \in \mathcal{M}$ **[By Result 2.6]**

Thus each maximal dual ideal is a prime dual ideal containing D(S).

Thus we get $\mathcal{M} \subseteq \wp$

Hence the result

***

Any set complement of a minimal prime ideal in a bounded distributive semilattice is maximal dual ideal is proved in the following.

Result : 2.21 : In a bounded distributive semilattice S, set comlement of a minimal prime ideal is a maximal dual ideal.

Proof : Let A be a minimal prime ideal of S we have to prove that set complement of A, denoted by cA is a maximal dual ideal

i) Let a, b, $\in$ cA . We get a, b $\notin$ A.

As A is prime, $a \wedge b \notin A$. Therefore $a \wedge b \in cA$.

Conversely, let $a \wedge b \in cA$. We get $a \wedge b \notin A$

As A is prime, $a \notin A$ and $b \notin A$

Therefore $a \in cA$ or $b \in cA$

Hence cA is a dual ideal.

ii) As cA is a dual ideal . Then it is contained in a maximal dual ideal M of S

[By Result 2.4]

i.e. $cA \subseteq M$

i.e. $A \supseteq cM$

Now we shall prove that cM is an ideal

iii) Let $x \leq y$ and $y \varepsilon cM$. We get $y \notin M$

As M is maximal dual ideal, we get $x \notin M$.

Therefore $x \varepsilon cM$

iv) Let $x, y \varepsilon cM$. We get $x,y \notin M$

As M is a maximal dual ideal ,there exist $z_1 \varepsilon M$ and $z_2 \varepsilon M$ such that $z_1 \wedge x = 0$ and

$z_2 \wedge y = 0$ **[By Result 2.5]**

As $z_1, z_2 \varepsilon M$ we get $z_1 \wedge z_2 \varepsilon M$.

Then $x \wedge (z_1 \wedge z_2) = 0$ and $y \wedge (z_1 \wedge z_2) = 0$

We get $x,y \varepsilon \{z_1 \wedge z_2\}^*$ **[By Resulst 2.1]**

As $\{z_1 \wedge z_2\}^*$ is an ideal, there exist $t \geq x, t \geq y$ such that $t \varepsilon \{z_1 \wedge z_2\}^*$

i.e. $t \wedge (z_1 \wedge z_2) = 0$ **[By Def.1.16]**

As $t \wedge (z_1 \wedge z_2) = 0$ and $z_1 \wedge z_2 \varepsilon M$ we get $t \notin M$ **[By Result 2.5]**

Hence $t \varepsilon cM$

Thus for given $x,y \varepsilon cM$ there exist $t \varepsilon cM$ such that $t \geq x$ and $t \geq y$

Hence cM is an ideal.

Now we prove that cM is prime

Let $a \wedge b \varepsilon cM$. We get $a \wedge b \notin M$

As M is a dual ideal, we get $a \notin M$ or $b \notin M$

i,e. $a \varepsilon cM$ or $b \varepsilon cM$. Therefore cM is prime

As $A \subseteq cM$ and cM is a prime ideal, we get by minimality of A, a contradiction.

Therefore $cA = M$. i.e. cA is a maximal dual ideal

Hence the result.

***

Venkatanarasimhan P.V. [19] has proved that in a pseudo complemented lattice L the first three of the following statements are equivalent and each of these is implied by the fourth.

I) Every prime ideal is minimal prime

ii) Every prime dual ideal is minimal prime

iii) Every prime dual ideal is maximal

iv) $D = [1)$

This result can be generalised to bounded distributive semilattice as follows.

Result :2.22: The following statement concerning a bounded distributive semilattice S are equivalent.

I) Every prime ideal is minimal prime

ii) Every prime dual ideal is minimal prime

iii) Every prime dual ideal is maximal

Proof :Suppose A is a prime dual ideal which is not minimal prime. Then there exist a prime dual ideal B such that $B \subset A$

We prove that cA and cB are prime ideals (cA,cB are set complements of A, B respectively)

i) Let $x \leq y$ and $y \in cA$ we get $y \notin A$

As A is dual ideal , we get $x \notin A$. Therefore $x \in cA$

ii) Let $x,y \in cA$. We get $x \notin A, y \notin A$.

