Chapter 2.

WEAK NOETHER LATTICES

2. WEAK NOETHER LATTICES

§ 1. INTRODUCTION.

In this chapter we study the concept of weak Noether lattices and some equivalent condition for.

The concept of weak Noether lattices is introduced by D. D. Anderson and C. Jayaram in [3] in 1993, of course, as a generalization of the concept of Noether lattices. Using this concept we study some equivalent conditions for a weak Noether lattice to be a principal element lattice. Before proceeding further let us familiar with some more basic facts, those we needed in the development of the study. We define some concept.

Definition 2.1. Factor of an element. [3]

Let L be a multiplicative lattice. An element $x \in L$ is said to be a factor of an element $a \in L$, if there exists an element $y \in L$ such that $a = xy$.

Now we prove an important result.

Lemma 2.2. Let x be a principal element of L and 0:x = 0. Then any factor of x is a principal element of L. [3]

Proof : Assume that, m is a factor of x. Then x = mk, for some k $\in$ L.

We first show that, m is a join principal element.

Let a, b $\in$ L. As (am$\vee$b):m $\leq$ (am$\vee$b):m, we have [(am$\vee$b):m]m $\leq$ am$\vee$b. So we have [(am$\vee$b):m]x = [(am$\vee$b):m]mk $\leq$ (am$\vee$b)k = amk$\vee$bk = ax$\vee$bk.

As [(am$\vee$b):m]x = [(amb):m]x, we have by (vii) of property 1.7, (am$\vee$b):m = [(am$\vee$b):m] x:x $\leq$ (ax$\vee$bk):x. As Lis a join principal element, we have (am$\vee$b):m $\leq$ (ax$\vee$bk):x = a$\vee$(bk):x = a$\vee$bk:mk, as x = mk.

Before proceeding further, we first prove that k is a weak join principal element. For that we need to prove:

Claim 1. For any y $\in$ L, y = yk:k

Let z $\leq$ y. Then zk $\leq$ yk and hence z $\leq$ yk:k. This gives that y $\leq$ yk:k.

Conversely, let z $\leq$ yk:k. Then we have zk $\leq$ yk. That is zmk $\leq$ ymk and hence zx $\leq$ yx. That is z $\leq$ yx:x = y$\vee$0:x, as x is a join principal element. Thus as 0:x = 0, we have z $\leq$ y.

This implies that, y = yk:k.

But as x = mk $\leq$ k and 0:x = 0, we have by property 1.7 that 0:k $\leq$ 0:x = 0 and hence y$\vee$0:k = y$\vee$0 = y = yk:k. Thus k is a weak join principal element.

Now we claim the following.

Claim 2. bk:mk = b:m.

Let $z \leq$ b:m. Then zm $\leq$ b and hence zmk $\leq$ bk. This gives that $z \leq$ bk:mk. Therefore b:m $\leq$ bk:mk.

Now let $z \leq$ bk:mk. Then zmk $\leq$ bk. As k is a weak join principal element, we have zm $\leq$ bk:k = $b \vee 0$:k = $b \vee 0$ = b and hence $z \leq$ b:m. Thus, bk:mk = b:m.

Consequently, we now have $(am \vee b):m \leq a \vee (bk:mk) = a \vee b:m$. But, trivially, $a \vee b:m \leq (am \vee b):m$. Thus, $(am \vee b):m = a \vee b:m$. This implies that, m is a join principal element.

Now we show that, m is a meet principal element.

Let a, b $\in$ L, Then by property 1.7 we have $(a \wedge mb)k \leq ak \wedge mkb$ = $ak \wedge xb \leq [(ak:x) \wedge b]x = [(ak:x) \wedge b]mk$. Thus $a \wedge mk \leq [(ak:x) \wedge b]mk:k$ = $[(ak:x) \wedge b]m$, by above claim (1).

Thus, by claim (2) we have $a \wedge mb \leq [ak:mk \wedge b]m = [(a:m) \wedge b]m$.

Obviously, we have the converse. Hence, we get $a \wedge mb = (a:m \wedge b)m$.

Q. E. D.

Now we turn a generalisation of the concept of Noether lattices, by dropping one axiom from the definition of Noether lattice.

§ 2. r-LATTICES.

Now here follows the special type of compactly generated multiplicative lattice, called r-lattice, which just an abstraction of the lattice of ideals of a commutative ring.

