CHAPTER II

CHAPTER - II

SINGLE STEP METHODS :

2.1 Introduction :

A singlestep method for the solution of the initial value problem

$$\frac{dy}{dt} = f(t,y), \quad y(t_o) = y_o, \quad t \in [t_o,b] \qquad (2.1)$$

is a related first order differential equation. A general singlestep method may be written as

$$y_{j+1} = y_j + h\, g(t_j,y_j,h), j=0,1,2,\ldots,N-1. \qquad (2.2)$$

where $g(t,y,h)$ is a function of the arguments t,h,y and in addition depends on $f(t,y)$. The function $g(t,y,h)$ is called the increment function. If y_{j+1} can be determined simply by evaluting the right hand side of (2.2), then the singelstep method is called explicit, otherwise it is called implicit. The local truncation error T_j given by

$$T_j = y(t_{j+1})-y(t_j)-h\, g(t_j,y(t_j),h), \quad j=0,1,2,\ldots,N-1. \qquad (2.3)$$

The largest integer p such that $|h^{-1}T_j|=O(h^p)$ is called the order of the singelstep method.

Now we determine specific forms for the increament function $g(t,y,h)$.

Let the solution $y(t)$ of equation (2.1) be expanded in Taylor seris about the point t_j and substituting $t=t_{j+1}$ we have

$$y(t_{j+1}) = y(t_j) + hy'(t_j) + h^2/2!y''(t_j) + h^3/3!y'''(t_j) + \ldots$$

Neglecting the terms of h^2 and higher powers, we have the approximate solution

$$y_{j+1} = y_j + hf(t_j, y_j), \qquad j = 0,1,2,\ldots,N-1.$$

This is called Euler's method.

2.2 Runge-Kutta Methods

We first explain the principle involved in the Runge-Kutta methods. By the mean value theorem any solution of equation (2.1) satisfies

$$y(t_{j+1}) = y(t_j) + h\, y'(t_j + \theta h, y(t_j + \theta h)), \quad 0 < \theta < 1.$$

We put $\theta=1/2$. Euler's method with spacing $h/2$, approximately given by

$$y(t_j + h/2) = y_j + h/2\, f(t_j, y_j).$$

Thus we have the approximation

$$y_{j+1} = y_j + hf(t_j+h/2, y_j+h/2f(t_j,y_j)) \qquad (2.4)$$

Alternatively,again using Euler method,we proceed as follows:

$$y'(t_j+h/2) = 1/2\ [y'(t_j)+y'(t_j+h)]$$
$$= 1/2\ [f(t_j,y_j)+f(t_j+h,y_j+hf_j)]$$

Thus we have the approximation

$$y_{j+1} = y_j + h/2[f(t_j,y_j)+f(t_{j+1},y_j+hf(t_j,y_j)] \qquad (2.5)$$

Either Eq.(2.4) or Eq.(2.5) can be written as

$$y_{j+1} = y_j + h \text{ (averge slope)}$$

This is the underlying idea of the Runge-Kutta approach.

In general,we find the slope at t_j and at several other points,average these slopes,multiply by h and add the result to y_j.

Thus the R-K method with m slopes can be written as

$$k_1 = h\ f(t_j,y_j)$$
$$k_2 = h\ f(t_j+c_2h, y_j+a_{21}k_1)$$
$$k_3 = h\ f(t_j+c_3h, y_j+a_{31}k_1+a_{32}k_2)$$
$$k_4 = h\ f(t_j+c_4h, y_j+a_{41}k_1+a_{42}k_2+a_{43}k_3)$$

⋮

⋮

⋮

$$k_m = h\ f(t_j + c_m h, y_j + \sum_{i=1}^{m-1} a_{mi} k_i)$$

and

$$y_{j+1} = y_j + w_1 k_1 + w_2 k_2 + \ldots\ldots + w_m k_m \qquad (2.6)$$

To determine the parameters in (2.6), we expand y_{j+1} in powers of h such that it agrees with the Taylors series expansion of the solution of the differential equation upto a certain number of terms. We explain the same for specific values of m = 2 and m = 3 which follows:

2.2.1 Second order methods

Consider the Runge-Kutta method with two slops: (m=2)

$$k_1 = hf(t_j, y_j)$$
$$k_2 = hf(t_j + c_2 h, y_j + a_{21} k_1)$$
$$y_{j+1} = y_j + w_1 k_1 + w_2 k_2 \qquad (2.7)$$

where the parameters c_2, a_{21}, w_1, w_2 are chosen to make y_{j+1} closer to $y(t_{j+1})$. By Taylor series,

$$y(t_{j+1}) = y(t_j) + hf(t_j, y(t_j)) + h^2/2!\,f'(t_j, y(t_j)) +$$

$$h^3/3!\,f''(t_j, y(t_j)) +$$

$$= y(t_j) + hf(t_j,y(t_j)) + h^2/2!(f_t+ff_y)_{t_j}$$

$$+ h^3/3![f_{tt}+2ff_{ty}+f^2f_{yy}+f_y(f_t+ff_y)]_{t_j}+\ldots \qquad (2.8)$$

We also have

$$k_1 = hf_j$$

$$k_2 = hf(t_j+c_2h,y_j+a_{21}hf_j)$$

$$= h\left\{\left[f_j+c_2hf_t+a_{21}hff_y\right]_{t_j}+h^2/2\left(c_2^2f_{tt}+2c_2a_{21}ff_{ty}+a_{21}^2f^2f_{yy}\right)_{t_j}+ \\ +\ldots\ldots \right\}$$

Using k_1 and k_2 in Eq.(2.7) we get

$$y_{j+1} = y_j + w_1hf_j + w_2h[f_j+h(c_2f_t+a_{21}ff_y)]_{t_j}+$$

$$w_2h^3/2[c_2^2f_{tt}+2c_2a_{21}ff_{ty}+a_{21}^2f^2f_{yy}]_{t_j} +\ldots\ldots$$

$$= y_j + (w_1+w_2)hf_j +h^2(w_2c_2f_t+w_2a_{21}f_{(t_j)}f_y) +$$

$$1/2h^3w_2(c_2^2f_{tt}+2c_2a_{21}f_{(t_j)}f_{ty}+a_{21}^2f_{(t_j)}^2f_{yy}) +\ldots(2.9)$$

Comparing the equations (2.8) and (2.9) and equating the coefficients of powers of h,we have

$$w_1+w_2 = 1$$
$$w_2c_2 = 1/2$$
$$a_{21}w_2=1/2 \qquad (2.10)$$

By taking c_2 as an arbitrary (non-zero) and solving these equations, we have

$$a_{21} = c_2,\ w_2 = 1/2c_2,\ w_1 = 1 - 1/2c_2$$

Generally, c_2 is chosen between 0 and 1.

By taking c_2 =1/2, implies w_1= 0 , w_2 = 1, a_{21} = 1/2, this known as imprpved tangent R-K method..

By taking c_2 = 2/3 , implies w_1 = 1/4 , w_2 = 3/4, it is the optimal second order R-K method.

