CHAPTER IV

CHAPTER - IV

PARTIAL DIFFERENTIAL EQUATONS :

4.1 INTRODUCTION :

The second order partial differential equation (P.D.E

is given by

$$L[u] \equiv P \frac{\partial^2 u}{\partial x^2} + 2Q \frac{\partial^2 u}{\partial x \partial y} + R \frac{\partial^2 u}{\partial y^2} - G\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad (4.1$$

Let P,Q,R be functions of x and y and G be linear function $u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ then (4.1) is said be linear. The most gene secon order P.D.E. in two independent variables x & y can written as

$$P(x,y) \frac{\partial^2 u}{\partial x^2} + 2Q(x,y) \frac{\partial^2 u}{\partial x \partial y} + R(x,y) \frac{\partial^2 u}{\partial y^2} + S \frac{\partial u}{\partial x} + T \frac{\partial u}{\partial y} + W u + Z = 0 \quad \ldots(4.2)$$

The P.D.E is said to be homogeneous if Z = 0, otherwise it is called inhomogeneous.

A solution of Eq.(4.1) & (4.2) will be of the form u u(x,y), which represents a surface in (x,y,u) space known integral surface.On this integral surface, there exis curves across which the partial derivatives $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial x \partial y}$ are discontinuous or indeterminate. Such curves are known characteristics. Let the solution of (4.1) be to pa through a curve Γ whose parametric equation is

$$x = x(r) \ , \ y = y(r) \ , \ u = u(r) \quad \ldots(4.3$$

Also assume that at each point (x,y,u) of Γ the parti derivative $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ are known . As solution will be of form $u = u(x,y)$ at each point of Γ , we have

$$\frac{du}{dr} = \frac{\partial u}{\partial x}\frac{dx}{dr} + \frac{\partial u}{\partial y}\frac{dy}{dr} \qquad \ldots(4.4)$$

Let us denote, $\frac{\partial u}{\partial x} = A(x,y), \frac{\partial u}{\partial y} = B(x,y)$,

We have

$$\frac{dA}{dr} = \frac{\partial A}{\partial x}\frac{dx}{dr} + \frac{\partial A}{\partial y}\frac{dy}{dr}$$

$$= \frac{\partial^2 u}{\partial x^2}\frac{dx}{dr} + \frac{\partial^2 u}{\partial x \partial y}\frac{dy}{dr} \qquad \ldots(4$$

Similarly ,

$$\frac{dB}{dr} = \frac{\partial^2 u}{\partial x \partial y}\frac{dx}{dy} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{dr} \qquad \ldots(4.6)$$

As $P, Q, R, G, \frac{dx}{dr}, \frac{dy}{dr}, a, b, \frac{dA}{dr}$ and $\frac{dB}{dr}$ are all known at each point of Γ. Then Eq. (4.1),(4.5) (4.6) are treated as three simultaneous equations for th unknowns $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ & $\frac{\partial^2 u}{\partial x \partial y}$ at each points of Γ . The solut of these equations exists and unique if

$$\begin{vmatrix} P & 2Q & R \\ \frac{dx}{dr} & \frac{dy}{dr} & 0 \\ 0 & \frac{dx}{dr} & \frac{dy}{dr} \end{vmatrix} = 0$$

Which implies

$$P\left(\frac{dy}{dr}\right)^2 - 2Q\left(\frac{dx}{dr}\right) \cdot \left(\frac{dy}{dr}\right) + R\left(\frac{dx}{dr}\right)^2 = 0$$

$$\text{or } P\left(\frac{dy}{dx}\right)^2 - 2Q\left(\frac{dy}{dx}\right) + R = 0$$

$$\text{or} \frac{dy}{dx} = \frac{2Q \pm \sqrt{4Q^2 - 4PR}}{2P}$$

$$\text{i.e. } \frac{dy}{dx} = \frac{1}{P}\left[Q \pm \sqrt{Q^2 - PR}\right]$$

i.e. We have two equations

$$\frac{dy}{dx} = \frac{1}{P}\left[Q + \sqrt{Q^2 - PR}\right]$$

$$\& \frac{dy}{dx} = \frac{1}{P}\left[Q - \sqrt{Q^2 - PR}\right] \qquad \ldots(4.7$$

whose solution can be represented by

$$V_1(x,y) = \alpha \text{ , } V_2(x,y) = \beta \qquad \ldots(4.8$$

where α, β are constants.

Thus there are two curves given by (4.8) on which second orde partial derivatives will not be calculated in a definite ar finite manner. These curves are known as characteristics ar these are either real and distinct or real & equal c imaginary according as

$Q^2 - PR > 0$, $Q^2 - PR = 0$, $Q^2 - PR < 0$ respectively.

If in the xy-plane, exists two real and distinct familie of characteristics or $Q^2 - PR > 0$, then P.D.E. (4.1) or (4.2 is said to be hyperbolic. If there exists real an conicident family of characteristics or $Q^2 - PR = 0$, ther parabolic and if no real characteristics exists or Q^2 -PR < 0, then elliptic type.