Definition 2.3. r-lattice. [1]

A multiplicative lattice L is called an r-lattice, if it is modular, principally generated, compactly generated with 1 compact.

The concept of r-lattices was first introduced by D. D. Anderson [1] in 1976 as an extension of the concept of Noether lattices. In fact, a Noether lattice is just an r-lattice satisfying the ascending chain condition (ACC).

According to [1], if R is a commutative ring, then the lattice of ideals of R, L(R) is an r-lattice. More generally, if R is a graded ring over a torsionless grading monoid, then $L_G(R)$, the lattice of graded ideals of R, is an r-lattice. In the same paper [1], it is proved further that, L(S), the lattice of ideals of an r-semigroup is also an r-lattice.

An important property of r-lattices is that they can be localized at sub-multiplicatively closed sets. The concept of localization was first introduced in multiplicative lattice by R. P. Dilworth [9] using primary decomposition.

We first study this concept.

Definition 2.4 : Multiplicatively Closed Set. [1]

Let L be a multiplicative lattice and S be a nonempty subset of L. Then S is said to be a multiplicatively closed set, if $ab \in S$, for every pair $a, b \in S$.

Definition 2.5 : Sub-multiplicatively Closed Set. [1]

Let L be a multiplicative lattice and S be a nonempty subset of L. Then S is said to be a multiplicatively closed set, if for every pair $a, b \in S$, there exists $c \in S$ such that $c \leq ab$.

Thus every multiplicatively closed set is a sub-muliplicatively closed set. Now we turn to the concept of localization.

Definition 2.6 : The Concept of Localization in r-lattices. [1]

Let L be an r-lattice and S be a sub-multiplicatively closed set of L. Define $a \leq b\,(S)$ for $a, b \in S$, if for every principal element $x \leq a$, there exists $t \in S$ such that $tx \leq b$.

Further, $a = b(S)$ iff $a \leq b\,(S)$ and $b \leq a\,(S)$.

According to D. D. Anderson [1], using the fact that S is a sub-multiplicatively closed (and hence consists of compact elements), the relation "$=(S)$" is an equivalence relation.

Thus for $a \in L$, define $a_S = \{ z \in L / z = a(S) \}$ and

$L_S = \{ a_S / a \in L \}$ is the set of all equivalence classes of elements of L.

Furthermore, L_S is a partially order set with the partial order relation:

$$a_S \leq b_S \text{ iff } a \leq b(S).$$

Now we study the concept of weak r-lattice which is a generalisation of the concept of r-lattice, obviously.

Definition 2.7 : Weak r-lattice. [2]

A multiplicative lattice L is said to be a weak r-lattice, if it is principally generated, compactly generated and has the greatest element 1 compact.

Thus a weak r-lattice is an r-lattice iff it is a modular lattice.

If N is the semiring of all non-negative integers under usual addition and multiplication, then the lattice L(N) of ideals of N is a weak r-lattice. But it is not an r-lattice, as L(N) is not a modular lattice (See [2]).

In [1], the concept of r-lattice is strongly introduced and studied well. The results established therein are mainly concerned with r-lattices. Note that, the difference between r-lattices and weak r-lattices is nothing but "modularity". In fact, many of the results proved in [1] don't require the condition of modularity and thus they can be carried over without any change to weak r-lattice. This fact is pointed out by D. D. Anderson himself with C. Jayaram in [1].

Thus, in this study these results will be called for directly to weak r-lattices without referring this fact.

Accordingly, the process of localization as given in section 2 of [1] (here, see section 2.6) for r-lattices is applicable without no harm to weak r-lattices which is used freely here in this study.

Throughout this chapter L represents a weak r-lattices unless otherwise stated.

Thus, now we recall the following result from [1]. Also we recall some basic concepts whenever we require.

Proposition 2.8 : Let S be a sub-multiplicatevely closed subset of L. Then we have,

(i) for any set $\{a_\alpha\} \subseteq L$, $(\vee_\alpha\ a_\alpha)_S = \vee_\alpha\ a_{\alpha S}$

(ii) $(a_1 \wedge ...... \wedge a_n)_S = a_{1S} \wedge .... \wedge a_{nS}$, for any finite subset of L.

(iii) the product $a_S b_S = (ab)_S$ makes L_S a multiplicative lattice.

