CHAPTER III

CHAPTER- III

MULTISTEP METHODS :

3.1 INTRODUCTION :

For solving differential equation if the value of $y(t)$ at $t = t_{j+1}$ uses the values of dependent variable and it's derivative at more than one grid or mesh points, then the numerical methods are called multistep methods.

consider the first order differential equation

$$y' = f(t,y), y(t_0) = y_0 \ , \ t \in [t_0 b].. \qquad ..(3.1)$$

Set the general multistep or P-step method for the solution of (3.1) as

$$y_{j+1} = c_1 y_j + c_2 y_{j-1} + \dots + c_p y_{j-p+1} + hg(t_{j+1}, t_j, \dots, t_{j-p+1}, y'_{j+1}, y'_j, \dots \ y'_{j-p+1} ; h) \qquad \dots(3.2)$$

Where h is constant stepsize and $c_1, c_2, \dots, c_p$ are real given constants. Also we know the approximate values of y and y' at the points $t_i = t_0 + ih$, $\qquad i = 0,1,2,,\dots.j.$

If g is independent of y'_{j+1} , then the general multistep method is called an explicit, open or predictor method otherwise an implicit, closed or corrector method.

The truncation or discretization error of the method (3.2) at $t = t_j$ is given by

$$T(y(t_j),h) = y(t_{j+1}) - c_1 y(t_j) \ldots - c_p y(t_{j-p+1})$$

$$-hg\,(t_{j+1}, t_j, \ldots t_{j-p+1}, y'(t_{j+1}), \ldots y'(t_{j-p+1}))] \qquad (3.3)$$

If P is the largest integer such that

$$\left| h^{-1} T(y(t_j),h) \right| = o(h^P),$$

then p is said to be the order of the general multistep method.

We will discuss the general linear multistep method of(3.2) which is given by

$$y_{j+1} = c_1 y_j + c_2 y_{j-1} + \qquad + c_k y_{j-k+1}$$
$$+h(b_o y'_{j+1} + b_1 y'_j + \qquad + b_k y'_{j-k+1}) \qquad (3.4)$$

The constants c_i's & b_j's are real and known. where $y_1, y_2, \ldots y_{k-1}$ are obtained by using singlestep methods to start the Eq.(3.4).

3.2 EXPLICIT MULTISTEP METHODS :

Consider, the integration of differential Eq. $y' = f(t,y)$ between the limits t_{j-k} and t_{j+1}, we get

$$y(t_{j+1}) = y(t_{j-k}) + \int_{t_{j-k}}^{t_{j+1}} f(t,y)dt \qquad (3.5)$$

For integration, consider a polynomial obtained by interpolating $f(t,y)$ at n points $t_j, t_{j-1}, \ldots\ldots, t_{j-n+1}$. For this, we will use Newton Backward difference formula of degree k-1.

If $f(t,y)$ has n continuous derivatives, then we have from integration formula.

$$P_{n-1}(t) = f_j + (t-t_j)\frac{\nabla f_i}{h} + \frac{(t-t_j)(t-t_{j-1})}{2!}\frac{\nabla^2 f_j}{h^2}$$

$$+\frac{(t-t_j)\ (t-t_{j-1})\ldots\ldots(t-t_{j-n+2})}{(n-1)!}\ \frac{\nabla^{n-1} f_j}{h^{n-1}}$$

$$+\frac{(t-t_j)\ (t-t_{j-1})\ldots.(t-t_{j-n+2})}{n!}\ \nabla^{(n)} f(\xi) \qquad (3.6)$$

Where n^{th} derivative of f at some ξ in an interval, $t \in (t_{j-n+1}, t_j)$ is $f^{(n)}(\xi)$.

Using $x = \dfrac{t-t_j}{h}$ in (3.6), we have

$$P_{n-1}\ (t_j+hx) = f_j + x\nabla f_j + \frac{x(x+1)}{2!}\nabla^2 f_j + \ldots..$$

$$+\ \frac{x(x+1)\ldots(x+n-2)}{(n-1)!}\ \nabla^{n-1} f_j$$

$$+\frac{x(x+1)\ldots.(x+n-1)}{n!}\ h^n f^{(n)}(\xi)$$

$$= \sum_{m=0}^{n-1} (-1)^m \binom{-x}{m} \nabla^m f_j + (-1)^n \binom{-x}{n} h^n f^{(n)}(\xi) \qquad (3.7)$$

where $\binom{-x}{m} = (-1)^m \dfrac{x(x+1)\ldots.(x+m-1)}{m!}$.