We state the Runge-Kutta method by giving the coefficients as follows:

c_2	a_{21}	
	w_1	w_2

1/2	1/2	
	0	1

Improved tangent

2/3	2/3	
	1/4	3/4

Optimal

1	1	
	1/2	1/2

Euler-Cauchy

2.2.2 Thired order method

Taking $m=3$ in Eq.(2.6), we have

$$y_{j+1} = y_j + w_1k_1 + w_2k_2 + w_3k_3 \qquad (2.11)$$

where $\qquad k_1 = hf_j$

$$k_2 = hf(t_j+c_2h, y_j+a_{21}k_1)$$

$$= hf(t_j+c_2h, y_j+a_{21}hf_j)$$

$$=h\ \{[f_j+c_2hf_t+a_{21}hff_y]_t + \ 1/2![c_2^2h^2f_{tt}+2c_2a_{21}h^2ff_{ty}$$

$$+a_{21}^2h^2f^2f_{yy}]_{t_j}+\ldots\}$$

$$=hf_j+h^2[c_2f_t+a_{21}f_yf_j]_{t_j}$$

$$+ h^3/2![c_2^2f_{tt}+2c_2a_{21}ff_{ty}+a_{21}^2f^2f_{yy}]_{t_j} + \ldots$$

$$k_3 = hf(t_j+c_3h, a_{31}k_1+a_{32}k_2+y_j)$$

$$= hf[t_j+c_3h, a_{31}hf_j + a_{32}(hf_j+h^2(c_2f_t+a_{21}f_yf_j)+\ldots)+y_j]$$

$$= hf_j + h^2[c_3f_t+(a_{31}+a_{32})f_jf_y] + h^3[c_2a_{32}f_tf_y+a_{21}a_{32}ff_y^2 +$$

$$1/2c_3^2f_{tt} + 1/2(a_{31}+a_{32})^2f_j^2\ f_{yy} + c_3(a_{31}+a_{32})ff_{ty}]+\ldots$$

Using these valuse in Eq.(2.11) we have

$$y_{j+1}= y_j +w_1hf_j+w_2(hf_j+h^2(c_2f_t+a_{21}f_jf_y)+$$

$$h^3/2(c_2^2\ f_{tt}+2c_2a_{21}f_jf_{ty}+a_{21}^2f_j^2f_{yy})) +$$

$$w_3(hf_j+h^2[c_3f_t+(a_{31}+a_{32})f_jf_y]+h^3[c_2a_{32}f_tf_y+a_{21}a_{32}ff_y^2+$$

$$c_3^2/2f_{tt}+1/2(a_{31}+a_{32})^2f_j^2f_{yy}+c_3(a_{31}+a_{32})f_jf_{ty}])+\ldots$$

$$y_{j+1} = y_j + (w_1+w_2+w_3)hf_j + h^2(w_2c_2+w_3c_3)f_t +$$

$$h^2(w_2a_{21}+w_3a_{31}+w_3a_{32})ff_y + h^3/2(w_2c_2^2+w_3c_3^2)f_{tt}+$$

$$h^3[w_2c_2a_{21}ff_{ty} + c_3w_3(a_{31}+a_{32})ff_{ty}]$$

$$+h^3[1/2a_{21}^2w_2+1/2(a_{31}+a_{32})^2w_3]f_j^2f_{yy}+w_3c_2a_{32}h^3f_tf_y$$

$$+w_3a_{21}a_{32}ff_y^2h^3+\ldots \qquad (2.12)$$

From Tayolar series, we have

$$y_{j+1} = y_j + hf_j + h^2/2!(f_t+f_jf_y)$$

$$+ h^3/3!(f_{tt}+2ff_{ty}+f^2f_{yy}+f_y(f_t+ff_y)) +\ldots. \qquad (2.13)$$

Comparing equations (2.12) and (2.13) for the coefficients of powers of h, we get

$w_1 + w_2 + w_3 = 1$ (I)

$w_2c_2 + w_3c_3 = 1/2$ (II)

$w_2c_2^2 + w_3c_3^2 = 1/3$ (III)

$w_2c_2a_{21} + w_3c_3a_{31} + c_3a_{32}w_3 = 1/3$ (IV)

$w_3(a_{31}+a_{32})^2 + a_{21}^2w_2 = 1/3$ (V)

$w_3c_2a_{32} = 1/6$

$w_3a_{21}a_{32} = 1/6 \Longrightarrow a_{21} = c_2$

Using in (IV),

$w_2a_{21}^2 + w_3c_3\,(a_{31}+a_{32}) = 1/3$

Using in (V), we have

$a_{31} + a_{32} = c_3$ (VI)

These gives the following six equations:

$a_{21} = c_2,$ $a_{31} + a_{32} = c_3,$

$w_1 + w_2 + w_3 = 1,$ $c_2w_2 + c_3w_3 = 1/2,$

$c_2^2w_2 + c_3^2w_3 = 1/3,$ $c_2a_{32}w_3 = 1/6,$ (2.14)

These equations are typical in R-K methods; the sum of a_{ij} in any row equals the corresponding c_i, and sum w_i's equals 1.

Thus the equations (2.14) are linear in w_2 and w_3 and have a solution for w_2 and w_3 if and only if

$$\begin{vmatrix} c_2 & c_3 & -1/2 \\ c_2^2 & c_3^2 & -1/3 \\ 0 & c_2 a_{32} & -1/6 \end{vmatrix} = 0$$

Therefore $c_2(2-3c_2)a_{32} - c_3(c_3-c_2) = 0,\ c_2 \neq 0$

(2.15)

$$a_{32} = \frac{c_3(c_3-c_2)}{c_2(2-3c_2)} ,$$

if $c_3 = 0$ or $c_2 = c_3$, then $c_2 = 2/3$ (For limiting case) and a_{32} is arbitrarily chosen (nonzero). By calculating w_i's and a_{ij}'s from equation (2.14) and represented in the following form:

$$\begin{array}{c|ccc} c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \hline & w_1 & w_2 & w_3 \end{array}$$

2/3	2/3		
2/3	0	2/3	
	2/8	3/8	3/8

Nystrom

1/2	1/2		
3/4	0	3/4	
	2/9	3/9	4/9

Nearly Optimal

1/2	1/2		
1	-1	2	
	1/6	4/6	1/6

Classical

1/3	1/3		
2/3	0	2/3	
	1/4	0	3/4

Hean

Fourth order R-K method : The most well known formula is classical R-K method of order four, which is given by

$y_{j+1} = y_j + 1/6(k_1+k_2+k_3+k_4)$,

where $k_1 = hf(t_j,y_j)$

$k_2 = hf(t_j+1/2h,y_j+1/2k_1)$,

$k_3 = hf(t_j+1/2h,y_j+1/2k_2)$,

$k_4 = hf(t_j+h,y_j+k_3)$.

The R-K method of order four is very much popular. It is a good choice for common purposes because it is quite accurate, stable and easy to program. It is not necessary to go to a higher order method because the increased accuracy is offset by additional computational effort. If more accuracy is required, then either a smaller step size or an adaptive method should be used.