(iv) for a, c $\in$ L with c compact, $(a{:}c)_S = a_S{:}c_S$.

(v) m principal in L implies m_S principal in L_S.

(vi) a is compact in L implies a_S is compact in L_S.

Now we recall the following concepts and then proceed further.

Definition 2.9 : Proper Element. [7]

An element a of a lattice with 1 is said to be a proper element, if $a \neq 1$.

Definition 2.10 : Maximal Element. [7]

A proper element m of a lattice is said to be maximal, if it is not contained properly in any another proper element.

Thus whenever $m \leq x$, we have either $x = m$ or $x = 1$.

Result 2.11 : If S is a multiplicative lattice with 1 compact, then S contains maximal elements. [1]

Theorem 2.12 : Let $a, b \in L$. Then $a = b$ iff $a_m = b_m$, for every maximal element m of L. [1]

Definition 2.13 : Prime Element.

A proper element p of a multiplicative lattice S is said to be a prime element of S, if for $a, b \in L$, $ab \leq p$ implies either $a \leq p$ or $b \leq p$. [9]

Obviously, every prime ideal of a commutative ring R is a prime element of L(R), the lattice of ideals of R.

In the example 1.34, the elements d and e are prime elements.

Evidently, we now have the following result which is abstract version of a very famous result: Every maximal ideal is a prime ideal in a commutative ring R with 1.

Result 2.14 : Every maximal element is a prime element, in a multiplicative lattice with 1 compact. [9]

Now analogous to the theorem 2.12 immediately we have the following result that we just note here.

Theorem 2.15 : Let $a, b \in L$. Then $a = b$ iff $a_p = b_p$, for every prime element p of L. [1]

Obviously, converse need not be true. Since nonmaximal prime ideals of a commutative ring R with unity are nonmaximal prime elements of the lattice L(R).

Now we recall a very important result that is useful extensively in the theory of multiplicative lattices. This result forms a natural abstraction of the result : Every principal ideal of a commutative ring R is compact. This fact is well discussed in the examples 1.20 and 1.9.

Result 2.16 : Let L be a multiplicative lattice in which 1 is compact. Suppose a is a weak principal element of L. Then $a = \vee_{\alpha} a_{\alpha}$ implies that $a = a_{\alpha_1} \vee \ldots\ldots \vee a_{\alpha_n}$, for some finite subset $\{\alpha_1, \ldots\ldots, \alpha_n\}$. [1]

This result implies directly the following fact that:

Result 2.17 : If L is compactly generated multiplicative lattice with 1 compact, then every weak principal element and hence every principal element is compact in L. [1]

All these basic results lead us to the following important result.

Lemma 2.18 : An element $a \in L$ is principal iff a is compact and a_p is principal in L_p, for every prime element p of L. [3]

Proof : Assume that, a is a principal element. Then by result 2.17, a is a principal element. Hence by proposition 2.8, for any x, we have $(x:a)_p = x_p:a_p$. Consequently, by theorem 2.15, it immediately follows that, a satisfies the meet and join principal laws in L iff a_p satisfies the same in L_p, for every prime element p of L. **Q. E. D.**

Now we need to recall another result from [1] to prove a next result. This is the only simplest result which gives characterization of weak meet and weak join principal elements. Interestingly, under quite mild hypothesis weak principal elements are principal elements. First recall some basic concepts.

Definition 2.19 : Interval.

In a lattice L, if $a, b \in L$ such that $a < b$, the *interval* a *to* b is the set $a/b = [a, b] = \{ x \in L / a \leq x \leq b \}$. [7]

Definition 2.20 : Sublattice L/a. [9]

For any lattice L, the set $L/a = \{ x \in L / x \geq a \} = [a, 1]$, is a lattice with respect to the binary operations defined on L and hence it is a *sublattice* of L.

Definition 2.21 : Multiplicative Lattice L/a.

For a multiplicative lattice L, the lattice L/a is a *multiplicative* lattice with multiplication $x \circ y = xy \vee a$. [9]

If R is commutative ring with unity, then the lattice L = L(R) is a multiplicative lattice. Furthermore, if I is an ideal of the ring R, then the lattice of the quotient ring R/I is nothing but the quotient lattice L/I. Now we the following result.

Proposition 2.22 : Let S be a multiplicative lattice and $a \in L$. Then

(i) a is a meet (join) principal element implies a is a weak meet (join) principal element.