Substituting (3.7) into (3.5) and putting dt = h dx.

t	t_{j-k}	t_{j+1}
x	-k	1

$$y(t_{j+1}) = y(t_{j-k}) + h\int_{-k}^{1} \left[\sum_{m=0}^{n-1} (-1)^m \binom{-x}{m} \nabla^m f_j \right.$$

$$\left. + (-1)^n \binom{-x}{n} h^n f^{(k)}(\xi) \right] dx$$

$$= y(t_{j-k}) + h \sum_{m=0}^{n-1} \alpha_{(m)}^{(k)} \nabla^m f_j + E_n^{(k)} \qquad (3.8)$$

where $\alpha_m^{(k)} = \int_{-k}^{1} (-1)^m \binom{-x}{m} dx\ldots\ldots$ (3.9)

$$E_n^{(k)} = h^{n+1} \int_{-k}^{1} (-1)^n \binom{-x}{n} f^{(n)}(\xi)\, dx.$$

If we neglect the remainder term $E_n^{(k)}$ from(3.8), then we have

$$y_{j+1} = y_{j-k} + h \sum_{m=0}^{n-1} \alpha_m^{(k)} \nabla^m f_j \qquad (3.10)$$

$$\alpha_m^{(k)} = \int_{-k}^{1} (-1)^m \binom{-x}{m} dx$$

Let us calculate $\alpha_m^{(k)}$ for m = 0,1,2,3,4,

Putting m = 0 in Eq.(3.9), we have

$$\alpha_0^{(k)} = \int_{-k}^{1} dx = 1 + k$$

For m= 1,

$$\alpha_1^{(k)} = \int_{-k}^{1} (-1)(-x)\, dx = \frac{1}{2} (1-k^2)$$

For m = 2,

$$\alpha_2^{(k)} = \int_{-k}^{1} (-1)^2 \binom{-x}{2} dx$$

$$= \int_{-k}^{1} \frac{-x\ (-x-1)}{2} dx$$

$$= \frac{1}{12} (5-3k^2+2k^3)$$

For m = 3,

$$\alpha_3^{(k)} = \int_{-k}^{1} (-1)^3 \binom{-x}{3} dx$$

$$= \int_{-k}^{1} (-1)(-1)^3 \frac{x(x+1)(x+2)}{3!} dx$$

$$= \frac{1}{24} (3-k)\ (3+k-k^2+k^3).$$

For m = 4

$$\alpha_4^{(k)} = \int_{-k}^{1} \frac{x(x+1)(x+2)(x+3)}{4!} dx$$

$$=\frac{1}{720}\left[251-90k^2+110k^3-45k^4+6k^5\right] \qquad (3.11)$$

From the Eq.(3.8),we calculate n values and thus we can find explicit multistep methods of order n. Here the truncation error is of the form ch^{n+1}, where c is indepedent of h. Let us obtain some different formula giving values for k.

3.2.1 ADAMS-BASHFORTH FORMULA (k=0).

Using the coefficients $\alpha_m^{(o)}$ from(3.11) into (3.10), we get,

From (3.11)

$$\alpha_0^{(o)} = 1,\ \alpha_1^{(o)} = \frac{1}{2},\ \alpha_2^{(o)} = \frac{5}{12},\ \alpha_3^{(o)} = \frac{3}{8},$$

$$\alpha_4^{(o)} = \frac{251}{720}.$$

$$\therefore y_{j+1} = y_j + h\sum_{m=0}^{(n-1)} \alpha_m^{(o)}\nabla^m f_j$$

$$= y_j + h\left[f_j + \frac{1}{2}\nabla f_j + \frac{5}{12}\nabla^2 f_j + \frac{3}{8}\nabla^3 f_j + \frac{251}{70}\nabla^4 f_j + \ldots\right].$$

The error term associated with truncation after $(n-1)^{th}$ ∇ is

$$E_n^{(o)} = h^{n+1}\int_0^1 (-1)^n\binom{-x}{n} f^{(n)}(\xi)\, dx$$

Since the coefficient of $f^{(n)}(\xi)$ unchanged it's sign in (0,1) ,thus we can write

$$E_n^{(0)} = \alpha_k^{(0)} h^{k+1} f^{(k)}(\xi).$$

Taking n=3, we have third order Adams-Bashforth method as

$$y_{j+1} = y_j + h\left[f_j + \frac{1}{2} \nabla f_j + \frac{5}{12} \nabla^2 f_j \right], \quad j \geq 2$$

$$= y_j + h\left[f_j + \frac{1}{2} (f_j - f_{j-1}) + \frac{5}{12} [f_j - 2f_{j-1} + f_{j-2}]\right.$$

$$= y_j + \frac{h}{12} \left[23f_j - 16f_{j-1} + 5f_{j-2}\right] \ldots \qquad (3.12)$$

To solve the differential Eq. ,

We require the values of $y_0, y_1, y_2 \ldots\ldots$ These values are obtained by using the singlestep method of order 3.

The error is given by

$$E_3^{(0)} = \alpha_3^{(3)} h^4 f'''(\xi)$$

$$= \frac{3}{8} h^4 f'''(\xi).$$

3.2.2 NYSTRYON FORMULAS (k = 1)

Putting k=1 in (3.11), we have $\alpha_0^{(1)} = 2$, $\alpha_1^{(1)} = 0$,

$$\alpha_2^{(1)} = \frac{1}{3}, \quad \alpha_3^{(1)} = \frac{1}{3}, \quad \alpha_4^{(1)} = \frac{29}{90}, \quad \alpha_5^{(1)} = \frac{14}{45}$$

Using these values in Eq.(3.10), we get

$$y_{j+1} = y_j + h \left[2f_j + \frac{1}{3} \nabla^2 f_j + \frac{1}{3} \nabla^3 f_j + \frac{29}{90} \nabla^4 f_j + \frac{14}{45} \nabla^5 f_j + \ldots \right]$$

