Chapter 3.

DEDEKIND DOMAINS

3. DEDEKIND DOMAINS

§ 1. INTRODUCTION.

In this chapter we study p-lattices, UFD lattices and Dedekind domains. Also we study their characterization.

Throughout this chapter, L denotes a multiplicative lattice with 1 compact.

Before proceeding further, we need to prove following important lemmas.

Lemma 3.1 : Let a be a weak principal element of L and $e \in L$ with $e{:}a = e$. If m is a factor of a, then for any $c \leq m$, $c \vee e = md \vee e$, for some $d \in L$. [3]

Proof : Assume that, m is a factor of a. Then $mk = a$, for some $k \in L$.

We first show that, for $x, y \in L$, $xk \leq yk$ implies $x \leq y \vee e$.

Let $xk \leq yk$, for some x, y. Then $xmk \leq ymk$, that is, $xa \leq ya$. As a is a weak meet principal element, by proposition 2.22, we have $x \leq y \vee 0{:}a$. But by property 1.7, we have $0{:}a \leq e{:}a = e$. So we get $x \leq y \vee e$.

Now assume that, $c \leq m$, for some $c \in L$. Then $ck \leq mk = a$. As a is a weak principal element, by proposition 2.22, we have $ck = ad$, for some $d \in L$. That is $ck = mkd = mdk$. Therefore by above argument, we have $c \leq md \vee e$ and $md \leq c \vee e$. Hence we get $c \vee e \leq md \vee e \leq c \vee e$. This implies that, $c \vee e = md \vee e$. **Q. E. D.**

Lemma 3.2 : Let L be a join principally generated, quasi-local multiplicative lattice. Let a be a weak principal element and q be a factor of a. Suppose that, $e:a = e$, where e is a join principal element and $q = \vee_\alpha x_\alpha$. If $q \not\leq e$, then $q \vee e = x_\alpha \vee e$, for some α. [3]

Proof : Define $a_\alpha = x_\alpha \vee e$, for each α. Then by lemma 3.1, we have $a_\alpha = x_\alpha \vee e = qd \vee e$, for some $d \in L$. Thus $qd \leq a_\alpha$. That is $d \leq a_\alpha{:}q$, by property 1.7. So $qd \leq (a_\alpha{:}q)q$. Consequently, we have $a_\alpha = qd \vee e \leq q(a_\alpha{:}q) \vee e$. But again by property 1.7, $(a_\alpha{:}q)q \leq a_\alpha$. This implies that, $(a_\alpha{:}q)q \vee e \leq a_\alpha \vee e = x_\alpha \vee e \vee e = x_\alpha \vee e = a_\alpha$. Thus, $a_\alpha = (a_\alpha{:}q)q \vee e$. Now note that, $q \vee e = (\vee_\alpha x_\alpha) \vee e = \vee_\alpha (x_\alpha \vee e) = \vee_\alpha a_\alpha = \vee_\alpha [q(a_\alpha{:}q) \vee e] = \vee_\alpha [q(a_\alpha{:}q)] \vee e = q[\vee_\alpha (a_\alpha{:}q)] \vee e$.

We now show that, $q \vee e = x_\alpha \vee e$, for some α.

If $\vee_\alpha (a_\alpha{:}q) = 1$, then $q \vee e = x_\alpha \vee e$, for some α.

Suppose $\vee_\alpha (a_\alpha{:}q) < 1$. As L is a quasi-local lattice, let m be the only maximal element of L. Then $\vee_\alpha (a_\alpha:q) \leq m$. Hence, $q[\vee_\alpha(a_\alpha:q)] \leq qm$ and hence $q\vee e \leq q[\vee(a:a)]e \leq qm\vee e$.

Thus, $q\vee e \leq qm\vee e$. But $qm \leq q$ and hence $qm\vee e \leq q\vee e$. This gives $qm\vee e = q\vee e$.

Now let b be a join principal element of L such that $b \leq q$. Put $a^* = b\vee e$. Then $a^* = q(a^*:q)\vee e = [(q\vee e)(a^*:q)]\vee e = [(qm\vee e)(a^*:q)]\vee e = m(a^*:q)q\vee e \leq ma^*\vee e$.

As b and e are principal elements and a* is a finite join of join principal element, by theorem 2.26, we have $a^* \leq e$ and so $b \leq e$. Consequently, $q \leq e$, which is a contradiction.