(ii) a is a weak meet principal element iff $x \leq a$ implies $x = ay$, for some $y \in L$.

(iii) a is a weak join principal element iff $ax \leq ay$ implies that $x \leq y \vee 0{:}a$, for some $x, y \in L$.

(iv) a is join principal, then a is (weak) meet principal implies ax is (weak) principal in L/x.

(v) If L is a modular lattice, then a is principal iff a is weak principal.

[1]

Lemma 2.23 : Let S be a multiplicative lattice with 1 compact. Suppose d is a finite join of join principal elements in S. If $d \leq b \vee cd$, then $c \vee b{:}d = 1$. Hence if $d = cd$, then $c \vee 0{:}d = 1$. [2]

Lemma 2.24 : Let S be a join principally generated multiplicative lattice with 1 compact. If every maximal element of S is a weak meet principal element, then every non-maximal prime element which is a finite join of join principal elements is a weak meet principal element. [3]

Proof : Assume that, every maximal element of S is a weak meet principal element. Let p be a non-maximal prime element which is a finite join of join principal elements.

Let $a \leq p$, for some $a \in S$.

We show that, $p \vee 0{:}a = 1$.

Suppose, if possible, $p \vee 0{:}a < 1$. Then there exists a maximal element $m \in S$ such that $p \vee 0{:}a \leq m$ and hence both $p, 0{:}a \leq m$. As p is a non-maximal prime element, we have $p < m$. Now as m is a weak meet principal element, we have $p = p \wedge m = m(p{:}m)$. But p is prime and $m \nleq p$. Hence $p{:}m \leq p$. But by (x) of property 1.7, we have $p \leq p{:}m$. So $p = p{:}m$.

This implies that, $p = m(p{:}m) = mp$. Consequently, by lemma 2.23, we have $m \vee 0{:}p = 1$. As $a \leq p$, we have by (iv) of property 1.7 that, $0{:}p \leq 0{:}a \leq m$.

Hence $1 = m \vee 0{:}p \leq m$. This gives $m = 1$. Which is a contradiction to the fact that, m is a maximal element. This implies that, $p \vee 0{:}a = 1$.

Therefore $a = a.1 = a(p \vee 0{:}a) = ap \vee a(0{:}a) = ap \vee 0 = ap$.

Thus ap implies $a = ap$.

Consequently, by proposition 2.22, p is a weak meet principal element. This completes the proof. **Q. E. D.**

§ 3. WEAK NOETHER LATTICES.

In this section, we now study the concept of weak Noether lattices which are natural generalisation of the concept of Noether lattices and then using this concepts we study some equivalent condition for a weak Noether lattices.

Definition 2.25 : Weak Noether Lattices. [3]

L is said to be a weak Noether lattice, if L is a weak r-lattice which satisfies the ascending chain condition.

Thus, weak Noether lattice is a Noether lattice iff it is a modular lattice.

According to [1], if N = (N, +, .) is the semiring of non-negative integers, then the lattice L(N) of all semiring ideals of N is a weak Noether lattice which is not a modular lattice. The fact that, L(N) is a non-modular lattice, follows from [20].

This shows that, a weak Noether lattice need not be a Noether lattice.

Now we consider some special type of multiplicative lattices which are abstract version of quasi-local rings, obviously.

Definition 2.25 : Quasi-local Lattices. [1]

A multiplicative lattice with 1 compact will be a quasi-local lattice, if it has unique maximal element.

If m is the only maximal element of a quasi-local lattice L, then such a lattice is denoted by (L, m).

Let us recall one more result which gives abstract version of Nakayama Lamma.

Theorem 2.26 : Let (L, m) be a quasi-local multiplicative lattice and suppose a is a finite join of join principal elements. Then for b and $d \neq 1$, $a \leq b \vee da$ implies $a \leq b$. In particular, $ad = a$ implies $d = 0$.

Definition 2.27 : Primary Element. [2]

An element $q \in L$ is said to be a *primary element,* if for compact elements x, y with $xy \leq q$ implies $x \leq q$ or $y^n \leq q$, for some $n \in Z^+$.

The concept, introduced by R. P. Dilworth [9], is obviously the abstraction of primary ideals of a commutative ring.