The general linear multistep methods can be expressed in the form

$$y_{j+1} = c_1 y_j + c_2 y_{j-1} + \ldots + c_k y_{j-k+1} + h (d_0 y'_{j+1} + d_1 y'_j + \ldots + d_k y'_{j-k+1})$$

$$\text{or } y_{j+1} = \sum_{i=1}^{k} c_i y_{j-i+1} + h \sum_{i=0}^{k} d_i y'_{j-i+1} \qquad (3.13)$$

Let us denote

$$\rho(\xi) = \xi^k - c_1 \xi^{k-1} - c_2 \xi^{k-2} - \ldots - c_k$$

$$\& \ \sigma(\xi) = d_0 \xi^k + d_1 \xi^{k-1} + \ldots + d_k$$

using these symbols the Eq.(3.12) can be written as

$$\rho(x) \ y_{j-k+1} - h \ \sigma(x) \ y'_{j-k+1} = 0$$

If we know the values of $y(t)$ & $y'(t)$ for succcessive k values of t, then generalised form (3.12) can be used.

The method (3.13) is said to be explicit or predictor formula, if $d_o = 0$. In this case only y_{j+1} exists on Left hand side of the equation and y_{j+1} can be calculated directly from right hand side.

Also the method (3.13) is said to be emplicit or corrector formula, if $d_o \neq 0$. In this y_{j+1} is depend on y'_{j+1}.

Here we assume that the polynomials $\rho(x)$ & $\sigma(x)$ have no common factors.

Let us define the difference operators L associated with th difference equation (3.12) as

$$L\ \{y(t),h\} = y(t_{j+1}) - \sum_{i=1}^{k} c_i y(t_{j-i+1}) - h \sum_{i=0}^{k} d_i y'(t_{j-i}) \quad (3.14)$$

Let us assume that the function y(t) has continuous deriva tive of sufficiently high order. Using Taylor series expansion to $y(t_{j-i+1})$ and $y'(t_{j-i+1})$, we have

$$y(t_{j-i+1}) = y(t_j) + (1-i)hy'(t_j) + \frac{(1-i)^2}{2!}y''(t_j) + \ldots\ldots$$

$$+ \frac{(1-i)^s}{s!} h^s y^{(s)}(t_j)$$

$$+ \frac{1}{s!} \int_{t_j}^{t_{j-i+1}} (t_{j-i+1}-u)^s y^{(s+1)}(u)\ du$$

$$y'(t_{j-i+1}) = y'(t_j) + (1-i)hy''(t_j) + \ldots$$

$$+ \frac{(1-i)^{s-1}}{(s-i)!} h^{s-1} y^{(s)}(t_j)$$

$$+ \frac{1}{(s-1)!} \int_{t_j}^{t_{j-i+1}} (t_{j-i+1}-u)^{s-1} y^{(s+1)}(u)du.$$

Using in (3.14), we have

$$L\{y(t),h\} = A_o y(t_j) + A_1 hy'(t_j) + A_2 h^2 y''(t_j) + \ldots$$

$$+ A_s h^s y^{(s)}(t_j) + d_n \ \ldots \qquad (3.15)$$

Where $A_o = 1 - \sum_{i=1}^{k} c_i$

$$A_p = \frac{1}{p!}\left[1 - \sum c_i (1-i)^p\right] - \frac{1}{(p-1)!} \sum d_i (1-i)^{p-1},$$

$$p = 1,2,\ldots,s.$$

and $$d_n = \frac{1}{s!}\left[\int_{t_j}^{t_{j+1}} (t_{j+1}-u)^s y^{(s+1)}(u)\ du\right.$$

$$-\sum c_i \int_{t_j}^{t_{j+1}} (t_{j-i+1}-u)^s y^{(s+1)}(u)\ du$$

$$-hs\int_{t_j}^{t_{j+1}} d_o(t_{j+1}-u)^{s-1}y^{(s+1)}(u)\ du.$$

$$-hs\sum d_i \int_{t_i}^{t_{j-i+1}} (t_{j-i+1}-u)^{s-1}y^{(s+1)}(u)\ du.\Big]$$

3.2.3 STABILITY ANALYSIS :

We will discuss the stability and convergence of th. order Adam-Bashforth method. For stability apply the th. order method (3.12) to the initial value differential Eq^n

$$y' = \lambda y, \qquad y(t_o) = y_o , \qquad t \in [t_o b].$$

Neglecting the round off error, we get

$$y_{j+1} = y_j + \frac{\lambda h}{12}[23y_j - 16y_{j-1} + 5y_{j-2}] \ldots\ldots (3.16)$$

The true solution will satify

$$y(t_{j+1}) = y(t_j) + \frac{\lambda h}{12}[23y(t_j) - 16y(t_{j-1}) + 5y(t_{j-2})] + 4L_j \qquad (3.17)$$

where L_j is the local truncation errors From (3.16) &(3.17) & using $\epsilon_j = y_j - y(t_j)$ we get

$$\epsilon_{j+1} = \epsilon_j + \frac{\lambda h}{12}[23\epsilon_j - 16\epsilon_{j-1} + 5\epsilon_{j-2}] - L_j \qquad (3.18)$$