Therefore $q\vee e = x_\alpha\vee e$, for some α. **Q. E. D.**

§ 2. π-LATTICES.

Now we need to recall following concepts for the development of the further theory.

Definition 3.3 : Minimal Prime Element over an element. [1]

Let $a \in L$. Then a prime element $p \in L$ is said to be a minimal prime element over a, if $a \leq p$ and whenever there is a prime element $q \in L$ with $x \leq q \leq p$, we have $p = q$.

Definition 3.4 : Minimal Prime. [1]

A prime element $p \in L$ is said to be a minimal prime element (of L), if p is a minimal prime element over 0.

Definition 3.5 : Dimension. [2]

Let p be a prime element of L. We say that, p has dimension n, if n is the suprimum of the lengths of the chains of distinct proper primes grater than p.

Where as the dimension of L is defined as dimension of minimal prime element of L or dimension of 0, if 0 is prime.

Thus, $\dim L = \sup \{ n \in Z^{+} / 0 \leq p_0 < p_1 < \ldots\ldots\ldots < p_n < 1, p_i\text{'s are prime elements in } L \}$

Definition 3.6 : Join Irreducible Element. [1]

An element $a \in L$ is said to be join irreducible, if $a = a_1 \vee a_2$ implies either $a = a_1$ or $a = a_2$.

Definition 3.7 : Completely Join Irreducible Element. [1]

An element $a \in L$ is said to be a completely join irreducible element, if $a = \vee_{\alpha} a_{\alpha}$ implies that $a = a_{\alpha}$, for some α.

We know that, a ring is said to be a π-ring, if every principal ideal is a product of prime ideals.

We now see an abstraction of this concept.

Definition 3.8 : π-Lattice. [1]

L is said to be a π-lattice, if there exists a set S of elements of L (not necessarily of principal elements) which generate L under joins such that every element of S is a finite product of prime elements of L.

The concept of π-lattices is first introduced in multiplicative lattices by D. D. Anderson [1] in 1976.

Obviously, every π-ring is an example of π-lattice.

Further, let $N = < N, +, . >$ be the semi-ring of nonnegative integers. Let L be the lattice of semi-ring ideals of N and S be the set of principal ideals of N. Then N is a non-modular quasi-local π-lattice. [3]

Now we turn to the following important result.

Theorem 3.9 : Let L be a quasi-local weak r-lattice with maximal element m. If L is a π-lattice, then either L is a domain or L has only finitely many minimal prime elements and every prime element is the join of the minimal prime elements contained in it. [3]

Proof : If dim L = 0, then we have nothing to prove.

Assume that, dim L > 0.

Then L contains a finite number of minimal primes $P_1, P_2, ..., P_n$ such that $m \nleq P_1, P_2, ---, P_n$. Note that, as L is a π-lattice, there exists a set S of elements of L which generates L under joins such that every element of S is a finite product of prime elements. Thus, if $x \in L$ is a principal element, then we have $x = \vee_\alpha x_\alpha, x_\alpha \in S$. If x is completely join irreducible, then obviously $x = x_\alpha$ for some $x_\alpha \in S$. Consequently, x itself is a product of primes. Thus every principal element which is completely join irreducible is a product of primes. Hence further note that, each p_i is a principal element.

Suppose L is not a domain.

Let q be a prime element of L and let $p_1, p_2, ---, p_m \leq q$ $(1 \leq m \leq n)$.

We show that $q = \vee_{i=1}^{m} p_i$.

Suppose, if possible $\vee_{i=1}^{m} p_i < q$. As L is principally generated, there exists a principal element $a \leq q$ such that $a \nleq p_j$ for each $j = 1, 2, --, m$. Define

$e = p_1 \wedge \text{---} \wedge p_m$. Then we have $e{:}a = (p_1 \wedge \text{---} \wedge p_m){:}a = p_1{:}a \wedge \text{----} \wedge p_m{:}a$, by property 1.7. Note that for each $j = 1, \text{--}, n$, we have $p_j : a = p_j$. Since, if $x \leq p_j : a$,

we have $xa \leq p_j$ and hence $x \leq p_j$, as $a \nleq p_j$. This shows that $p_j : a \leq p_j$. That is $p_j{:}a = p_j$, by property 1.7. Hence, $e : a = p_1 \wedge p_2 \wedge \text{---} \wedge p_m = e$.