Throughout our discussion assume that the mathematical prolbems are well posed i.e. If solution exists, then it is unique and depends continuously on the given data. The method of solution of P.D.E. is the finite difference method. The numerical solution of P.D.E was implimented in 1950 with the advent of automatic digital computers. Now a days by means of modern high performance computers, the numerical sol of P.D.E. is carried out extensively and often on a very large scale for problems in physics, engineering and other fields of applied analysis, in order to obtain approximate solution of rigorous equations or to simulate real phenomena by means of numerical experiments.

Generally, in the solution of P.D.E.the region of integration is covered with a net, usually of square or recta- ngular mesh and values of the dependent variables are deter mined at nodes of this net. The partial derivatives in the P.D.E. are replaced by suitable difference quotients, conver- ting differential equation to a difference equation at each nodal point. Usually the mesh lengths are sufficiently small for the higher difference terms to be neglected, although they are sometimes included in the integration proces. The network and nodes are shown in Fig. (4.1).

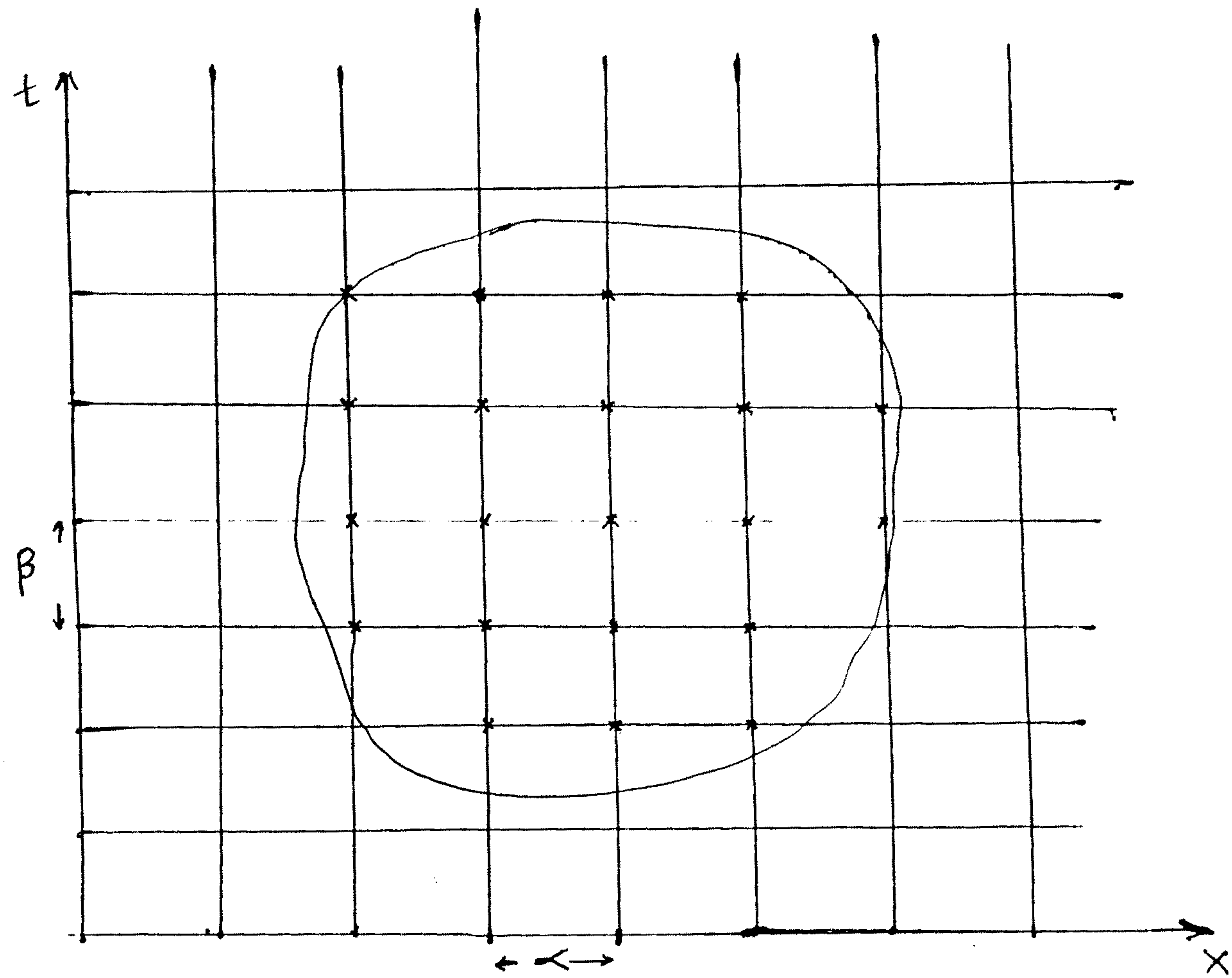

Fig. 4·1 The Region R and Node Points.