2.2.3 Convergence

Definition (2.1):The singlestep method (2.2) is said to be regular if the function $g(t,y,h)$ is defined and continuous in the domain $t_0 \le t \le b$, $-\infty < y < \infty$, $0 \le h \le h_0$ and if there exists a constnt L such that

$$|g(t,y,h) - g(t,z,h)| \le L\ |y-z| \qquad (2.16)$$

for every $t \in [t_0,b]$, $y,z \in (-\infty,\infty)$, $h \in (0,h_0)$.

Definition 2.2: A singlestep method of the form(2.2) is said to be consistent if $g(t,y,0) = f(t,y)$.

Theorem 2.1: A necessary and sufficint condition for convergence of a regular singlestep method of order $p \ge 1$ is consistency.

We will use this theorem to state the convergen c e of Runge-Kutta methods.

(a) Convergence of second order R-K method:

The R-K method of order two be

$$y_{j+1} = y_j + w_1 k_1 + w_2 k_2 \qquad \text{(From (2.7))}$$

and the corresponding increament function is given by

$$g(t_j,y_j,h) = h^{-1}(w_1k_1+w_2k_2) \qquad (2.17)$$

As $f(t,y)$ satisfies Lipschitz condition (From Theorem 1.1).

Thus k_1 and k_2 satisfy

For $k_1 = hf(t_j,y_j)$,

$$|k_1 - k_1^*| = h\,|f(t_j,y_j) - f(t_j,y_j^*)|$$

$$\leq hL\,|y_j - y_j^*|$$

$$k_2 = hf(t_j+c_2h, y_j+a_{21}k_1)$$

$$|k_2 - k_2^*| = h\,|f(t_j+c_2h, y_j+a_{21}k_1) - f(t_j+c_2h, y_j^*+a_{21}k_1^*)|$$

$$\leq hL\,[|y_j+a_{21}k_1-y_j^*-a_{21}k_1^*|]$$

$$\leq hL\,[|y_j-y_j^*| + a_{21}\,|k_1-k_1^*|]$$

$$\leq hL\,[|y_j-y_j^*| + a_{21}Lh|y_j-y_j^*|]$$

$$\leq hL\,|y_j-y_j^*|\,(1+a_{21}Lh)$$

Using in (2.17), the increment function satisfies

$$|g(t_j,y_j,h)-g(t_j,y_j^*,h)|$$

$$= h^{-1}\,|w_1k_1+w_2k_2-w_1k_1^*-w_2k_2^*|$$

$$\leq h^{-1}\,[w_1\,|k_1-k_1^*| + w_2|k_2-k_2^*|]$$

$$\le h^{-1}\,[w_1Lh\mid y_j-y_j^*\mid + w_2Lh\mid y_j-y_j^*\mid(1+a_{21}Lh)]$$

$$\le L\mid y_j-y_j^*\mid[w_1+w_2+w_2a_{21}hL]$$

$$\le L\mid y_j-y_j^*\mid[1+Lh/2] \qquad \text{(From (2.10)}$$

The increment function g satisfies a Lipschitz condition in y and it is also continuous in h. Thus R-K method is consistent, hence it is convergent (by Theorem 2.1).

(b) Convergence Of Thired Order R-K Method

The R-K method of order three from (2.11),be

$$y_{j+1} = y_j+w_1k_1+w_2k_2+w_3k_3$$

and the correspondig increament function is given by

$$g(t_j,y_j,h) = h^{-1}(w_1k_1+w_2k_2+w_3k_3)$$

We know that f(t,y) satisfies Lipschitz condition hence k_1,k_2,k_3 also satify

$$k_1 = hf(t_j,y_j)$$

$$|k_1-k_1^*| \le hL|\,y_j-y_j^*\,|$$

$$k_2 = hf(t_j+c_2h, y_j+a_{21}k_1)$$

$$| k_2-k_2^* | \le Lh \left[| y_j-y_j^* | + a_{21} | k_1-k_1^* |\right]$$

$$\le Lh \left[| y_j-y_j^* | + a_{21}Lh| y_j-y_j^* |\right]$$

$$\le Lh | y_j-y_j^* | [1+a_{21}Lh]$$

$$k_3 = hf(t_j+c_3h, y_j+a_{31}k_1+a_{32}k_2)$$

$$| k_3-k_3^* | \le Lh \{| y_j-y_j^* | + a_{31}| k_1-k_1^* | + a_{32} | k_2-k_2^* |\}$$

$$\le Lh \{1+a_{31}Lh+a_{32}Lh(1+a_{21}Lh)\} |y_j-y_j|$$

Therefore the increament function satisfies

$$|g(t_j,y_j,h)-g(t_j,y_j^*,h)|$$

$$= h^{-1} | w_1k_1+w_2k_2+w_3k_3-w_1k_1-w_2k_2-w_3k_3^* |$$

$$\le h^{-1} \{w_1|k_1-k_1^*| +w_2|k_2-k_2^*| +w_3|k_3-k_3^*|\}$$

$$\le h^{-1} \{w_1Lh| y_j-y_j^* |+w_2Lh(1+a_{21}Lh)|y_j-y_j^*|$$

$$+Lhw_3(1+a_{31}Lh+a_{32}Lh(1+a_{21}Lh))|y_j-y_j^*|\}$$

$$\leq L \{w_1+w_2+w_3+(w_2a_{21}+w_3a_{31}+w_3a_{32})Lh + w_3a_{21}a_{32}(Lh)^2\} \; | y_j - y_j^* |$$

$$\leq L [1+1/2Lh+1/6(Lh)^2] \; |y_j - y_j^*| \qquad \text{(From2.14)}$$

Therefore the increment function g satisfies a Lipschitz condition in y and it is also continuous in h. Thus we conclude that the third order Runge-Kutta method is also convergent.

2.2.4 STABILITY ANALYSIS

Now we discuss the stability of R-K methods. First, consider the first order differential equation

$$y' = \lambda y, \qquad y(t_o) = y_o \tag{2.18}$$

where λ is a constant.

It has the exact solution, given by

$$y(t) = y(t_o)\, e^{\lambda(t-t_o)}$$

which at $t_j = t_o + jh$, becomes

$$y(t_j) = y(t_o)\, e^{\lambda jh} = y_o\, (e^{\lambda h})^j$$

But applying a singlestep method on equation (2.18), gives a difference equation with solution of the form

$$y_j = a\, [E(\lambda h)]^j$$

where a is a constant to be determined from initial condition and $E(\lambda h)$ is an approximation to $e^{\lambda h}$.

Definition(2.3):A singelstep method is absolutely stable if $|E(\lambda h)| \le 1$ and relatively stable if $|E(\lambda h)| \le e^{\lambda h}$.