Consider the lattice of ideals of the ring $< Z_{12}, +, . >$ in which the only ideals are $0 = (0)$, $1 = (1)$, $a = (4)$, $b = (6)$, $c = (2)$ and $d = (3)$.

The element a is a primary element. But, 0 is not a primary element, since $ad = 0$ but neither $a = 0$ nor $d^n = 0$, for any positive integer n. Note that, a and d are both idempotent elements in L.

From the definition of prime element, it is obvious that every prime element is a primary element. Converse need not be true. Since in the example 1.26, a is primary element, but it is not a prime element

Further, if q is a primary element, then $\sqrt{q}$ is a prime element. Because, for compact elements x and y such that $xy \leq \sqrt{q}$, then $x^n y^n = (xy)^n \leq \sqrt{q}$, for some $n \in Z^+$. As q is a primary element, either $x^n \leq q$ or $(y^n)^m = y^{nm} \leq q$, for some $m \in Z^+$. Consequently, $x \leq \sqrt{q}$ or $y \leq \sqrt{q}$. Thus, $\sqrt{q}$ is a prime element.

From this it can be easily observed that, the condition x, y to be compact is necessary in the definition 2.27 of primary elements, which is not required in the definition of prime elements.

Definition 2.28 : p-Primary Element. [2]

An element $q \in L$ is said to be p-primary element, if q is primary and $\sqrt{q} = p$ is a prime element.

In the above discussed example, the ideal (4) is a (2)-prime element, whereas 0 is not so.

Lemma 2.29 : Let L be a weak Noether lattice in which m is the only prime element. Then for any proper elements $a, c \in L$, $\wedge_{n=1}^{\infty}(a^n \vee c) = c$. In particular, $\wedge_{n=1}^{\infty} a^n = 0$. [3]

Proof : Let $b = \wedge_{n=1}^{\infty}(a^n \vee c)$. We first show that, $b \leq c \vee cb$. As m is the only prime element, m must be the only maximal element, as every maximal element is a prime element. Thus L is a quasi-local lattice and every element of L is a m-primary element.

Now as L is a weak Noether lattice, we have a, b are compact elements and hence they are finite join of join principal elements. Since $ab \leq c \vee ab$ and $c \vee ab$ is a primary element, it follows that $b \leq c \vee ab$ or $a^n \leq c \vee ab$, for some $n \in Z^+$. Hence, $b \leq a^n \vee c \leq c \vee ab$. Thus, we have $b \leq c \vee ab$. Therefore by theorem 2.26, we have $b \leq c$. Of course, $c \leq b$. This gives that, $b=c$.

In particular, if we put $c = 0$, we have $b = 0$. This completes the proof of the theorem. **Q. E. D.**

We note the following important lemmas which we need.

Lemma 2.30 : Let L be a multiplicative lattice and $a, b \in L$. Then

(i) if a, b are join principal, then ab is so.

(ii) if a, b are meet principal, then ab is so.

(iii) if a, b are principal, then ab is so. [9]

Lemma 2.31 : Let p be a non-maximal principal element of L and $q = \wedge_{n=1}^{\infty} p^n$. Then (i) q is prime, (ii) $pq = q$ and (iii) any prime element properly contained in p is contained in q. [3]

These results leads us to understand the following important lemma.

Lemma 2.32 : Let L be a quasi-local weak Noether lattice. If the maximal element m of L is a principal element, then every non-zero element is a power m^k $(k \geq 0)$ of m.

Proof : Assume that, the only maximal element m of L is a principal element of L. We first show that, $\wedge_{n=1}^{\infty} m^n = 0$. If m is the only prime element, then directly by lemma 2.29, we have $\wedge_{n=1}^{\infty} m^n = 0$. Suppose there exists some prime elements in L which are different from the maximal element m. Then m is a non-maximal principal prime element of L. Consequently, by lemma 1.4, we have $m(\wedge_{n=1}^{\infty} m^n) = m^n$.

As L is the quasi-local weak Noether lattice, every element is a finite join of join principal elements and hence by theorem 2.26, we have $\wedge_{n=1}^{\infty} m^n = 0$.

Thus, in any case, we have $\wedge_{n=1}^{\infty} m^n = 0$.