4.2 DIFFERENCE METHODS FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS :

One Space Dimention :

Here we will discuss the parabolic equation i.e the equation of heat flow in one dimentional,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \ldots(4.9)$$

Consider the arbitrary region R x [o,T] with suitable initi and boundary conditions, where $R = (a \leq x \leq b)$ & $0 \leq t \leq$ We superimpose on the region Rx [o,T] a rectangular grid wi grid lines parallel to the co-ordinate axes. With spacing & β in space and time directions respectively.

Let us define the grid points on corresponding region as

$$t_j = j\beta, \qquad j = 0,1,2,\ldots\ldots,N.$$

$$x_i = i\alpha, \qquad i = 0,1,2,\ldots\ldots,M.$$

Where $x_o = a$, $x_M = b$, $\alpha = \frac{b-a}{M}$ & $T = N\beta$

Denote the solution at (x_i, t_j) by U_i^j and its approximate value by u_i^j, the differential Eq.(4.9) becomes

$$\left(\frac{\partial u}{\partial t}\right)_{(x_i,t_j)} = \left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i,t_j)}$$

We have

$$\left(\frac{\partial u}{\partial t}\right)_{(x_i,t_j)} = \frac{1}{\beta} \log(1+\Delta t)\, u_i^j$$

$$\text{Where } \frac{\partial}{\partial t} \equiv \frac{1}{\beta}\log(1+\Delta t)$$

$$\equiv \frac{-1}{\beta}\log(1-\Delta t)$$

$$\left(\frac{\partial u}{\partial t}\right)_{(x_i,t_j)} = \frac{1}{\beta}\Delta_t U_i^j + O(\beta)$$

$$= \frac{1}{\beta}\left[U_i^{j+1} - U_i^j\right] + o(\beta) \qquad \ldots(4.10a)$$

Similarly,

$$\left(\frac{\partial u}{\partial t}\right)_{(x_i,t_j)} = \frac{-1}{\beta} \log(1-\nabla_t)\, u_i^j$$

$$= \frac{-1}{\beta} \nabla_t U_i^j + o(\beta)$$

$$= \frac{1}{\beta}(U_i^j - U_i^{j-1}) + O(\beta) \qquad (4.10b)$$

$$\& \left(\frac{\partial u}{\partial t}\right)_{(x_i,t_j)} = \frac{1}{2\beta} \delta_{2t} U_i^j$$

$$= \frac{1}{2\beta}(U_i^{j+1} - U_i^{j-1}) + o(\beta^2) \qquad (4.10c)$$

Now consider R.H.S. of Eq.(4.9) Which can be written as :

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_{(x_i,t_j)} = \frac{4}{\alpha^2}\left[\sinh \frac{\delta x}{2}\right]^2 U_i^j$$

$$= \frac{1}{\alpha^2} \delta_x^2 U_i^j + o(\alpha^2) \qquad (4.11)$$

Using this ,we will discuss the following different methods:

a) Neglecting the error terms and using Eq.(4.10a) &(4.11) i (4.9), we have

$$u_i^{j+1} = (1-2\lambda)\, u_i^j + \lambda\,(u_{i-1}^j + u_{i+1}^j) \qquad (4.12)$$

Where $\lambda = \frac{\beta}{\alpha^2}$

Which is known as Schmidt method. As the method gives the relation between the function values at the two levels (j+1) & j. Thus it is called two level formula.

The schemetic form is shown in Fig (4.2)

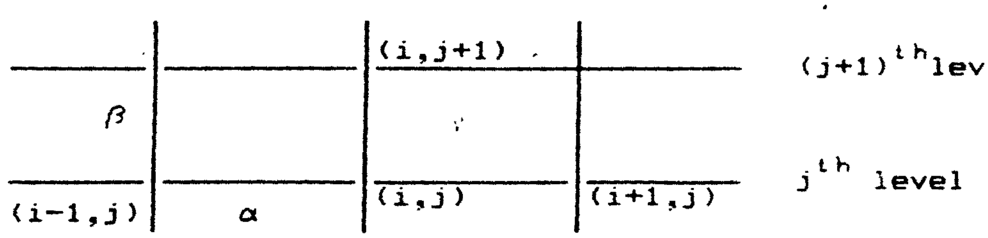

Fig.(4.2)

The solution value at any point (i,j+1) on the $(j+1)^{th}$ lev is expressed in term of the solution values at the poin (i-1,j), (i,j)&(i+1,j) on the j^{th} level. Such a method called an explict method.

The truncation error T_i^j at the node (i,j+1) is given by

$$T_i^j = U_i^{j+1} - U_i^j - \lambda\ (U_{i+1}^j - U_i^j + U_{i-1}^j\ ) \qquad (4.13)$$

Using Taylor series expansion at each term about (x_i, t_j) on the right hand side of (4.13), we get.