We will apply the Euler-Cauchy second order R-K method (From 2.2.1) to equation(2.18) and we get

$k_1 = h\ f(t_j,y_j) = \lambda h\ y_j$

$k_2 = h\ f(t_j+h,y_j+k_1)$

$= \lambda h\ [1+\lambda h]\ y_j = [(\lambda h)+(\lambda h^2)]\ y_j$

$y_{j+1} = y_j + 1/2\ [w_1+w_2]$

$= [1+\lambda h+1/2(\lambda h)^2]\ y_j$

Thus, the growth factor for the second order method is

$$E(\lambda h) = 1 + \lambda h + 1/2(\lambda h)^2$$

and for this exact solution is $e^{\lambda h}$.

Also $e^{\lambda h} = 1 + \lambda h + (\lambda h)^2/2! + (\lambda h)^3/3!+...$

If $\lambda h>0$, then $|E(\lambda h)| \le e^{\lambda h}$; so that the second order R-K method is always relatively stable.

If $\lambda h < 0$, then consider the following table to find the interval of absolute stability.

λh	o	- 0.5	-1.0	-1.5	-2.0
$E(\lambda h)$	1	0.625	0.5	o.525	1

From the above table and Fig.(2.1). the interval of absolute stability is $-2 < \lambda h < 0$.

Similarly, apply the classical thired order R-K method to the equation(2.18)and we will get

$$k_1 = \lambda h y_j,$$

$$k_2 = h\ f(t_j+1/2h, y_j+1/2k_1)$$

$$= [\lambda h+1/2(\lambda h)^2]\ y_j$$

$$k_3 = h\ f(t_j+h, y_j-k_1+2k_2)$$

$$= [(\lambda h)+(\lambda h)^2+(\lambda h)^3]\ y_j$$

$$y_{j+1} = y_j + 1/6(k_1+4k_2+k_3)$$

$$= [1+\lambda h+1/2(\lambda h)^2+1/6(\lambda h)^3]\ y_j$$

with growth factor of thired order R-K method is

$$E(\lambda h) = 1+\lambda h+1/2(\lambda h)^2+1/6(\lambda h)^3.$$

If $\lambda h > 0$, then $E(\lambda h) \leq e^{\lambda h}$ hence the thired order R-K method is also relatively stable.

If $\lambda h < 0$, then from Fig.(2.1) the interval of absolute stability is $-2.5 < \lambda h < 0$.

Also apply on Eq.(2.18)the fourth order method, we have the growth factor

$E(\lambda h) = 1+\lambda h+1/2(\lambda h)^2+1/3!(\lambda h)^3+1/4!(\lambda h)^4$.

Therfore fourth order R-K method is also relatively stable and from Fig.(2.1), the interval of absolute stability is $-2.78 < \lambda h < 0$.

The existance of rounding errors in R-K methods, will depend in some way on the coefficints of the method. The negative signs appearing amongst the coefficients of the method, especially $w_1, w_2, \ldots, w_k$ is a sign of trouble. Many high order methods where negative signs occur have large values for some $|a_{ij}|$ and this will lead to loss of accuracy through cancelation of significant digits.

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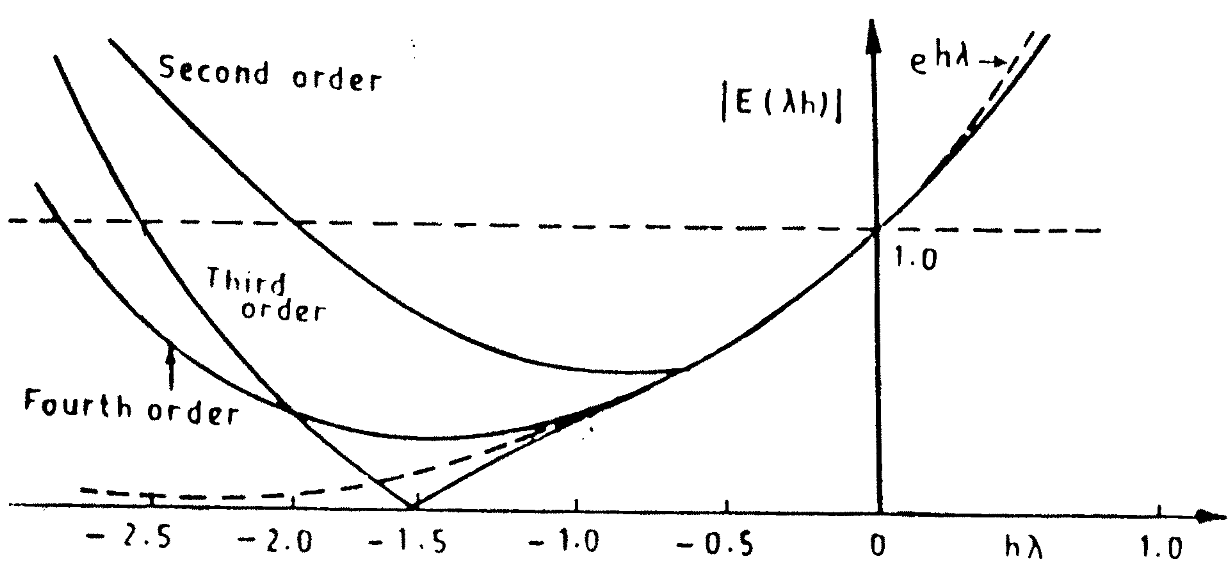

Fig. 2.1 Stability of Runge-Kutta method

2.3 Higher order differential equations:

The higher order differential equations can be solved by taking system of equivalent first order equations. Now we will form direct singelstep methods to solve higher order equations.

Consider a general second order equation

$$y'' = f(t,y,y'), \quad t \in [t_o,b] \tag{2.19}$$

with the initial conditions

$$y(t_o) = y_o, \; y'(t_o) = y_o'.$$

There is Runge-Kutta method, given by

$$k_1 = h^2/2! \; f(t_j,y_j,y_j'),$$

$$k_2 = h^2/2! f(t_j+2/3h, y_j+2/3hy_j'+2/3k_1, y_j'+4/3h\; k_1)$$

$$y_{j+1} = y_j + hy_j' + 1/2(k_1+k_2)$$

$$y'_{j+1} = y'_j + 1/2h\;(k_1+3k_2).$$

2.3.1: Here we dicuss only second order differential equation in which, the function f is independent of y'. We can construct the Runge-Kutta method in which the local truncation error in y and y′ is $o(h^4)$.