Now assume that, $a = q_1 ... q_k$ for some prime elements $q_1, \text{---}, q_k \in L$. As $a \leq q$ and q is a prime element, we have $q_i \leq q$, for some i, say $q_1 \leq q$. Now as p_i's are the only finite number of minimal prime elements, and q_1 is prime, we have $p_j \leq q$, for some $j = 1, 2, \text{---}, n$. Again for the sake of convenience, say $p_1 \leq q$,

Also note that $e \leq p_1$ and hence $p_1 = p_1 \vee e$.

Now by lemma 2,1, we have $p_1 = p_1 \vee e = q_1 d \vee e$, for some $d \in L$. Thus $q_1 d \leq p_1$. Also note that, $p_1 < q_1$. Since, if $p_1 = q_1$, we have $a \leq q_1 = p_1$, which is a

contradiction to the fact that $a \nleq p_j$. Thus, $q_1 d \leq p_1$ and $p_1 < q$, i.e., $q_1 \nleq p_1$. As p_1 is prime, we have $d \leq p_1$ and hence $p_1 = q_1 d \vee e \leq q_1 p_1 \vee e$. Therefore by Theorem 2.26, we have $p_1 \leq e$. As $e = p_1 \wedge \text{-----} \wedge p_m \leq p_1$, we get $p_1 = e$. Thus, $p_1 = e = e{:}a = p_1{:}\ a$. But $p_1{:}a = a$. Thus, $p_1 = a \leq q$. This shows that q contains only one minimal prime element. Let us say it is nothing but p_1. Hence by lemma 2.2, $q_1 = q_1 \vee p_1 = x_\alpha \vee p_1$, for some principal element $x_\alpha \leq q_1$.

We now claim that, $q_1 = x_\alpha$.

As L is a π-lattice, $x_\alpha = s_1 s_2 \ldots s_r$, for some prime elements $s_1, ---, s_r \in L$. Since $q_1 = x_\alpha \vee p_1$, we have $x_\alpha \leq q_1$. As q_1 is prime, we have $s_i \leq q_1$ for some i, say $s_1 \leq q_1$. But p_1 is the only minimal prime element contained in q_1. It follows that, $p_1 \leq s_1$ and hence $q_1 = x_\alpha \vee p_1 \leq s_1 \vee s_1 = s_1$. This implies that $q_1 = s_1$.

Therefore, $x_\alpha = s_1 \ldots s_1 = q_1 s_2 \ldots s_r = q_1 d$, for some $d \in L$. Again, as $q_1 = x_\alpha \vee p_1 = q_1 d \vee p_1$, and $q_1 > p_1$, by theorem 2.39, we have $d = 1$. This shows that, $q_1 = x$.

Now as $p_1 < q_1$ and q_1 is a weak meet principal element, we get $p_1 = p_1 \wedge q_1 = q_1(p_1 : q_1)$. But $p_1 : q_1 = q_1$. Hence we have $p_1 = q_1 p_1$. Consequently, by 2.26, we have $p_1 = 0$. This implies that, 0 is a prime element.

Thus, whenever $ab = 0$, we have either $a = 0$ or $b = 0$. Which shows that, $0 : x = 0$, for every x in L. That is L is a domain. Which contradicts to the fact that, L is not a domain.

Thus, $q = \vee_{i=1}^{n} p_i$. Thus every prime element is the join of minimal prime elements.

Thus, now we have following obvious corollary.

Corollary 3.10 : Let L be a quasi-local weak r-lattice. If L is a π-lattice, then L is either a domain or L has only finitely many prime elements and every prime element is compact.

Proof : In virtue of theorem 3.9, we have either L is a domain or L has only finitely many minimal prime elements and every prime element is the finite join of the minimal prime elements which are principal elements (refer to the proof of the above theorem). But by 2.17, each principal element is compact, as L is compactly generated.

Consequently, every prime element is compact, as every finite join of compact elements is compact.

§3. UFD LATTICE.

Now we study the concept of UFD lattices.

Definition 3.11 : UFD Lattice.

A principally generated multiplicative lattice domain is said to be a UFD lattice, if every principal element ~~in~~ is a product of principal primes.

The concept of a UFD Lattice is introduced by D. D. Anderson [1] in 1976, as an abstract concept of the lattice of ideals of UFD ring (by UFD we mean Unique Factorization Domain).

According to [1], if R is a commutattive domain ring, then the lattice L(R) of ideals of R can be UFD without R being a UFD.