$$T_i^j = U(x_i, t_{j+1}) - U(x_i, t_j) - \lambda \Big[U(x_{i+1}, t_j) - 2U(x_i, t_j) + U(x_{i-1}, t_j) \Big]$$

$$= U(x_i, t_j) + \beta\ U_t + \frac{\beta^2}{2!} U_{tt} + \frac{\beta^3}{3!} U_{ttt} + \ldots - U(x_i, t_j)$$

$$-\lambda \Big[U(x_i, t_j) + \alpha U_x + \frac{\alpha^2}{2!} U_{xx} + \frac{\alpha^3}{3!} U_{xxx} + \frac{\alpha^4}{4!} \frac{\partial^4 U}{\partial x^4} + \frac{\alpha^5}{5!} \frac{\partial^5 U}{\partial x^5} + \frac{\alpha^6}{6!} \frac{\partial^6 U}{\partial x^6} + \ldots$$

$$-2U(x_i, t_j) + U(x_i, t_j) - \alpha U_x + \frac{\alpha^2}{2!} U_{xx} - \frac{\alpha^3}{3!} \frac{\partial^3 U}{\partial x^3} + \frac{\alpha^4}{4!} \frac{\partial^4 U}{\partial x^4} + \frac{\alpha^2}{5!} \frac{\partial^5 U}{\partial x^5} + \frac{\alpha^6}{6!} \frac{\partial^6 U}{\partial x^6} \ldots\ldots \Big]$$

$$T_i^j = \beta \frac{\partial (U_i^j)}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2}{\partial t^2} U_i^j + \frac{\alpha^3}{6} \frac{\partial^3}{\partial t^3} U_i^j + \ldots$$

$$- \lambda \Big[\alpha^2 \frac{\partial^2}{\partial x^2} U_i^j + \frac{\alpha^4}{12} \frac{\partial^4}{\partial x^4} U_i^j + \frac{\alpha^6}{360} \frac{\partial^5}{\partial x^5} U_i^j + \ldots \Big]$$

$$= \beta \left(\frac{\partial}{\partial t} U_i^j - \frac{\partial^2}{\partial x^2} U_i^j \right) + \frac{\beta^2}{2} \frac{\partial^2}{\partial t^2} U_i^j + \frac{\beta^3}{6} \frac{\partial^3}{\partial t^2} U_i^j$$

$$- \frac{\beta\ \alpha^2}{12} \frac{\partial^4}{\partial x^4} U_i^j - \frac{\beta \alpha^4}{360} \frac{\partial^5}{\partial x^5} U_i^j \qquad \ldots(4.14)$$

From (4.9)

$$\frac{\partial}{\partial t} u_i^j \equiv \frac{\partial^2}{\partial t^2} u_i^j$$

Similarly $\dfrac{\partial^2}{\partial t^2} \equiv \dfrac{\partial^2}{\partial x^4}$, $\dfrac{\partial^3}{\partial t^3} \equiv \dfrac{\partial^6}{\partial x^6}$

Using in(4.14) we have

$$T_i^j = \beta\left[\frac{\alpha^2}{2} (\lambda - \frac{1}{6}) \frac{\partial^4}{\partial x^4} U_i^j + \frac{\alpha^4}{6} (\lambda - \frac{1}{60}) \frac{\partial^6}{\partial x^6} U_i^j + \ldots \right]$$

$$= \frac{\alpha^2 \beta}{2} (\lambda - \frac{1}{6}) \frac{\partial^4}{\partial x^4} U_i^j + \frac{\alpha^4 \beta}{6} (\lambda^2 - \frac{1}{60}) \frac{\partial^6}{\partial x^6} U_i^j + \ldots.$$

Thus the method (4.12) ,i.e. Schmidt method is of order $(\beta + \alpha^2)$ When $\lambda = \frac{1}{6}$, the method is of order $(\alpha^4 + \beta^2)$.

(b) Now neglecting the error terms and using Eq.(4.10b) & (4.11) in (4.9), we have

$$-\lambda\ u^{j}_{i-1} + (1 + 2\lambda)\ u^{j}_{i} - \lambda\ u^{j}_{i+1} = u^{j-1}_{i} \qquad (4.15)$$

This method is called Laasonen method. In this the solution value at any point (i,j+1) on the $(j+1)^{th}$ level is dependent on the same level and one value on j^{th} level. As solution values at $(j+1)^{th}$ level are evaluated implicitly, the method (4.15) is called an implicit method. It is also a two level method. This method can be expressed in schemetic form given in Fig (4.3)

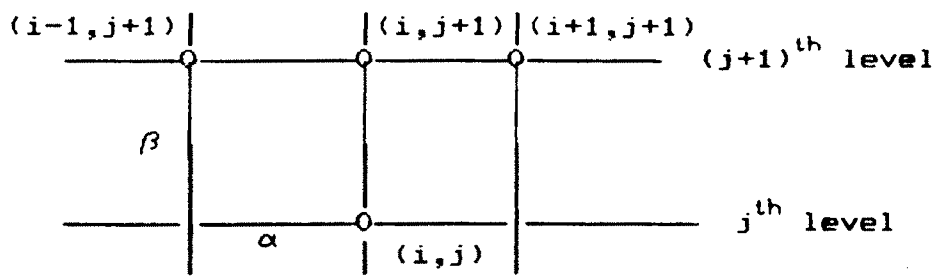

Fig.(4.3)

The truncation error T^{j}_{i} at the node (i,j+1) is given by

$$T^{j}_{i} = -\lambda\ u^{j+1}_{i-1} + (1 + 2\lambda)\ u^{j+1}_{i+1} - \lambda U^{j+1}_{i+1} - u^{j}_{i} \qquad (4.16)$$