This gives an inhomogeneous third order linear difference equation with constant coefficient. The solution of this equation consists of a particular solution and a linear combination of the three independent solutions of the homogeneous equations with $L_j = 0$ in (3.18)

The homogeneous equation is

$$\epsilon_{j+1} = \epsilon_j + \frac{\lambda h}{12}\left(23\epsilon_j - 16\epsilon_{j-1} + 5\epsilon_{j-2}\right) \qquad (3.19)$$

We consider the solution of the Eq.(3.19) of the form $\epsilon_j = A\,\xi^j$, Where $A \neq 0$ and ξ is an arbitrary number have to be determined by using this solution in (3.19).

$$A\xi^{j+1} = A\xi^j + \frac{\lambda h}{12}\left(23A\xi^j - 16A\xi^{j-1} + 5A\xi^{j-2}\right)$$

$$\xi = 1 + \frac{\alpha}{12}\left(23 - 16\xi^{-1} + 5\xi^{-2}\right), \qquad \text{where}, \alpha = \lambda h$$

$$\xi^3 = \xi^2 + \frac{\alpha}{12}\left(23\xi^2 - 16\xi + 5\right)$$

$$\xi^3 = (1 + \frac{23\alpha}{12})\xi^2 + \frac{4\alpha}{3}\xi - \frac{5}{12}\alpha = 0 \qquad (3.20)$$

Let the roots of the equation (3.20) are ξ_1, ξ_2,& ξ_3(distinc

Then the solution of the difference equation (3.18) is of the form

$$a_1\xi_1^j + a_2\xi_2^j + \alpha_3\xi_3^j$$

Now to find particular solution of inhomogenous equation (3.19), we take $L_j=L$, a constant. Thus the particular solution is L/α.

Hence the general solution of difference equation (3.19) with distinct roots gives

$$\epsilon_j = a_1\xi_1^j + a_2\xi_2^j + a_3\xi_3^j + L/\alpha \qquad (3.21)$$

Here a_1, a_2, a_3 are arbitrary constants & are obtained from initial errors.

For stability $|\epsilon_j| < \infty$ as $j \to \infty$ and if $|\xi_i| > 1$, the error $|\epsilon_j|$ increases unboundedly.

Definition : A linear multistep method, is said to be strongly stable if $|\xi_i| < 1$ for $i \neq 1$, and it is said to be absolutely stable if

$$|\xi_i| \leq 1 \ , \qquad i = 1,2,\ldots,k.$$

The region in λL plane, vhere the method is absolutely stable is called the region of absolutely stable. The largest $|\xi_i|$ of the Eq.(3.20) is shown in Fig (3.1)