Consider the second order initial value differential equation

$$y'' = f(t,y),\ y(t_o) = y_o,\ y'(t_o) = y'_o. \tag{2.20}$$

Let us define

$$k_1 = h^2/2!\ f(t_j,y_j),$$

$$k_2 = h^2 1/2!\ f(t_j+a_2h, y_j+a_2hy'_j+a_{21}k_1)$$

$$y_{j+1} = y_j + hy'_j + w_1k_1 + w_2k_2,$$

$$y'_{j+1} = y'_j + 1/h(w_1^*k_1+w_2^*k_2). \tag{2.21}$$

The Taylor series expansion gives

$$y_{j+1} = y_j+hy'_j+ h^2/2!y''_j+ h^3/3!y'''_j+ h^4/4!y_j^{IV}+....$$

$$y'_{j+1} = y'_j+ hy''_j+ h^2/2!y'''_j+h^3/3!y_j^{IV}+.... \tag{2.22}$$

where $y''_j = f(t_j,y(t_j)) = f_j,$

$$y'''_j = (f_t+f_yy')_{t_j} = Df_j,\ \text{where}\ D \equiv \partial/\partial t+y'\partial/\partial y$$

$$y_j^{IV} = (f_{tt}+2y'f_{ty}+(y')^2f_{yy}+ff_y)_{t_j}$$

$$= D^2f_j + f_jf_y$$

Using in Eq.(2.22), we get

$$y_{j+1} = y_j + hy'_j + h^2/2!f_j + h^3/3!\ Df_j + h^4/4!(D^2f_j + f_jf_y) + \ldots$$

$$y'_{j+1} = y'_j + hf_j + h^2/2!(Df_j) + h^3/3!(D^2f_j + f_jf_y) + \ldots . \qquad (2.23)$$

Now ,

$$k_1 = h^2/2!\ f(t_j, y_j) = h^2/2!f_j,$$

$$2/h^2k_2 = f_j + h[a_2f_t + a_2y'_jf_j] +$$

$$h^2[a_{21}/2f_jf_y + a_2^2/2f_{tt} + a_2^2/2(y'_j)^2f_{yy} + a_2^2y'_jf_{ty}] + o(h^3)$$

$$k_2 = h^2/2\ f_j + h^3/2[a_2f_t + a_2y'_jf_j] +$$
$$h^4/4[a_{21}f_jf_y + a^2{}_2f_{tt} + a_2^2(y'_j)^2f_{yy} + 2a_2^2y'_jf_{ty}] + o(h^5)$$

$$= h^2/2\ f_j + h^3/2\ a_2Df_j + h^4/4[a_2^2D^2f_j + a_{21}f_jf_y] + o(h^5).$$

Using the values of k_1 & k_2 in Eq.(2.21), we have

$$y_{j+1} = y_j + hy'_j + \frac{h^2}{2}f_j(w_1 + w_2) + \frac{w_2a_2}{4}\ h^3(Df_j)$$
$$+ \frac{h^4}{4}(a_2^2 + w_2D^2f_j + w_2a_{21}f_jf_y) +$$

$$\& \ y'_{j+1} = + y'_j + \frac{h}{2}w_1^*f_j + w_2^*\frac{h}{2}f_j + \frac{v_2^*h^2a_2}{2}\ Df_j$$
$$+ \frac{h^3}{4}w_2^*(a_2^2D^2f_j + a_{21}\ f_jf_y) + \ldots$$

Comparing these equations with Eq.(2.23) for the coefficients of h, h^2 & h^3, we have

$$w_1 + w_2 = 1 \qquad\qquad w_1^* + w_2^* = 2$$

$$w_2 a_2 = \frac{1}{3} \qquad\qquad w_2^* a_2^2 = 2/3$$

$$w_2^* a_{21} = 2/3$$

$$w_2^* a_2 = 1$$

Solving these set of equations,we have

$$a_2 = 2/3, \qquad a_{21} = 4/9 \;, \qquad w_1 = w_2 = \frac{1}{2} \;,$$

$$w_1^* = 1/2 \;, \qquad w_2^* = 3/2$$

Thus finally the R-K method for second order initial value problem, Eq.(2.19) becomes

$$k_1 = \frac{h^2}{2!} f(t_j, y_j),$$

$$k_2 = \frac{h^2}{2!} (t_j + 2/3h, y_j + 2/3hy'_j + 4/9k_1)$$

and

$$y_{j+1} = y_j + hy'_j + 1/2(k_1 + k_2)$$

$$y'_{j+1} = y'_j + \frac{1}{2h} (k_1 + 3k_2) \qquad (2.24)$$

2.3.2 STABILITY ANALYSIS :

Now we will discuss the stability and the error analysis of the Range-Kutta Method (2.24).

Let us consider the differential equations

$$y'' = \lambda y \tag{2.25}$$

subject to the conditions

$$y(t_o) = y_o \quad , \; y'(t_o) = y'_o , \qquad t \in [t_o b]$$

where λ is a real number.

Here we discuss the three cases.

$$\lambda = 0, \qquad \lambda = -K^2 , \qquad \lambda = k^2$$

We will find the values of k_1, k_2 for the Eq. (2.25)

we get $\quad k_1 = \frac{h^2}{2!} f(t_j, y_j)$

$$k_1 = \frac{h^2}{2} \lambda y_j ,$$

$$k_2 = \frac{h^2}{2} [\lambda (y_j + 2/3hy'_j + 4/9h^2/2 \lambda \; y_j)]$$

$$= \frac{\lambda h^2}{2} [(1 + 2/9h^2\lambda) y_j + 2/3hy'_j]$$

Using these values of k_1 & k_2 into y_{j+1} & y'_{j+1}, we have

$$y_{j+1} = y_j + hy'_j + \frac{\lambda h^2}{4} [y_j + (1 + \frac{2\lambda h^2}{9}) y_j + 2/3h \; y'_j]$$

$$y_{j+1} = (1 + \frac{\lambda h^2}{2} + \frac{\lambda^2 h^4}{18}) y_j + (h + \frac{\lambda h^3}{6}) y'_j \tag{2.26a}$$

$$\& \; y'_{j+1} = y'_j + \frac{1}{2h} [\frac{h^2 \lambda}{2} y_j + \frac{3\lambda h^2}{2} [(1+2/9\lambda h^2) y_j + 2h/3 \; y'_j]$$

$$= y_j' + \frac{1}{2h}\left(\frac{\lambda h^2}{2} + \frac{3\lambda h^2}{2} + \frac{\lambda^2 h^4}{3}\right) y_j + 1/2\ \lambda h^2 y_j'$$

$$= \left(\frac{\lambda h}{4} + \frac{3\lambda h}{4} + \frac{\lambda^2 h^3}{6}\right) y_j + \left(1 + \frac{\lambda h^2}{2}\right) y_j'$$

$$y'_{j+1} = \left(\lambda h + \frac{\lambda^2 h^3}{6}\right) y_j + \left(1 + \frac{\lambda h^2}{2}\right) y_j' \qquad (2.26b)$$

Rewriting into matrix equation form,

$$\begin{bmatrix} y_{j+1} \\ y'_{j+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_j \\ y_j' \end{bmatrix} \qquad (2.27)$$

Where $a_{11} = 1 + \frac{\lambda h^2}{2} + \frac{\lambda^2 h^4}{18}$, $a_{12} = h + \frac{h^3\lambda}{6}$

$a_{21} = \lambda h + \frac{\lambda^2 h^3}{6}$, $a_{22} = 1 + \frac{\lambda h^2}{2}$.

(i) case $\lambda = 0$, we have

$$\begin{bmatrix} y_{j+1} \\ y'_{j+1} \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_j \\ y_j' \end{bmatrix}$$

$y_{j+1} = y_j + hy_j'$

& $y'_{j+1} = y_j'$

Putting j = 0,1,2,... in last equation, we have

$y'_1 = y'_0$, $y'_2 = y'_1 = y'_0$ & so on

$y'_j = y'_0$

and from first equation

$$y_1 = y_0 + hy'_0$$

$$y_2 = y_1 + hy'_1 = (y_0 + hy'_0) + hy'_0 = y_0 + 2hy'_0$$

and so on $y_j = y_0 + jhy'_0$

which is required result .