Expanding each term on right hand side by Taylor series about (x_i, t_j) and simplifying we get

$$T^{j}_{i} = \beta \left[\frac{\partial}{\partial t} u^{j}_{i} - \frac{\partial^2}{\partial x^2} u^{j}_{i} \right] + \frac{1}{2} \beta^2 \frac{\partial^2}{\partial t^2} u^{j}_{i}$$

$$- \beta^2 \frac{\partial^3}{\partial x^2 \partial t} u^{j}_{i} - \frac{\alpha^2 \beta}{12} \frac{\partial^4}{\partial x^4} u^{j}_{i} - \frac{1}{2} \beta^3 \frac{\partial^4}{\partial x^2 \partial t^2} u^{j}_{i+\ldots}$$

Using Eq.(4.9), we have

$$\beta^{-1} T_i^j = \frac{\beta}{2}\left(\frac{\partial^2}{\partial t^2} u_i^j - \frac{2\partial^3}{\partial x^2 \partial t} u_i^j \right) - \frac{\alpha^2}{12} \frac{\partial^4}{\partial x^4} u_i^j + \dots$$

Thus method (4.15) is of order $(\beta + \alpha^2)$

(c) First converting methods (4.12) and (4.15) into same function values at $(j+1)^{th}$ and j^{th} level and then averaging these methods, we have

$$\frac{1}{2}\Big[-\lambda u_{i-1}^{j+1} - \lambda u_{i+1}^{j+1} - \lambda (u_{i-1}^{j} + u_{j+1}^{j}) + (2 + 2\lambda) u_i^{j+1}$$

$$- (2-2\lambda) u_i^j \Big] = 0$$

$$\text{i.e. } \frac{-\lambda}{2} u_{i-1}^{j+1} - \frac{\lambda}{2} u_{i+1}^{j+1} + (1+\lambda) u_i^{j+1}$$

$$= \frac{\lambda}{2} u_{i-1}^{j} + \frac{\lambda}{2} u_{i+1}^{j} + (1-\lambda) u_i^j \qquad (4.17)$$

which can be expressed as

$$\left(1 - \frac{\lambda}{2}\delta_x^2\right) u_i^{j+1} = \left(1 + \frac{\lambda}{2}\delta_x^2\right) u_i^j \qquad (4.18)$$

The method (4.17) or (4.18) is called Crank-Nicolson method.

The Schematic representaiton. is shown in Fig (4.4)

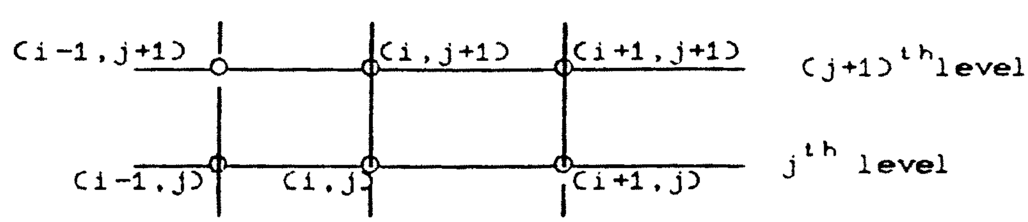

Fig.(4.4)

The truncation error is given by

$$T_i^j = U_i^{j+1} - U_i^j - \frac{\lambda}{2}\left[U_{i+1}^j - 2U_i^j + U_{i-1}^j + U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}\right]$$

$$= \beta \left[\frac{\partial}{\partial t} U_i^j - \frac{\partial^2}{\partial x^2} U_i^j \right] + \frac{1}{2} \beta^2 \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} U_i^j - \frac{\partial^2}{\partial x^2} U_i^j \right]$$

$$- \frac{\alpha^2 \beta}{12} \frac{\partial^4}{\partial x^4} U_i^j - \frac{\beta^3}{4} \frac{\partial^4}{\partial x^2 \partial t^2} U_i^j \ldots$$

Using (4.9), we find

$$\beta^{-1} T_i^j = O(\beta^2 + \alpha^2) \quad .$$

(d) Use Eq.(4.10c) & (4.11) in (4.9), & neglecting the error terms, we have

$$u_i^{j+1} = u_i^{j-1} + 2\lambda \left(u_{i-1}^j - 2u_i^j + u_{i+1}^j \right) \qquad (4.19)$$

This is an explicit three level method. It is called <u>Richardson method</u>. The Schematic representation of this meth od is in Fig(4.5)

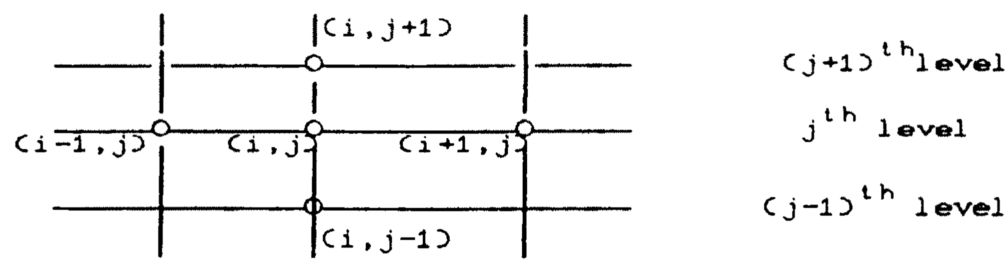

Fig.(4.5)