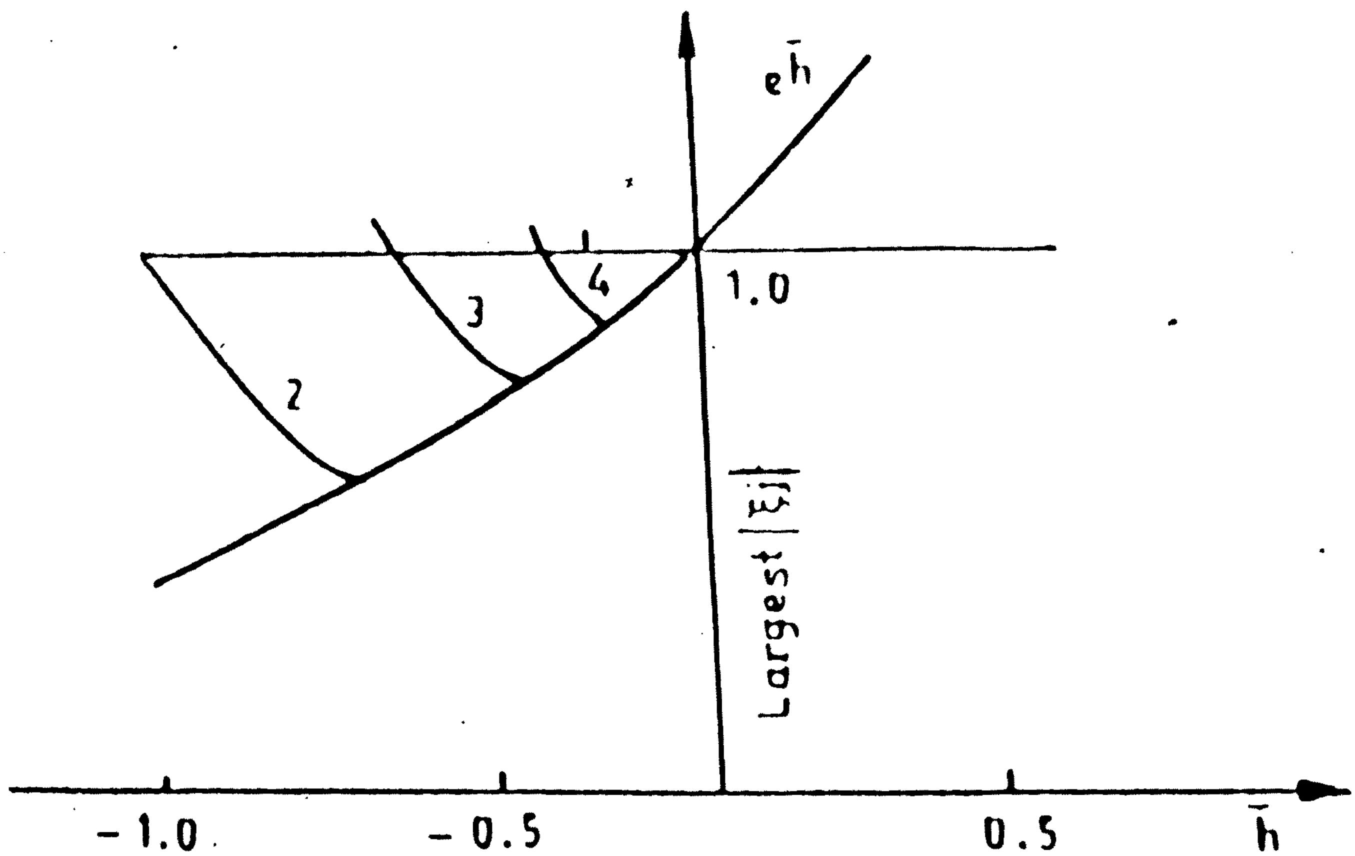

Fig. 3.1 Dominant root in Adams-Bashforth methods

The interval of Adams-Bashforth absolute stability is given below :

k	1	2	3	4	5
(β,0)	-2	-1.33	-0.55	-0.3	-0.2

Convergence of Adam-Bashforth third order method :

Let us obtain the constants a_0, a_1, a_2 For, consider

$$B_j = \epsilon_j - L/\alpha \,, \qquad j = 0,1,2.$$

The constants a_1, a_2, a_3 be obtained by

$$B_0 = a_1 + a_2 + a_3$$

$$B_1 = a_1\xi_1 + a_2\xi_2 + a_3\xi_3$$

$$B_2 = a_1\xi_1^2 + a_2\xi_2^2 + a_3\xi_3^2$$

Assume that the initial errors $\epsilon_0, \epsilon_1, \epsilon_2$ are constant and equa to ϵ .

$$a_1 + a_2 + a_3 = \epsilon - L/\alpha$$

$$a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = \epsilon - L/\alpha$$

$$a_1\xi_1^2 + a_2\xi_2^2 + a_3\xi_3^2 = \epsilon - L/\alpha$$

solving these three equations for a_1, a_2, a_3 (using matrix method) we have

$$a_1 = (\epsilon - L/\alpha)\,\frac{(1-\xi_3)(1-\xi_2)}{(\xi_1-\xi_3)(\xi_1-\xi_2)}$$

$$a_2 = \left(\epsilon - \frac{L}{\alpha}\right)\frac{(1-\xi_1)(1-\xi_3)}{(\xi_1-\xi_2)(\xi_2-\xi_3)}$$

$$a_3 = (\epsilon - \frac{L}{\alpha}) \frac{(1-\xi_1)(1-\xi_2)}{(\xi_1-\xi_3)(\xi_2-\xi_3)}$$

Using these values in Eq.(3.21),we have

$$\epsilon_j = (\epsilon - \frac{L}{\alpha}) \left[\frac{(1-\xi_2)(1-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} \xi_1^j - \frac{(1-\xi_1)(1-\xi_3)}{(\xi_1-\xi_2)(\xi_2-\xi_3)} \xi_2^j \right.$$

$$\left. + \frac{(1-\xi_1)(1-\xi_2)}{(\xi_1-\xi_3)(\xi_2-\xi_3)} \xi_3^j \right] + \frac{L}{\alpha} \qquad (3.22)$$

If $h \longrightarrow 0$, $\xi_1 \longrightarrow 1$, ξ_2 ,ξ_3 approach to zero, the method is stable.

If $|\lambda h|$ is small, ξ_1 behaves like $e^{\lambda h}$ and ξ_2, ξ_3 are less than one.

Then Eq^n (3.22) can be written as

$$\epsilon_j \leq (\epsilon - \frac{L}{\alpha}) e^{\lambda jh} + \frac{L}{\alpha}$$

$$\text{or } \epsilon_j \leq \epsilon e^{\lambda(t_j-t_o)} + \frac{L}{\alpha} (1-e^{\lambda(t_j-t_o)}) \qquad (3.23)$$

by taking $|\epsilon| = 0$, $| L| \leq \frac{3}{8} h^4 M_4$,

$$M_4 = \max_{(t_{j-2},t_{j+1})} |y^{(4)}(\xi)|$$

Then Eq.(3.23) becoms

$$\epsilon_j \leq \frac{3h^3}{8} \frac{M_4}{\lambda} (1-e^{\lambda(t_j - t_o)})$$

This shows that $|\epsilon_j|$ -> 0 as h -> 0 like ch^3.

Rutishauser observed in his famous paper that high order and a small local error are not sufficient for a useful multistep method. The numerical solution can be 'unstable', even though the stepsize h is taken very small.