For example, for any Dedekind domain R, L(R) is a UFD lattice. In fact, L(R) is a UFD if R is a π-domain.

Many results on UFD lattices are discussed in [1], of which some of them we need recall here.

The following theorem has already been proved by D. D. Anderson in [1] but in r-lattice. As pointed out by D. D. Anderson and C. Jayaram in [3] (see 2.7), this result is also valid in weak r-lattices, as it does not require the condition of modularity. Let us study this result.

But, before proceeding further we need to recall following result form [1], which is proved by D. D. Anderson in 1976, but in r-lattices. Later on the generalisation of this result is given by N. K. Thakare, C. S. Manjarekar and S. Maeda [16] in 1988, but in compactly generated multiplicative lattices. This generalisation is now very famous by the name "Separation Lemma". We recall here the result from [1].

Theorem 3.12 : Let L be an r-lattice and let S be a sub-multiplicatively closed subset of L. Suppose $a \in L$ and $t \nleq a$, for every $t \in S$. Then there exists an element $b \nleq a$ maximal with respect to the property that $t \nleq b$, for every $t \in S$. Further, any such b is a prime element. [1]

As pointed out by D. D. Anderson and C. Jayaram in [3], this result still holds good in weak r-lattices, as it does not require the condition that the lattice to be modular.

Theorem 3.13 : Let L be a weak r-lattice. Then L is a UFD iff every non-zero prime of L contains a non-zero principal prime. [3]

Proof : Assume that, L is a UFD lattice. Let p be a non-zero prime element. As L is generated by principal elements, there exists a non-zero principal element $x \in L$ such that $x \leq p$. As L is UFD, $x = p_1 p_2 \ldots p_n$ where p_i's are nonzero principal prime elements.

Thus, as p is prime, $p_i \leq p$, for some i. This shows that, every non-zero prime of L contains a non-zero principal prime.

Conversely, assume that, every non-zero prime of L contains a non-zero principal prime.

Define, $S = \{ 0 \neq x \in L \,/\, x \text{ is a product of principal primes} \}$.

As L is a domain, $S \neq \emptyset$ and we have S is a multiplicatively closed set.

Obviously $0 \notin S$. Thus by Theorem 3.12, there exists a prime element p maximal with respect to $t \not\leq p$, for each $t \in S$.

By assumption, if p is a nonzero prime element, then p must contain a non-zero principal prime element, which contradicts to the fact that $p \notin S$. Therefore $p = 0$.

Thus, $0(= p)$ is such a largest element. This implies that, if $x > 0$ is a principal element, then x must contain a nonzero principal element $y \in S$ such that y is a product of principal primes.

Let $x \in L$ be non-zero principal element. Then $x \geq p_1 p_2 \ldots\ldots p_n$, where p_i's are principal prime elements. Now as x is weak meet principal, by proposition 2.22, we have $xa = p_1 \ldots\ldots p_n$ for some $a \in L$.

If $n = 1$, we have $xa = p_1$ and hence $p_1 \leq x, a$ (by property 1.7). But as p_1 is prime, either $x \leq p_1$ or $a \leq p_1$. This gives either $x = p_1$ or $a = p_1$.

If $x = p_1$, then we are through.

Now if $a = p_1$, then $xp_1 = p_1$ and hence again by proposition 2.22, we have $1 = x \vee 0{:}\, p_1$. As L is a domain, $0{:}\, p_1 = 0$. Which gives $x = 1$. Which is a trivial case.

Thus for $n = 1$, the result is clear.

Now suppose $n > 1$. Then $xa = p_1 \ldots p_n \leq p_1$ and as p_1 is prime, say $a \leq p_1$. Then as p_1 is principal, $a = bp_1$, for some $a \in L$. Hence $p_1 p_2 \ldots\ldots p_n = xa = xbp_1$ and hence $xb = p_2 \ldots\ldots p_n$. Thus by induction on x, we have $xz = p_n$ for some $z \in L$. Consequently we have the result.

Corollary 3.14 : Let L be a weak r-lattice. Then L is a UFD iff L is a π-domain. [3]

Theorem 3.15 : Suppose L is a principally generated multiplicative lattice domain. Then L is a UFD lattice iff every principal element is a product of prime elements of L. [3]

Proof : obvious.

Let us recall some more concepts.

Definition 3.16 : Proper Element. [7]

An element a of L is said to be a proper element, if $a \neq 1$.

Definition 3.17 : Non - Trivial Element. [3]

An element a of L is said to be a trivial element, if $a \neq 0, 1$.