Taking the average of u_i^{j+1} & u_i^{j-1} and replacing this for u_i^j in (4.19), we have

$$u_i^{j+1} = u_i^{j-1} + 2\lambda \left[u_{i-1}^j - (u_i^{j+1} + u_i^{j-1}) + u_{i+1}^j \right]$$

$$u_i^{j+1} = \frac{1-2\lambda}{1+2\lambda} u_i^{j-1} \frac{2\lambda}{1+2\lambda} \left(u_{i-1}^j + u_{i+1}^j \right) \qquad (4.20)$$

This is also three level an explicit method. It is known as <u>DuFort</u> and <u>Frankel</u> <u>method</u>. The schematic representation is shown in Fig (4.6)

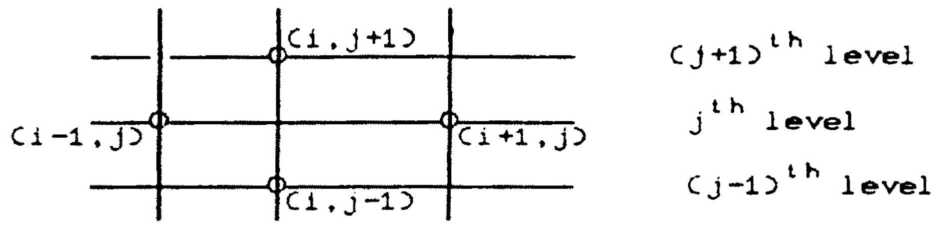

Fig.(4.6)

The truncation error of this method is given by

$$T_i^j = (1 + 2\lambda)u_i^{j+1} - (1-2\lambda)u_i^{j-1} - 2\lambda\,(u_{i-1}^j + u_{i+1}^j) \qquad (4.21)$$

Expanding every term in Taylor series, the errors in (4.21) about the point (x_i, t_j), we have

$$T_i^j = (1+2\lambda)\left[u_i^j + \beta\frac{\partial}{\partial t}u_i^j + \frac{\beta^2}{2}\frac{\partial^2}{\partial t^2}u_i^j + \ldots\ldots\right]$$

$$-(1-2\lambda)\left[u_i^j - \beta\frac{\partial}{\partial t}u_i^j + \frac{\beta^2}{2}\frac{\partial^2}{\partial t^2}u_i^j + \ldots\right]$$

$$-2\lambda\left[2u_i^j + \alpha^2\frac{\partial^2}{\partial x^2}u_i^j + \frac{\alpha^4}{12}\frac{\partial^4}{\partial x^4}u_i^j + \ldots\right]$$

$$2\beta = \left[\frac{\partial}{\partial t}u_i^j + 2\lambda\,\beta^2\frac{\partial^2}{\partial t^2}u_i^j - 2\lambda\,\alpha^2\frac{\partial^2}{\partial x^2}u_i^j - \frac{\lambda\alpha^4\partial^4}{6\partial x^4}u_i^j + \ldots\right]$$

$$\text{and } \beta^{-1}\,T_i^j = 2\left[\frac{\partial}{\partial t}u_i^j - \frac{\partial^2}{\partial x^2}u_i^j\right]$$

$$+ 2\left(\frac{\beta}{\alpha}\right)^2\frac{\partial^2}{\partial t^2}u_i^j - \frac{\alpha^2}{6}\frac{\partial^4}{\partial x^4}u_i^j + \ldots$$

Thus we have the following cases.

I) If $\beta/\alpha \to 0$ as $\alpha \to 0$ then $\beta^{-1}\,T_i^j \to 0$

$$\& \quad \frac{\partial}{\partial t}u_i^j = \frac{\partial^2}{\partial x^2}u_i^j$$

This shows that the difference scheme (4.20) is consistent with differential equation (4.9). In this case the order of method is $\beta^2 + \alpha^2 + (\frac{\beta}{\alpha})^2$.

II) If $\frac{\beta}{\alpha} \to c$, as $\alpha \to 0$, then $\beta^{-1} T_i^j \to 0$

$$\& \; \frac{\partial}{\partial t} u_i^j - \frac{\partial^2}{\partial x^2} u_i^j + c^2 \frac{\partial^2}{\partial t^2} u_i^j = 0,$$

This shows that the method (4.20) approximates the hyperbolic equation $\frac{\partial u}{\partial t} + c^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$

4.3 STABILITY AND CONVERGENCE ANALYSIS :

Let us denote that the analytic solution of differential equation by $u(x_i, t_j)$, difference solution of difference equation by u_i^j and the numerical solution by $\bar{u}_i^j$, These are related by

$$\left|u(x_i, t_j) - \bar{u}_i^j\right| \le \left|u(x_i, t_j) - u_i^j\right| + \left|u_i^j - \bar{u}_i^j\right| \qquad (4.22)$$

In practice, we require the to have left hand side of (4.22) to be small. But this depends on two terms, the first value of R.H.S. arises because the differential equation is replaced by difference equation and it is called local truncation error. By convergence of difference scheme, this truncation error converges to zero when α & β both tend to zero. Other term on R.H.S. of (4.22) is the numerical error. This arises because in actual calculation we cannot solve the difference equation exactly because of round-off errors. If the difference scheme is stable, then the second term in (4.22) is practically equal to zero.