As A is a dual ideal we get $[x) \nsubseteq A$ and $[y) \nsubseteq A$

Therefore $[x) \cap [y) \nsubseteq A$. Then there exist $t \in [x) \cap [y)$ such that $t \notin A$

This shows that there exist $t \in cA$ such that $t \geq x$ and $t \geq y$

From (i) and (ii) we get cA is an ideal

Now let $a \wedge b \in cA$. We get $a \wedge b \notin A$

Therefore $a \notin A$ or $b \notin A$

Therefore $a \in cA$ or $b \in cA$. Therefore cA is prime ideal

On the similar line we can prove that cB is a prime ideal and cB is not minimal .

Thus (I) $\Rightarrow$ (II) .

Let C be a prime dual ideal which is not maximal. Then there exist maximal dual ideal say M such that $C \subset M$. As S is distributive M is prime **[By Result 2.6]**

There M is a prime dual ideal which is not minimal prime.

Hence (II) $\Rightarrow$ (III)

Let A be a prime ideal which is not minimal prime. Then there is a minimal prime ideal B such that $B \subset A$. Clearly cA and cB are proper prime dual ideals and $cA \subset cB$

[By Result 2.21]

Thus cA is a prime dual ideal which is not maximal.

Hence (III) $\Rightarrow$(I).

***

Stone characterized distributive lattices by means of the following separation property: a lattice is distributive if and only if when a dual ideal D and an ideal I are disjoint, there exists a prime dual ideal containing D and disjoint from I. This result can be generalized to semilattices as follows:

Result :2.23 : An up directed semilattice is distributive if and only if for any dual ideal D and any ideal I, such that $D \cap I = \phi$, there exists a prime dual ideal containing D and disjoint from I.

Proof :(I) Only if part: Let S be an updirected distributive semilattice . Then for any dual ideal D and any ideal I in S such that $I \cap D = \phi$, we have to prove that there exists a prime dual ideal containing D and disjoint from I.

Define,

$K = \{J/J$ is a dual ideal in S such that $I \cap J = \phi$ and $D \subseteq J\}$

Since D is a dual ideal in S such that $I \cap D = \phi$ and $D \subseteq D$ Therefore $D \in K$ and hence

$K \neq \phi$

Let ζ be any chain in K

Define $X = \bigcup_{C \in \zeta} C$

We have to prove that X is dual ideal in S such that $X \cap I = \phi$ and $D \subseteq X$. (i.e. $X \in K$)

i) Let $a \leq b$

For $a \in X$ we get $a \in C$, for some $C \in \zeta$.

Therefore $b \varepsilon C$ [As C is dual ideal]

Hence $b \varepsilon X$

ii) For $a \wedge b \varepsilon X$, we get $a \wedge b \varepsilon C$ for some $C \varepsilon \zeta$

Therefore $a \varepsilon C$ & $b \varepsilon C$ (As C is a dual ideal)

Hence $a \varepsilon X$ & $b \varepsilon X$.

Conversely,

For $a,b \varepsilon x$ we get $a \varepsilon C_1$ and $b \varepsilon C_2$ for $C_1, C_1 \varepsilon \zeta$

As ζ is a chain we have $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$

Assume that $C_1 \subseteq C_2$

Then $a, b \varepsilon C_2$ and C_2 is a dual ideal, we get $a \wedge b \varepsilon C_2$ and hence $a \wedge b \varepsilon X$.

Therefore from (i), (ii) and (iii) we get ζ is a dual ideal X

Now as $\zeta \subseteq K$ We get $I \cap C = \phi$, for all $C \varepsilon \zeta$

Therefore $I \cap [\bigcup_{C \in \zeta} C] = \phi$

That is, $I \cap X = \phi$ [As $X = \bigcup_{C \in \zeta} C$]

As $\zeta \subseteq K$ we get $D \subseteq C$, for all $C \varepsilon \zeta$

Therefore $D \subseteq X$.

Thus weget X is a dual ideal in S such that $I \cap X = \phi$ and $D \subseteq X$, i.e. $X \varepsilon K$

Hence by Zorns Lemma, K contains a maximal element say M

We prove that M is prime.

Let $D_1 \cap D_2 \subseteq M$

Assume that $D_1 \nsubseteq M$ and $D_2 \nsubseteq M$

As $D_1 \nsubseteq M$ we get $x_1 \varepsilon D_1$ such that $x_1 \notin M$

Similarlly as $D_2 \nsubseteq M$ we get $x_2 \varepsilon D_2$ such that $x_2 \notin M$

As $x_1 \notin M$ we get $[M \vee [x_1)] \cap I \neq \phi$.