3.3 EXTRAPOLATION METHODS:

3.3.1 Approximation of truncation error:

To adjust the step size we required the local discretization (or truncation) errors of the solutions at each step. A method to calculate discretization error is called extrapolation or Richardson's extrapolation. Here we are ignoring the rounding arror. Theorem(3.4.1): If $f(t,y)$ is sufficiently differentiable, and if P is order of the numerical method, then ϵ_j will satisfy

$$\epsilon_j = h^P \psi(t_j) + o(h^{P+1}) \qquad (3.24)$$

where $\psi(t_j)$ is the solution of the initial value problem

$$\psi'(t) = f_y(t,y(t))\,\psi(t) - \frac{1}{(p+1)!} y^{(p+1)}(t), \qquad \psi(t_0) = 0.$$

Let us denote $\epsilon_j = y_j(h)-y(t_j)$ where $y_j(h)$ be the approximation to $y(t_j)$ at $t=h$; then

$$\epsilon_j = y_j(h)-y(t_j) = h^P\psi(t_j) + o(h^{(p+1)}) \qquad (3.25)$$

For $h/2$ we can write

$$y_j(h/2)-y_j(t) = (h/2)^P \psi(t_j) + o(h^{p+1}) \qquad (3.26)$$

Thus on subtracting (3.26) and (3.25), we have

$$y_j(h) - y_j(h/2) = [\, h^P-(h/2)^P \,]\, \psi(t_j) + o(h^{P+1})$$

$$= \left(1 - \frac{1}{2^P} \right) h^P \psi(t_j) + o(h^{P+1}) \qquad (3.27)$$

From equations (3.24) and (3.27), we have

$$y_j(h) - y_j(h/2) = \left(1 - \frac{1}{2^P} \right) \epsilon_j + o(h^{P+1})$$

$$\epsilon_j = \frac{2^P}{2^P - 1} [\, y_j(h) - y_j(h/2) \,] + o(h^{P+1}) \qquad (3.28)$$

Hence, we find the Richardson extrapolation to the true solution at t_j, from (3.28) and using

$$\epsilon_j = y_j(h) - y(t_j), \quad y(t_j)$$

$$= y_j(h) - \frac{2^P}{2^P - 1} [\, y_j(h) - y_j(h/2) \,] + o(h^{P+1}) \; y(t)$$

$$= \frac{2^P y_j(h/2) - y_j(h)}{2^P - 1} + o(h^{P+1}) \qquad (3.29)$$

This is known as Richardson extrapolation to the true solution at $t=t_j$. From the equations (3.28) and (3.29), we express the accumulated truncation error at t_j as

$$d_j = \frac{2^P}{2^P - 1} [\, y_j(h) - y_j(h/2) \,]$$

and actual error in the extrapolated solution E_j as

$$E_j = \frac{2^p y_j(h/2) - y_j(h)}{2^p - 1} - y(t_j)$$

This error helps us to decide whether the chosen step h is right, too big or too small.

3.3.2 Richardson Extrapolation :

Consider initial value differential equation

$$y' = f(t,y), \qquad t \in [t_0, b].$$

Let y(t) be true solution and y(t,h) be approximate solution, using any suitable numerical method with step size h. Thus y(t,h) contain some error. Let y(t,h) have asymptotic expansion in h of the following form

$$y(t,h) = y(t) + \alpha_1 h + \alpha_2 h^2 + \alpha_3 h^3 + \ldots \qquad (3.30)$$

Evaluate y(t,h) for $h_0 > h_1 > h_2 > ..$ and eliminating α_1, α_2, $\alpha_3, \ldots$, we get

$$y(t,h_0) = y(t) + \alpha_1 h_0 + \alpha_2 h_0^2 + \alpha_3 h_0^3 + \ldots.$$

$$y(t,h_1) = y(t) + \alpha_1 h_1 + \alpha_2 h_1^2 + \alpha_3 h_1^3 + \ldots.$$

$$y(t,h_2) = y(t) + \alpha_1 h_2 + \alpha_2 h_2^2 + \alpha_3 h_2^3 + \ldots.$$

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Eliminating α_1 from these equations, we have

$$\frac{h_0 y(t,h_1) - h_1 y(t,h_0)}{h_0 - h_1} = y(t) - h_0 h_1 \alpha_2 - h_0 h_1 (h_0 + h_1) \alpha_3 - \ldots$$

$$\frac{h_1 y(t,h_2)-h_2 y(t,h_1)}{h_1 - h_2} = y(t) - h_1 h_2 \alpha_2 - h_1 h_2 (h_1+h_2)\alpha_3 - \dots$$

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$$\frac{h_{i-1} y(t,h_i)-h_i y(t,h_{i-1})}{h_{i-1}-h_i} = y(t) - h_{i-1} h_i \alpha_2$$

$$- h_{i-1} h_i (h_{i-1}+h_i)\alpha_3 - \dots$$

Here we use the notation

$$P_i^{(j)} = \frac{h_j P_{i-1}^{(j+1)} - h_{i+j} P_{i-1}^{(j)}}{h_i - h_{i+j}} ,$$

$$\text{and } P_i^{(o)} = y(t,h_i). \qquad (3.31)$$

This calculatin of $P_i^{(j)}$ can be simplified by using the following traingular array,