4.3.1 MATRIX STABILITY ANALYSIS :

Using the given boundary conditions every two level difference method for solving Eq.(4.9), can be written as

$$A_o\, u^{j+1} = A_1 u^j + B^j \;,\; j = 0,1,2,,\ldots \qquad (4.23)$$

where B^n contains the boundary conditions & $|A_o| \neq 0$.

For $A_o = I$, the difference scheme (4.23) is called explicit scheme, otherwise it is an implicit scheme. The stability of difference scheme (4.23) so that $|\lambda_i| \le 1$ for all i, where

λ_i are eigen values of $A_0^{-1}A_1$. Let us discuss the stability of some of the above methods.

I) Consider first Crank-Nicolson method (4.17)

Expressing this into matrix form (2.23), we can write this as,

$$A_0\, u^{j+1} = A_1 u^j + B^j$$

$$\text{Where} \quad A_0 = \begin{bmatrix} 1+\lambda & -\lambda/2 & 0 & \cdots & 0 & 0 & 0 \\ -\lambda/2 & 1+\lambda & -\lambda/2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & -\lambda/2 & 1+\lambda & -\lambda/2 \\ 0 & 0 & 0 & & 0 & -\lambda/2 & 1+\lambda \end{bmatrix}$$

$$A_0 = I + \frac{\lambda}{2} P$$

$$\& \; A_1 = \begin{bmatrix} 1-\lambda & \lambda/2 & 0 & \cdots & 0 & 0 & 0 \\ \lambda/2 & 1-\lambda & \lambda/2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda/2 & 1-\lambda & \lambda/2 \\ 0 & 0 & 0 & \cdots & 0 & \lambda/2 & 1-\lambda \end{bmatrix}$$

$$A_1 = I - \frac{\lambda}{2} P$$

$$\text{Where} \quad P = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Let λ_j be the eigen values and $V^{(j)}$ be the corresponding eigen vectors of the matrix P. Then

$$\lambda_j = 4 \sin^2 \left(\frac{q\pi}{2m}\right), \quad q = 1,2,\ldots,M-1.$$

$$\& \; V^{(j)} = \left[\sin \frac{q\pi}{2M} .\sin \frac{2q\pi}{M} \quad \sin \frac{3q\pi}{M} \ldots\sin \frac{(M-1)q\pi}{M}\right]^T,$$

$$1 \le q \le M-1.$$

From this we can write eigen values of

$$A_o = I + \frac{\lambda}{2} P \text{ as}$$

$$1 + \frac{\lambda}{2} 4 \sin^2\left(\frac{q\pi}{2M} \right) = 1 + 2\lambda \sin^2 \frac{q\pi}{2M} .$$

and that of $A_o^{-1}A_1 = A_o^{-1} (I - \frac{\lambda}{2} P)$

$$= A_o^{-1}[I + \frac{\lambda}{2} P - \lambda P]$$

$$= A_o^{-1}[A_o - \lambda P]$$

$$= I - \lambda A_o^{-1}P$$

as
$$1 - \frac{\lambda.4 \sin^2 \frac{q\pi}{2M}}{1+2\lambda \sin^2\frac{q\pi}{2M}} = \frac{1-2\lambda \sin^2 \frac{q\pi}{2M}}{1+2\lambda \sin^2 \frac{q\pi}{2M}} = \mu_q$$

& $|\mu_q| \leq 1$

This is possible for all positive λ. Hence Crank-Nicolson method is unconditionally stable.

II) Now we will discuss the stability of Schmidt method (4.12

We can express this method into matrix form

$$A_o u^{j+1} = A_1 u^j + B^j$$

Where
$$A_o = I, \quad A_1 = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 1-2\lambda & \lambda & & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ 0 & 0 & 0 & \cdots & \lambda & 1-2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & 0 & \lambda & 1-2\lambda \end{bmatrix}$$

$$= I - \lambda P$$

$$\& \quad A_o^{-1}A_1 = I - \lambda P.$$

Thus the eigen values μ_q of $A_o^{-1} A_1$ are

$$1 - 4\lambda \sin^2 \frac{q\pi}{2M} , \qquad q = 1,2,\ldots, M-1.$$

Hence the condition for the stability of this method is

$$-1 \le 1-4\lambda \sin^2 \frac{q\pi}{2M} \le .$$

Hence if $0 < \lambda \le \frac{1}{2}$, then the method is unconditionaly stable.