Then there exists $i \varepsilon I$ such that $i \varepsilon (M \vee [x_1)$

i.e. $i \varepsilon I$ such that $i \geq m_1 \wedge x_1$, $m_1 \varepsilon M$

Then we get $m_1 \wedge x_1 \varepsilon I$ [As I is an ideal)

Similarlly, as $x_2 \notin M$ we get $[MV[x_2)] \cap I \neq \phi$

This gives $m_2 \wedge x_2 \varepsilon I$, $m_2 \varepsilon M$

Therefore $(m_1 \wedge m_2) \wedge x_1 \varepsilon I$ and $(m_1 \wedge m_2) \wedge x_2 \varepsilon I$

Therefore for any $t \geq x_1$ and $t \geq x_2$ (t exists as S is updirected) we get $(m_1 \wedge m_2) \wedge t \varepsilon I$.

But as $t \geq x_1$ and $t \geq x_2$ we get $t \varepsilon D_1$ and $t \varepsilon D_2$

Therefore $t \varepsilon D_1 \cap D_2 \subseteq M$.

Therefore $t \varepsilon M$ and $(m_1 \wedge m_2) \varepsilon M$

Therefore $(m_1 \wedge m_2) \wedge t \varepsilon M$

Thus $(m_1 \wedge m_2) \wedge t \varepsilon I \cap M = \phi$

This is a contradiction.

Therefore as $D_1 \cap D_2 \subseteq M$, we get $D_1 \subseteq M$ or $D_2 \subseteq M$

This shows that M is a prime dual ideal.

II) If Part :Consider, for any dual ideal D and any ideal I such that $D \cap I = \phi$ there exists a prime dual ideal containing D and disjoint from I in an updirected semilattice S, then we have to prove that S is distributive.

Let us consider any three elements a, b, c ε S such that $c \geq a \wedge b$.

We have to find $a_1 \geq a$, $b_1 \geq b$ such that $c = a_1 \wedge b_1$

We denote by D_1 (rcsp. D_2) the (non-empty) set of upper bounds of a and c (rcsp.b and c)

That is, $D_1 = \{a_1 c\}^u = \{x \varepsilon S/x \geq a, x \geq c\}$

and $D_2 = \{b,c\}^u = \{x \varepsilon S/x \geq b, x \geq c\}$

D_1 and D_2 are dual ideals as well as

$D = \{z/ z \geq x \wedge y, x \varepsilon D_1 \text{ and } y \varepsilon D_2\}$

Let us suppose D does not cantain c.

i.e. $c \notin D$.

Therefore $(c] \cap D = \phi$

As $D \subseteq P$, where P is prime dual ideal in S,

we get $(c] \cap p = \phi$

Thus as $D \subseteq p$ and $c \notin p$

And as $D_1 \subseteq D \subseteq P$ we get $D_1 \subseteq P$

Therefore $[a) \cap [c) \subseteq P$

As P is prime dual ideal we get $a \varepsilon P$

on the similar line we can show sthat $b \varepsilon P$.

Therefore $a \wedge b \varepsilon P$ and as $a \wedge b \leq c$ we get $c \varepsilon P$

This is contradiction.

Therefore $c \in D = \{z / z \geq x \wedge y, x \in D_1 \text{ and } y \in D_2\}$

i.e. $c \geq a_1 \wedge b_1$ where $a_1 \in D_1$ and $b_1 \in D_2$

Now as $a_1 \in D_1$ we get $a_1 \geq a$, $a_1 \geq c$

and as $b_1 \in D_2$ we get $b_1 \geq b$, $b_1 \geq c$

Thus as $a_1 \geq c$ and $b_1 \geq c$ we get $a_1 \wedge b_1 \geq c$

Thus as $c \geq a_1 \wedge b_1$ and $c \leq a_1 \wedge b_1$ we get

$c = a_1 \wedge b_1$ where $a_1 \geq a$ and $b_1 \geq b$.

Therefore S is distributive.

Hence the result

***

Result 2.23 provides us with sufficient condition for an updirected semilattice to be distributive. Let us consider the following separation properties of the semilattice S.

Corollary :2.24 : When an ideal and a dual ideal are disjoint they can be separated by a prime dual ideal.

Corollary :2.25 : a dual ideal and an element not belonging to it can be separated by a prime dual ideal.

Corollary :2.26 : an ideal and an element not belonging to it can be separated by a prime dual ideal.

Corollary : 2.27 : any two distinct elements can be separated by a prime dual ideal.