(ii) case $\lambda = -k^2$, in this case the solutions are oscillating.

let us consider the eigen values of the matrix represented by Eg.(2.27).

Let α be eigen value then characteristic equation of matrix be

$$\alpha^2 - (a_{11} + a_{22})\alpha - a_{12}\, a_{21} = 0$$

$$\alpha = \frac{1}{2}\,[a_{11} + a_{22} \pm \{ (a_{11} + a_{22})^2 + 4a_{12}a_{21}\}^{1/2}]$$

$$\alpha_1, \alpha_2 = \frac{1}{2}\,[(a_{11} + a_{22}) \pm \{(a_{11} + a_{22})^2 + 4a_{12}a_{21}\}^{1/2}]$$

Using $\lambda = -k^2$ in a_{ij} & substituting in above result, we have

$$\alpha_1, \alpha_2 = \frac{1}{2}\,\{2 - h^2k^2 + \frac{h^4k^4}{18} \pm [(\frac{hk}{18})^2(h^6k^6 - 36h^4k^4 + 432h^2k^2 - 1296)]^{1/2}\}$$

Taking $h^2k^2 = z$, $z^3 - 36z^2 + 432z - 1296$ has one root approximately 4.44045 by Newton's Method.

$$\alpha_1, \alpha_2 = \frac{1}{2}\{ 2-h^2k^2+\frac{h^4k^4}{18} \pm [(\frac{hk}{18})^2(h^2k^2 -4.44045)$$

$$(h^4k^4- 2P_1 \; h^2k^2+P_1^2+ P_2^2)]^{1/2}\}$$

Where $P_1 = 15.779763$ & $P_2 = 6.5467418$

Calculating α_1 & α_2 as functions of h^2k^2, we find that the roots have unit modulus for $0 \leq h^2k^2 \leq 4.44$. Thus stability interval of the R-K Method is

$$0 < h^2k^2 < 4.44.$$

(iii) case $\lambda = k^2$, the solution of (2.25) are exponential in nature and the solution canbe written in matrix form as.

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \cosh k(t-t_o) & \frac{1}{k}\sinh k\,(t-t_o) \\ \sinh k(t-t_o) & \cosh k\,(t-t_o) \end{bmatrix} \begin{bmatrix} y_o \\ y_o' \end{bmatrix}$$

For the point $t = t_o + nh = t_n$, this solution becomes

$$\begin{bmatrix} y(t_n) \\ y'(t_n) \end{bmatrix} = \begin{bmatrix} \cosh(nkh) & \frac{1}{k}\sinh(nkh) \\ \sinh(nkh) & \cosh(nkh) \end{bmatrix} \begin{bmatrix} y_o \\ y_o' \end{bmatrix}$$

The maximum eigen value of the matrix is obtained by the characteristic equation,

$$\cosh^2(nkh) - 2\cosh(nkh)\,\alpha + \alpha^2 - \sinh^2(nkh) = 0.$$

$$\alpha^2 - 2\cosh(nkh) + 1 = 0$$

$$\alpha_1,\ \alpha_2 = \frac{2\cosh(nkh) \pm [4\cosh^2(nkh) - 4]^{1/2}}{2}$$

$$\alpha_1, \alpha_2 = \cosh(nkh) \pm \sinh(nkh)$$

$$\alpha_1 = \cosh nkh + \sinh nkh = \frac{e^{nkh} + e^{-nkh}}{2} + \frac{e^{nkh} - e^{-nkh}}{2}$$

$$\alpha_1 = e^{nkh}$$

$$\& \quad \alpha_2 = e^{-nkh}$$

2.4 ADAPTIVE NUMERICAL METHODS :

The numerical methods which contain arbitrary parameters so as to tailor the numerical methods to fit the particular problems are known as adaptive numerical methods. This is used to stabilize the numerical methods. In this method the starting point is homogeneous linear form of the givendifferential equation and using the analytic solution of linear differential equation, obtain a difference equation which is identical to that of differential equation. Now we study the singlestep methods, how to stabilize with a small modification.

2.4.1 RUNGE-KUTTA-TREANOR METHOD :-

The first order initial value problem (2.1)

$$y' = f(t,y), \qquad y(t_o) = y_o$$

can be written in the form

$$y' + py = \emptyset(t,y) \qquad (2.28)$$

where $\emptyset(t,y) = f(t,y) + py$

and $p > 0$ is an arbitrary parameter to be obtain

It is assumed that Eq.(2.28) can be approximated by

$$\frac{dy}{dt} = f(t,y) = -P(y-y_j) + A + B(t-t_j) + \frac{c}{2}(t-t_j)^2 \qquad (2.29)$$

where (t_j,y_j) is contained in appropriate interval The four constants A,B,C & P can be evaluated by determining the value of f(t,y) at four points(t_i,y_i) in the interval

$[t_j, t_j+h]$ and solve the resulting equations.

We select classical Range -Kutta nodes t_j , $t_j+h/2$, $t_j+h/2$ and t_j+h and putting, we have

$$k_1 = hf(t_j, y_j)$$

$$k_2 = hf(t_j + h/2, z_1) , \quad z_1 = y_j + 1/2k_1$$

$$k_3 = hf(t_j+h/2, z_2) , \quad z_2 = y_j + 1/2k_2$$

$$k_4 = hf(t_j + h, z_3) , \quad z_3 = y_j + k_3$$

The four equations are

$$k_1 = Ah$$

$$k_2 + phz_1 = phy_j + Ah + 1/2h^2B + 1/8h^3c \qquad (2.30a)$$

$$k_3 + phz_2 = phy_j + Ah + 1/2h^2B + 1/8h^3c \qquad (2.30b)$$

$$k_4 + phz_3 = phy_j + Ah + h^2B + 1/2h^3c \qquad (2.30c)$$

Solving these equations :

$$hA = k_1$$

$$ph = - \left[\frac{k_3 - k_2}{z_2 - z_1}\right] \qquad \text{From(2.30a \& 2.30b)}$$

$$h^2B = [-3(k_1 + phy_j) + 2(k_2 + phz_1) + 2(k_3 + phz_2) - (k_4 + phz_3)]$$

$$h^3C = 4 [(k_1+phy_j) - (k_2+phz_1) - (k_3+phz_2) + (k_4+phz_3)] \qquad (2.31)$$

On integrating (2.29) between the limits t_j and t_{j+1} i.e. on solving Linear differential equation.