4.3.2 CONVERGENCE ANALYSIS :

If the solution of the difference equation tends to the solution of differential equation as $\alpha \longrightarrow 0$, $\beta \longrightarrow 0$ then the difference equation is said to be convergent.

(I) Let us discuss the convergence of Schmidt method (4.12). The exact solution satisfies.

$$u(x_i, t_{j+1}) = u(x_i, t_j) - 2\lambda\, u(x_i, t_j) + \lambda \left[u(x_{i-1}, t_j) + u(x_{i+1}, t_j) \right] + O(\beta^2 + \alpha^2) \qquad \ldots (4.24)$$

Define $E_i^j = u(x_i, t_j) - u_i^j$ & $E^j = \max |E_i^j|$

Substracting (4.12) from (4.24), we get

$$E_i^{j+1} = (1-2\lambda) E_i^j + \lambda (E_{i-1}^j + E_{i+1}^j) + M(\beta^2 + \beta\alpha^2)$$

where M is a constant independent of α.

When $\lambda < \frac{1}{2}$, the coefficients are positive and

$$\max_i |E_i^{j+1}| \le (1-2\lambda + \lambda + \lambda) \max_i (E_i^j) + M(\beta^2 + \beta\alpha^2)$$

Hence $E^{j+1} \le E^j + M(\beta^2 + \beta\alpha^2)$

$$\le E^{j-1} + 2M(\beta^2 + \beta\alpha^2)$$

....

$$\le E^0 + (j+1) M (\beta^2 + \beta\alpha^2)$$

As $t_{j+1} = (j+1)\beta$ and $E^{o} = 0$, we have

$E^{j+1} \leq t_{j+1} M(\beta + \alpha)$.

As $\alpha \to 0$, $\beta \to 0$, we get

$E^{j+1} \to 0$

and $u_i^j \to u(x_i, t_j)$

Hence the method is convergent for $0<\lambda < \frac{1}{2}$.

II) Now we will see the convergence of

Crank-Nicolson method (4.17).

Using $E_i^j = u(x_i, t_j) - u_i^j$ in (4.17)

For we will express into matrix form

$$A_o u^{j+1} = A_1 u^j + B^j$$

as $$\left[I + \frac{\lambda}{2} P \right] u^{j+1} = \left[I - \frac{\lambda}{2} P \right] u^j + B^j$$

$$\left[I + \frac{\lambda}{2} P \right] E^{j+1} = \left[I - \frac{\lambda}{2} P \right] E^j + T^j \qquad \ldots(4.25)$$

$j = 0,1,2,\ldots.$

Where $$E^j = \left[E_1^j \; E_2^j \; \ldots\ldots \; E_{m-1}^j \right]^T$$

& $$T^j = \left[T_1^j \; T_2^j \; \ldots\ldots E_{m-1}^j \right]^T$$

& $$T_i^j = 0\,(\beta^3 + \beta\alpha^2)$$

The initial condition gives $E^{o} = 0$. The expression (4.25) can be written as

$$E^{j+1} = H E^j + \sigma^j \qquad \ldots(4.26)$$

$j = 0,1,2,\ldots.$

Where $H = (I + \frac{\lambda}{2} P)^{-1} \; (I - \frac{\lambda}{2} P)$

$$\sigma^j = (I + \frac{\lambda}{2} P)^{-1} T^j$$

Applying (4.26) reccursively, we obtion

$$E^j = H^j E^o + \sum_{i=1}^{j} H^{j-1} \sigma^{i-1}$$

or $|| E^j|| \leq ||H||^j \, ||E^o|| + \sum_{i=1}^{j} ||H||^{j-1} || \sigma^{i-1}|| \leq ||H||^j$

$$|| E^o|| + \frac{1-||H||^j}{1-||H||} \left(\max_{0 \leq k \leq j-1} \sigma^k \right)$$

As the matrix P is symmetric, hence the matrix H. Using spectral norm. $|| H|| = \text{Max} \, |u_j|$,

where u_j are the eigen values of H (j = 1 to M-1)

The eigen values of matrix H are given by

$$\mu_j = (1 + \frac{\lambda}{2} \lambda j)^{-1} (1 - \frac{\lambda}{2} \lambda j)$$

Where λ_j are eigen values of P,

$$\& \quad \lambda_j = 4 \sin^2 \frac{j\pi}{2M} , \quad 1 \leq j \leq M-1.$$

$$\mu_j = \frac{1 - 2\lambda \sin^2 \frac{j\pi}{2M}}{1+2\lambda \sin^2 \frac{j\pi}{2M}} \quad \ldots .$$

Hence $||H|| = \max_j ||u_j|| < 1$, for $\lambda > 0$.

$$|| E^j|| \leq || E^o|| + \frac{1}{1-||H||} \left(\max_{o \leq k \leq j-1} || T^k|| \right)$$

we also have $|| T^k || = c(\beta^2+\alpha^2)$,

where c is a constant independent of β & α . Hence ,we have

$|| E^j|| \leq || E^o|| + c \, (\beta^2+ \alpha^2)$

Hence we conclude that there is conditional convergence as $\beta \to 0$, $\alpha \to 0$.