***

Further we have

Result :2.28 : Let I be an ideal and let D be a dual ideal of a distributive semilattice S. If $I \cap D = \phi$ then there exist a prime ideal P of S with $I \subseteq P$ and $P \cap D = \phi$

Proof : Let I be an ideal and let D be a dual ideal of distributive semilattice S such that $I \cap D = \phi$ Define

$K = \{J/J$ is an ideal in S such that $I \subseteq J$ and $J \cap D = \phi\}$

Since I is an ideal in S such that $I \subseteq I$ and $I \cap D = \phi$

Therefore $I \in K$ and hence $K \neq \phi$

Let ζ be any chain in K

Define $M = \bigcup_{C \in \zeta} C$

We shall prove that $M \in K$

i) Since $C \subseteq M$ for some $C \in \zeta$, we get $M \neq \phi$

ii) Let $a \leq b$ and $b \in M$. We get $b \in C$, for some $C \in \zeta$

Therefore $a \in C$ [As C is an ideal]

Therefore $a \in M$

iii) Let $a,b \in M$. We get $a \in X$ and $b \in Y$, for $X,Y \in \zeta$

As ζ is a chain, we have $X \subseteq Y$ or $Y \subseteq X$

Consider $X \subseteq Y$

Therefore $a,b \in Y$ and Y is an ideal.

Then there exist $c \in Y$ such that $c \geq a$, $c \geq b$

Thus for a,b ε M there exist c ε M such that $c \geq a$ and $c \geq b$.

Therefore from (i),(ii) and (iii) we get M is an ideal

As $\zeta \subseteq K$, $I \subseteq C$ for each $C \varepsilon \zeta$

Therefore $I \subseteq \bigcup_{C \in \zeta} C$

i.e. $I \subseteq M$ [As $M = \bigcup_{C \in \zeta} C$]

Now $M \cap D = (\bigcup_{C \in \zeta} C) \cap D$

$= \bigcup_{C \in \zeta} (C \cap D)$ [As $C \varepsilon \zeta \subseteq K$ we get $C \cap D = \phi$)

$= \bigcup_{C \varepsilon \zeta} \Phi$

Therefore $M \cap D = \phi$

Thus we get $M = \bigcup_{C \in \zeta} C \ \varepsilon K$

Hence by zorn's Lemma, K contains a maximal element, say P.

We shall prove that P is a prime ideal in S such that $I \subseteq P$ and $I \cap D = \phi$

Let $I_1 \cap I_2 \subseteq P$.

Assume that $I_1 \not\subseteq P$ or $I_2 \not\subseteq P$

As $I_1 \not\subseteq P$ we get $x_1 \varepsilon I_1$ such that $x_1 \notin P_1$

and as $I_2 \not\subseteq P$ we get $x_2 \varepsilon I_2$ such that $x_2 \notin P$

As $x_1 \notin P$ we get $[PV(x_1]] \cap D \neq \phi$

Then there exist $d \varepsilon D$ such that $d \varepsilon P V (x_1]$

Thus $d \varepsilon D$ such that $d \leq p_1 \wedge x_1$

As D is a dual ideal, we get $p_1 \wedge x_1 \varepsilon D$.

On the similar line, as $x_2 \notin P$ we get

$[PV(x_2]] \cap D \neq \phi$. Hence we get $p_2 \wedge x_2 \varepsilon D$

Now as D is dual ideal we get $(p_1 \wedge p_2) \wedge x_1 \varepsilon D$ and $(p_1 \wedge p_2) \wedge x_2 \varepsilon D$

Therefore for any $t \geq x_1$ and $t \geq x_2$ [t exist as S is updirected semilattice],

we get $(p_1 \wedge p_2) \wedge t \varepsilon D$.

But as $t \geq x_1$ and $t \geq x_2$ we get $t \varepsilon I_1$ and $t \varepsilon I_2$

Therefore $t \varepsilon I_1 \cap I_2 \subseteq P$

Therefore $t \varepsilon P$ and $p_1 \wedge p_2 \varepsilon P$.

Thus we get $(p_1 \wedge p_2) \wedge t \varepsilon P$ and $(p_1 \wedge p_2) \wedge t \varepsilon D$

That is $(p_1 \wedge p_2) \wedge t \varepsilon P \cap D = \phi$, a contradiction. Hence $I_1 \cap I_2 \subseteq P$ gives $I_1 \subseteq P$ or $I_2 \subseteq P$. Hence the result.

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