$$\left[y.e^{pt} \right]_{t_j}^{t_{j+1}} = (A + py_j) \left[\frac{e^{pt}}{p}\right]_{t_j}^{t_{j+1}} + B \left[(t-t_n)\frac{e^{pt}}{p} - \frac{e^{pt}}{p^2}\right]_{t_j}^{t_{j+1}} +$$

$$\frac{c}{2}\left[(t-t_n)^2 \frac{e^{pt}}{p} - 2(t-t_n)\frac{e^{pt}}{p^2} + \frac{2e^{pt}}{p^3}\right]_{t_j}^{t_{j+1}}$$

$$y_{j+1}\, e^{p(t_j+h)} - y_j\, e^{pt_j} = \frac{(A + py_j)}{p}\, e^{pt_j}(e^{ph}-1)$$

$$+ B\left[\frac{he^{p(t_j+h)}}{p} - e^{pt_j}\frac{(e^{ph}-1)}{p^2}\right]$$

$$+ \frac{c}{2}\left[\frac{h^2 e^{p(t_j+h)}}{p} - \frac{2h}{p^2}\, e^{p(t_j+h)} + \frac{2}{p^3}\, e^{pt_j}(e^{ph}-1)\right]$$

$$y_{j+1}\, e^{ph} - y_j = \left(\frac{A+py_j}{p}\right)(e^{ph}-1) + B\left(\frac{phe^{ph} - (p^{ph}-1)}{p^2}\right)$$

$$+ \frac{c}{2p^3}\left[h^2p^2e^{ph} - 2ph\, e^{ph} + 2(e^{ph}-1)\right]$$

$$y_{j+1} = y_j + hA\,\frac{(1-e^{-ph})}{p} + h^2B\,\frac{[ph-(1-e^{-ph})]}{(ph)^2}$$

$$+ \frac{h^3c}{2}\,\frac{[h^2p^2 - 2hp + 2(1-e^{-ph})]}{(ph)^3}$$

$$y_{j+1} = y_j + hA\, F_1 + h^2BF_2 + h^3cF_3 \qquad (2.32)$$

where $F_1 = \dfrac{e^{-ph}-1}{-(ph)}$, $F_2 = \dfrac{e^{-ph}-ph+1}{(-ph)^2}$

& $F_3 = \dfrac{e^{-ph}-1/2(ph)^2+ph-1}{(-ph)^3}$

$$F_{n+1} = \frac{F_n - \frac{1}{n!}}{(-ph)} , \qquad n = 3,4 \tag{2.33}$$

Using (2.31) in (2.32), we have

$$\begin{aligned} y_{j+1} &= y_j + k_1F_1 + F_2[-3(k_1 + phy_j) + 2(k_2+phz_1) \\ &\quad +2(k_3 + phz_2) - (k_4 + phz_3)] \\ &\quad +4F_3[(k_1+phy_j) - (k_2+ phz_1)-(k_3+ phz_3) + (k_4+phz_4)] \end{aligned} \tag{2.34}$$

which is known as Runge-Kutta-Treanor Method.

Using the values of z_1, z_2, z_3 from (2.30) into the equation (2.34), we have

$$\begin{aligned} y_{j+1} &= y_j + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ &\quad -(ph)^2 [(k_2 - k_3) F_3 + (k_1 - 4k_2 + 2k_3 + k_4) F_4 \\ &\quad -4 (k_1 - k_2 - k_3 + k_4) F_5] \end{aligned} \tag{2.35}$$

The value of P is given by

$$ph = -2 \left(\frac{k_3 - k_2}{k_2 - k_1}\right)$$

In the equation (2.35), the first part is due to the fourth order Runge-Kutta method and the remaining part is fifth order and higher in h. Thus when equation(2.34) is used to integrate over an interval where ph is small, the result will be identical with Runge-Kutta. If ph is large, then equation (2.34) gives a far superior solution.

As the value of P is evaluated by the difference of two values of k_i,i.e.the difference of two values of f_i,it may happens that the significant figures in calculation of p loses, if the change in k_i is very small. It may possible to get negative value of p. Thus in practical use the sign of p should be tested and if it is negative,it should be set zero, so that Eq.(2.34) reverts to R-K method.

2.4.2 RUNGE-KUTTA-NYSTROM-TREANOR METHOD :

Consider second order differential equations

$y'' = f(t,y), \quad y(t_o) = y_o, \quad y'(t_o) = y'_o.$

We develop singlestep methods of the form

$$y_{j+1} = y_j + hy'_j + h^2\phi_1(t_j,y_j,h),$$

$$y'_{j+1} = y'_j + h\phi_2(t_j,y_j,h) \qquad (2.36)$$

to obtain a numerical solution of the above differential equation. Here $\phi_1(t_j,y_j,h)$ and $\phi_2(t_j,y_j,h)$ are increment functions. Let us apply method (2.36) to the initial value problem

$y'' = \lambda y, \quad \lambda > 0, \; y(t_o) = y_o, \quad y'(t_o) = y_o'.$

and assume that it can be written in the form

$$\begin{bmatrix} y_{j+1} \\ h\, y'_{j+1} \end{bmatrix} = A(\sqrt{\lambda} h) \begin{bmatrix} y_j \\ h y'_j \end{bmatrix}$$

where $A(\sqrt{\lambda}h)$ is a 2 x 2 matrix.

Definition (2.4) Method (2.36) is said to have interval of periodicity $(0,H_0^2)$ if, for all $H^2 \in (0,H_0^2)$, $H = \sqrt{\lambda}\, h$, h being the step length, all the eigen values of $A(\sqrt{\lambda}h)$ are complex and lie on the unit circle.

Definition (2.5) The single step method defined by (2.36) is said to be p-stable if its interval of periodicity is $(0,\infty)$.

Let us consider adaptive numerical method for the initial value problem

$$y''=f(t,y), y(t_o) = y_o \;, \; y'(t_o) = y_o'$$

we write this in the form

$$y'' + py = g(t,y) \qquad (2.37)$$

where $g(t,y) = f(t,y) + py$ and $P>0$ is an arbitrary parameter to be obtained.

Let us write equation (2.37) as

$$y''= f(t,y) =-p(y-y_j) + A + B(t-t_j) + \frac{c}{2} (t-t_j)^2$$

denote

$$\phi(t) = py_j + A + B(t-t_j) + \frac{C}{2}(t-t_j)^2 \tag{2.38}$$

$$y'' + py = \phi(t) \tag{2.39}$$

where $\phi(t)$ is an approximation of $g(t,y)$. The general solution of (2.39) will consist of a complementary function and a par ticular integral.

herefore $y(t) = C.F. + P.I.$

As $P > 0$, $C.F = A \cos\sqrt{p}t + B \sin\sqrt{p}t$.

$$\& \ P.I = \frac{1}{D^2+P} \phi(t) \quad , \text{ Where } D \equiv \frac{d}{dt}$$

$$= \frac{1}{\sqrt{p}} \int_j^t \sin \sqrt{p}\,(t-J)\, \phi(J)\, dJ.$$

The complete solution is given by

$$y(t) = A \cos\sqrt{p}t + B \sin\sqrt{p}t$$

$$+ \frac{1}{\sqrt{p}} \int_{t_j}^{t} \sin\sqrt{p}(t-J)\, \phi(J) \tag{2.40}$$

Where A & B are arbitrary constants.