Now let $a \neq 0 \in L$. Then $\wedge_{n=1}^{\infty} m^n < a$. This implies that, $a \leq m^n$ but $a \not\leq m^{n+1}$, for some $n \in Z^+$. Now as m is a principal element, we have m^n is also a principal element, by result 2.30. Consequently, m^n is a weak meet principal element and hence $a = a \wedge m^n = m^n(a:m^n)$. Obviously, $a:m^n = 1$, since if $a:m^n < 1$, then $a:m^n \leq m$, as (L, m) is a quasi-local lattice. This in turn implies that, $a \leq m^{n+1}$, by property 1.7. But this is a contradiction. Thus $a:m^n = 1$ and hence we have $a = m^n(a:m^n) = m^n.1 = m^n$.

§ 4. PRINCIPAL ELEMENT LATTICES.

In this section, we study an equivalent condition for L to be a principal element lattice.

Definition 2.33 : Principal Element Lattices.

A multiplicative lattice is said to be a principal element lattice, if every element of it is a principal element.

The lattice in the example 1.2 (i) is a principal element lattice.

We know that, each ideal of the ring of integers Z is a principal ideal. Thus, the lattice L(Z) of ideals of Z is an example of a principal element lattice.

However it is obvious that, the lattice in the example 1.34 is not a principal element lattices.

The following theorem gives an equivalent condition for L to be a principal element lattices. But for this we need to recall following result.

Result 2.34 : Let L be a compactly generated multiplicative lattice in which finite product of compact elements is compact. If L is join principally generated weak meet principal primes. Then L is a principal element lattices.

Theorem 2.35 : L is a principal element lattice iff L is a weak Noether lattice in which every maximal element is a weak meet principal element. [3]

Proof : Suppose L is a weak Noether lattice in which every maximal element is a weak meet principal element. Then by lemma 2.24, every non-maximal prime element which is a finite join of join principal elements is a weak meet principal element. Consequently, by result 2.34, L is a principal element lattice. **Q. E. D.**

Before proceeding further we now recall the following concepts, that we need in the study.

Definition 2.36 : Divisors of Zero or Zero Divisor. [1]

Let L be a multiplicative lattice and $a \in L$. Then a is called divisor of zero, if there exists a non-zero element b in L such that $ab = 0$.

Definition 2.37 : Zero Annihilator. [1]

An element of a multiplicative lattice L is said to have zero annihilator, if $0:a = 0$.

Thus, an element a is said to have zero annihilator, if it is not a divisor of zero.

Definition 2.38 : Domain.

Let L be a multiplicative lattice. Then L is called a domain, if it is without divisors of zero. [1]

Obviously, the lattice L(Z) of the ideals of the ring of integers Z is a domain. Whereas the lattice $L(Z_{12})$ of ideals of Z_{12} is not a domain, as it contains divisors of zero e. g., $(4).(3) = 0$.

It should be noted carefully that, if a multiplicative lattice L is a domain, then the element 0 is a prime element and $0:x = 0$, for each x. Since whenever we have $ab = 0$ in a domain we have either $a = 0$ or $b = 0$.

Now we recall the following theorem from [1].

Theorem 2.39 : Let (L, m) be a quasi-local multiplicative lattice and suppose a is a join principal element in L and a finite join of principal elements each of which has zero annihilator. Then a is principal.

In particular, if a is a join principal and is finite join of principal elements, then $a \vee p$ is principal in L/p, for every prime p of L. [1]

The following theorem is an extension of theorem 2 of [15].

Theorem 2.40 : Suppose L is a domain and for every prime element p of L, L_p is a weak Noether lattice. If every maximal element is compact and join principal, then every element is principal, i. e., L is a principal element lattice.

[3]

Proof : By theorem 2.39, it follows that every maximal element is locally principal and hence a principal element.

By lemma 2.32, it is clear that every prime element is nothing but a maximal element. Consequently, every prime element is a principal element.

Thus, every element is a principal element, since again by lemma 2.32, each element is a power of maximal element containing it and finite product of principal element is again a principal element. (see 2.30).

The following is an obvious corollary follows from the facts that, if 0 is a prime element, then no element of L is a zero divisor and hence L is a domain.

Secondly, if L is a weak Noether lattice, it satisfies the ascending chain condition and consequently, each element of L is compact. Thus, the following corollary follows:

Corollary 2.41 : If L is a weak Noether lattice in which 0 is a prime element and every maximal element is a join principal element. Then every element of L is a principal element or L is a principal element lattice. [3]