P-scheme

$$\begin{array}{llll}
P_0^{(0)} & P_1^{(0)} & & \\
 & & P_2^{(0)} & \\
P_0^{(1)} & P_1^{(1)} & & P_3^{(0)} \\
 & & P_2^{(1)} & \\
P_0^{(2)} & P_1^{(2)} & & \\
P_0^{(3)} & & &
\end{array}$$

The entries in the table other than first columan are computed by

$P_{i-1}^{(j)}$

$P_{i}^{(j)}$

$P_{i-1}^{(j+1)}$

Putting $h_i = h_o(1/2)^k$ in equation (3.31), we have

$$P_i^{(j)} = \frac{2^i P_{i-1}^{(j+1)} \pm P_{i-1}^{(j)}}{2^i - 1} \qquad (3.32)$$

From (3.31), we have each $P_i^{(j)}$ is a linear combination o- $y(t,h_k)$, $k = j, j+1, \ldots, j+i$. This combination can be written as

$$P_i^{(j)} = \sum_{k=o}^{i} a_{i,i-k} P_o^{(j+k)} \qquad (3.33)$$

where $a_{i,i-k}$ are constant coefficients.Using this in (3.31), we have

$$\sum a_{i,i-k} P_o^{(j+k)} = \frac{h_j \sum_{o}^{i} a_{i-1,i-1-k} P_o^{(j+1+k)} - h_{i+j} \sum_{o}^{i} a_{i-1,i-1-k} P_o^{(j+k)}}{h_j - h_{i+j}}$$

Comparing both sides, we get

$$a_{i,i-k} = \frac{h_j a_{i-1} - h_{i+j} a_{i-1,i-1-k}}{h_j - h_{j+i}},$$

$$a_{m-1,m} = a_{m-1,-1} = 0 .$$

The successes of extrapolation algorithm (3.31) depends on existance of the series expansion (3.30). The error in the extrapolation algorithm (3.31) is given by

$$y(t) - P_i^{(j)} = h_i h_{i+1} h_{i+2} \ldots h_{i+j} E_i^{(j)} .$$

The coefficients $E_i^{(j)}$ in the remainder term can be expessed in many ways as a divided difference or as

$$\frac{y^{(j+1)}(\xi^{(j)})}{(j+1)}$$

where $\xi_i^{(j)}$ is contained in $(0, \max.(h_i, \ldots, h_{i+j}))$.
The sequance of $\{h_i\}$ proposed by Bulirsch and Stoer(1964) are

$$\{h_o, h_o/2, h_o/3, h_o/4, h_o/6, h_o/8, h_o/12, \ldots\}.$$

As it leads to a stable algorithm and cheaper to compute. most of numerical integration methods for ordinary differential equations have an error expansion of the form (3.30). In the next section, we will present Euler extrapolation method.

3.3.3 EULAR EXTRAPOLATION METHOD:

Using Euler's method, the approximate value $y_j(h)$ is obtained as

$$y_{j+1} = y_j + hf(t_j, y_j), \qquad j = 0,1,2,\ldots$$

From (3.30), the approximate value $y_j(h)$ to $y(t_j)$ has the asymptotic expansion of the form

$$y(t_j,h) = y(t_j) + \alpha_1(t_j)h + \alpha_2(t_j)h^2 + \ldots.$$

Using step lengths $h_o, h_o/2, h_o/2^2, \ldots, h_o/2^m$ and generate $P_o^{(m)}$. WE take $t_{j+1} - t_j = h_o$ and obtain y_{j+1} with step length h_o and denote it by $P_o^{(O)}$, i.e. $P_o^{(O)} = y_j + h_o f_j$.
Next put $h_1 = h_O/2^2$, so that applying Euler's method four times and so on, for $h_m = h_o/2^m$ we use Euler's method 2^m times to obtain $P_o^{(m)}$.

Let us apply this procedure to the initial value differential equation,

$$y' = \lambda y, \qquad y(t_o) = y_o.$$

We have

$$P_o^{(O)} = y_j + h_o f_j = y_o + \lambda h_o y_j = (1+\lambda h_o)y_j$$

$$P_o^{(1)} = y(t_{j+1},h_1) = (1 + \lambda h_o/2)y_j, \text{ as } h = h_o/2$$

.

.

$$P_o^{(m)} = (1 + \frac{\lambda h_o}{2^m})y_j \tag{3.34}$$

After calculating the first column $P_o^{(m)}$ of P-scheme, other columns are obtained with the help of (3.32).

Using this procedure the P-table is easily generated column by column. Thus a convenient convergence

test is applied and that adjucent elements is a column ac
within some prescribed tolerance ϵ

$$\left| \frac{P_i^{(j)} - P_i^{(j-1)}}{P_i^{(j-1)}} \right| \leq \epsilon$$

When this cnvergence exists, then $P_i^{(j)}$ is used as the v
y_{j+1} and the procedure is repeated to obtain y_{j+2}.

This Euler algorithm has the advantages that it
simple recursion relation and it gives automatic control
accuracy. This algorithm is carried out at each step ir
integration, thus it is based on local extrapolation.