Differentiating (2.40) w.r.to t (using differentiation under integral sign)

$$y'(t) = -\sqrt{p} A\sin \sqrt{p}t + B \sqrt{p} \cos\sqrt{p}t.$$

$$+\frac{1}{\sqrt{P}} \int_{t_j}^{t} \sqrt{P} \cos \sqrt{P}(t-J)\ \phi(J)\ dJ + \frac{1}{\sqrt{P}} (t-t)\phi(t)(1)$$

$$y'(t) = -A\sqrt{P} \sin \sqrt{P}t + B\sqrt{P} \cos \sqrt{P}t$$

$$+\int_{t_j}^{t} \cos\sqrt{p}(t-J)\ \phi(J)\ dJ \qquad (2.41)$$

calculating the value of (2.40) at t_{j+1} , t_j and (2.41) at t_j and eliminating A and B from the resulting equations we obtain

$$y(t_{j+1}) = \cos\sqrt{P}h.y(t_j) + \frac{1}{\sqrt{P}} \sin\sqrt{P}h\ y'(t_j)$$

$$+\frac{1}{\sqrt{P}} \int_{t_j}^{t_{j+1}} \sin\sqrt{P}\ (t_{j+1}-J)\ \phi(J)dJ. \qquad (2.42)$$

calculating (2.40) at t_j and (2.41) at t_{j+1},t_j and eliminating A and B from the resulting equations, we obtain

$$y'(t_{j+1}) = -\sqrt{P} \sin \sqrt{P}h\ y(t_j) + \cos \sqrt{P}h\ y'(t_n)$$

$$+ \int_{t_j}^{t_{j+1}} \cos \sqrt{P}(t_{j+1}-J)\ \phi(J)\ dJ \qquad (2.43)$$

We know the values of y(t) and y'(t) at initial point $t = t_i$. Thus we can obtain singlestep methods for the numerical integration of Eq.(2.39), by replacing $\phi(J)$ in (2.42) & (2.43) of an appropriate interpolating polynomial at $t = t_j$

$$-\frac{B}{P\sqrt{P}}\sin\sqrt{P}h - \frac{C}{P^2}(1-\cos\sqrt{P}h$$

$$y_{j+1} = y_j\cos(w) + y'_j\, h\frac{\sin w}{w} + y_j + \frac{A}{P} + \frac{Bh}{P} + \frac{Ch^2}{2P} - y_j\cos w$$

$$-\frac{A}{P}\cos w - \frac{B}{P\sqrt{P}}\sin w - \frac{C}{p^2}(1-\cos w)$$

$$y_{j+1} = y_j + hy'_j\frac{\sin w}{w} + \frac{Ah^2}{w^2}(1-\cos w) + \frac{Bh^3}{w^2}\left[1-\frac{\sin w}{w}\right]$$

$$+ Ch^4\left[\frac{1}{2w^2} - \left[\frac{1-\cos w}{w^2}\right]\right]$$

$$y_{j+1} = y_j + hy'_jF_1 + Ah^2F_2 + Bh^3F_3 + ch^4F_4 \qquad (2.45)$$

Similarly, using Eq.(2.44) in Eq.(2.41), we have at $t=t_{j+1}$,

$$hy'_{j+1} = hy'_jF_o + Ah^2F_1 + Bh^3F_2 + ch^4F_3 \qquad (2.46)$$

Where $w = \sqrt{P}h$, $F_o = \cos w$, $F_1 = \frac{\sin w}{w}$,

$$w^2F_{m+2} = \frac{1}{m!} - F_m, \quad m = o,1,2,\ldots.$$

we approximate (2.38) $\phi(t,y)$ by Taycvr's series of degree 2 at $t = t_j$ & putting for $\phi(J)$ in (2.42) & (2.43).

The approximate polynimial is

$$\phi(J) = \phi(t_j) + (J - t_j)\, \phi'(t_j) + \frac{(J-t_j)^2}{2!}\, \phi''(t_j) + \ldots$$

$$\phi(J) = py_j + A + (J-t_j)\, B + \frac{(J-t_j)^2}{2!}\, C. \qquad (2.44)$$

Using Eq.(2.44) in Eq. (2.40), we have at $t = t_{j+1}$

$$y(t_{j+1}) = \cos\sqrt{P}h\; y(t_j) + \frac{1}{\sqrt{P}} \sin\sqrt{P}h\; y'(t_j) +$$

$$\frac{1}{\sqrt{P}} \int_{t_j}^{t_{j+1}} \sin\sqrt{P}(t_{j+1}-J) \left[py_j + A + B(J-t_j) + \frac{C}{2}(J-t_j) \right] dJ$$

Let $I = \frac{1}{\sqrt{P}} \int_{t_j}^{t_{j+1}} \sin \sqrt{P}(t_{j+1}-J)\; \phi(J)\; dJ$

$$= \frac{1}{\sqrt{P}} \Bigg[\left(py_j + A + B(J-t_j) + \frac{C}{2}(J-t_j)^2\right) \left(\frac{-\cos\sqrt{P}(t_{j+1}-J)}{-\sqrt{P}}\right.$$

$$\left. -(B + C(J-t_j)\right] \cdot \left[\frac{-\sin\sqrt{P}(t_{j+1}-J)}{(\sqrt{P})^2}\right] + C\left[\frac{\cos\sqrt{p}(t_{j+1}-J)}{-\sqrt{P}\;)^3}\right]\Bigg]_{t_j}^{t_{j+1}}$$

$$= \left[py_j + A + Bh + \frac{Ch^2}{2}\right] \frac{1}{P} - \frac{(py_j + A)}{P} \cos(h\sqrt{p})$$

⋮

STABILITY :

Applying the single step method developed above to the equation

$$y'' = -\lambda y, \quad \lambda > 0, \qquad y(t_o) = y_o, \; y'(t_o) = y'_o \tag{2.47}$$

when P is chosen as the square of the frequency of the solution of the linear homogeneous problem in (2.37), Here $P = \lambda$,

$$\begin{aligned} \phi_i &= f_i + py_i \\ &= -\lambda y_i + py_i \\ &= 0, \quad \text{for } i = 1,2,3,4. \end{aligned}$$

and so comparing with Eq.(2,44), we have

$$\begin{aligned} y_{j+1} &= y_j \cos w + hy'_j \frac{\sin w}{w} \\ hy'_{j+1} &= -y_j w \sin w + hy'_j \cos w \end{aligned}$$

which can be written as

$$\begin{bmatrix} y_j \\ hy'_{j+1} \end{bmatrix} = \begin{bmatrix} \cos w & \frac{\sin w}{w} \\ -w \sin w & \cos w \end{bmatrix} \begin{bmatrix} y_j \\ y'_j \end{bmatrix}$$

The characteristic equation of the method is

$$\alpha^2 - 2 \cos w \, \alpha + 1 = 0$$

The roots are

$$\alpha = e^{\pm iv}$$

$$|\alpha| = 1$$

The eigen values are complex and of unit modulii and hence by the definition (2.5), the method is P-stable.