3.3.4 STABILITY ANALYSIS:

The column of P-table is generated by equation (3

$$P_o^{(m)} = \left(1 + \frac{\lambda h_o}{2^m} \right)^{2^m} y_j$$

and other columns are obtained by the relation (3.32), w
can be expressed as a linear combination of the first co
elements,

$$P_i^{(j)} = \sum_{k=0}^{j} a_{i,i-k} \; P_c^{(j+k)}$$

$$= \left(\sum a \, (1 + \frac{\lambda h_o}{2^{j+k}})^{2^{j+k}} \right) y_j \qquad (3.35)$$

The coeffcients satisfies the relations

$$a_{i,i-k} = \frac{2^{i}\, a_{i-1,i-k} - a_{i-1,i-1-k}}{2^{i}-1}$$

$$a_{i-1,i} = a_{i-1,-1} = 0 \qquad (3.36)$$

If for some $i=I$ and $j=J$, the extrapolated value $P_I^{(J)}$ is as y_{j+1}, then we have

$$y_{j+1} = E[\lambda h_o, I, J]\, y_j$$

where $E(\lambda h_o, I, J) y_j$ is the characteristic root, given by

$$E(\lambda h_o, I, J) = \sum_{k=o}^{J} a \left(1 + \frac{\lambda h_o}{2^{J+k}}\right)^{2^{J+k}}$$

This Euler extrapolation method is absolutely stable if

$$| E(\lambda h_o, I, J) | \le 1.$$

The principle diagonal of P-scheme converges faster than other diagonal or column and so we find the interval absolute stability for I=0 and different values of J. H for J=2, we have

$$P_o^{(0)} = p_o^{(0)}$$
$$P_1^{(0)} = -P_o^{(0)} + 2P_o^{(1)}$$
$$P_2^{(0)} = 1/3P^{(0)}_o - 2P_o^{(1)} + 8/3P_o^{(2)}$$

[From (3.35) & (3.36) putting I=0 and J=2]

Thus $P_1^{(0)} = \left[-(1+\lambda h_o)+2(1+\lambda h_o/2)^2\right]y_j$

$$= \left(1+\lambda h_o+\lambda^2 h_o^2/2\right)y_j$$

which gives the explicit second order R-K method. N

consider J=2

$$P_2^{(0)} = 1/3 \left[P_o^{(0)} - 6P_o^{(1)} + 8P_o^{(2)}\right]$$

$$= 1/3 \left[(1+\lambda h_o)-6(1+\lambda h_o/2)^2+8(1+\lambda h_o/4)^4\right]y_j$$

$$= \left[1+\lambda h_o+(\lambda h_o)^2/2+(\lambda h_o)^3/6+(\lambda h_o)^4/96 \right] y_j$$

Thus the characteristic root is given by

$$E(\lambda h_o,0,2) = 1+\lambda h_o+(\lambda h_o)^2/2+(\lambda h_o)^3/6+(\lambda h_o)^4/96.$$

which gives the stability interval

$-2.785 < \lambda h_o < 0.$

As the number of extrapolations is icreased, the ove all algorithm becomes more stable. In the limit of a infinite number of extrapolations the method approche A-stability.

In extrapolation method the order (column c P-scheme), step size (i.e. basic step size h_o) and number c stages (rows of P-scheme)are all variables. Thu implementation contain a choice of increasing the order increasing the

number of stages or decreasing the stepsize to achi required accuracy. The first implementation (Bulirch a Stoer 1966) which was developed on a machine with 40 t matissa, limited the order to 6. After procedures (F 1971, Gear 1971) also use this limit as a standared.[H_2]

Thus we conclude that, the extrapolation algorithms gives good estimates of the local error and are exteme flexible with regard to variation of the step h_o.

3.4 COMPARISON OF METHODS :

The numerical methods for solving initial value di ferential equations fall into three parts (i) One-ste methods, (ii) multistep methods and (iii) extrapolatio methods.

All these methods are allowed a change of stepleng in each integration step. The modern multistep methods extrapolation methods are not applied for fixed orders. extrapolation methods,we can increase the order easily attaching another column to the table value of extrapolati values. RAnge-Kutta-Fehlberg type one-step methods are tie to a fixed order. Extrapolation methods have the leas amount of overhead tim . The relability of extrapolatic method is some what high, but for modest accuracy requiremer they are too expesive.

For multistep methods, least amount of computatic is measured in evaluation of the right hand side of th differential equation. In a predictor method the right han side of differential equation must be evaluated only once pe step, where as in corrector method this number is equal t the number of iterations. The expense caused by the ste control in multistep methods can be a truble. Multiste methods have the largest amount of overhead time. The multistep methods are cheapest in direct costs, but force th store of (k-1) previous results and complicate step changing, so their indirect costs are very high.

The evidence from Enright et. al.(1974) is that the divided difference form is the most efficient way to use the Adams methods.

At present R-K methods are the most efficient in terms of total time, unless derivative evaluations are expen sive. The fourth order R-K method has the advantage of simp licity and is the most efficient method for many routine low accuracy calculations.

Of the other methods those of Englad for p=4 are cheapest and should be used for that case, otherwise embedd ing method should be used, unless it is found that the extra reliability of extrapolation methods outweighs their extra cost in a particular application.