Now we have the following result.

Theorem 3.18 : Suppose L is a principally generated. If L is a UFD and every non-trivial prime element is maximal, then every element is principal. [3]

Proof : We first show that, every non-trivial prime element is a principal element.

Let p be a nontrivial prime element. Let $a \leq p$ be a non-zero principal element. As L is a UFD lattice, $a = p_1 \dots p_n$, where p_i's are principal primes. As a is non-zero and L is a domain, we must have each $p_i \neq 0$. Hence by assumption, each p_i is maximal.

Also note that $a = p_1 \dots p_n \leq p$ and p is prime. Hence $p_i \leq p$ for some i. Consequently, $p_i = p$, as p_i is a maximal element. This shows that p is a principal element.

Let $a \in L$. If $a = 1$, we have nothing to prove. Assume that, $a < 1$. Then $a \leq m$, for some maximal element $m \in L$. Then m is principal and hence by proposition 2.22. We have $a = mb$, for some $b \in L$. **Q. E. D.**

Theorem 3.19 : Suppose L is principally generated. If L is a domain in which every non-trivial principal element of L is the product of a finite number of maximal elements, then every element is principal. [3]

Proof : Let p be a non-trivial prime element. Then there exists a nonzero principal element $a \leq p$.

By hypothesis $a = p_1 \ldots\ldots p_1$, where p_i's are maximal elements of L. Then as $a = p_1 \ldots p_n \leq p$ we must have $p_i \leq p$, for some i. Consequently as p_i is maximal, $p = p_i$, say $p = p_1$. Thus $a = p_1 p_2 \ldots p_n = p(p_2 \ldots p_n)$. That is p is a factor of a. Hence by Lemma 2.2, p is principal.

§ 4. DEDEKIND DOMAIN.

In this section, we study Dedekind domain. First recall the definition.

Definition 3.20 : Dedekind Domain [3]

A domain L is a Dedekind domain, if every element of L is a finite product of prime elements.

We will see that, if L is a Dedekind domain, if every element of L is a finite product of prime elements.

Lemma 3.21 : For i = 1, 2, .., k, let p_i be a weak join principal non-trivial prime element of a domain L. Let $a = p_1 \ldots p_k$. Then this is the only way of writing the element a as a product of non-trivial prime elements of L except for the order of the factors. [3]

Proof : Let $a = q_1 \ldots\ldots q_n$ where q_i's are nontrivial prime elements of L. Without loss of generality, assume p_2 is minimal among $p_2, \ldots p_k$. Since $q_1q_2 \ldots q_n \leq p_1$ and p_1 is prime, we have $q_i \leq p_1$, for some i, say $q_1 \leq p_1$. Since $p_1 p_2 \ldots p_k \leq q_1$, we have $p_j \leq q_1$, for some j and hence j = 1. Thus $p_1 = q_1$. Again since $p_1 \ p_2 \ldots p_k = p_1q_2 \ldots q_n$ and p_1 is weak join principal, we get $p_2 \ldots p_k = p_2' \ldots p_n'$. Continuing like this we eventually get n=k and $j \leq k$.

Lemma 3.22 : Suppose L is principally generated. If L is a domain in which every nontrivial principal element of L is the product of a finite number of maximal elements, then every element is principal. That is L is a principal element lattice. [3]

Proof : Assume that, $p \leq (p \vee a)^2$, $a \not\leq p$, p is prime.

Then $p \leq (p \vee a)(p \vee a) = p^2 \vee pa \vee ap \vee a^2 = p^2 \vee pa \vee a^2 = p^2 \vee a(p \vee a)$. Hence by property 1.7, $[p^2 \vee a(p \vee a)] : p = 1$.

As p is principal, $1 = p \vee [a(a \vee p):p]$. Also $a(a \vee p):p = (a^2 \vee ap):p = a \vee a^2:p$, due to principality of p. Therefore, we have $1 = p \vee [a \vee a^2:p] = p \vee a \vee a^2:p$.

We now show that $a^2: p \leq a$. Let $x \leq a^2:p$. Then $xp \leq a^2$. Hence $xp \leq a$. As a is weak meet principal, xp = ad, for some $d \in L$. Thus, $ad \leq p$. But as p is prime and $d \not\leq p$, we have $d \leq p$. Thus $xp = ad \leq ap$.

That is $x \leq ap:p$. As p is a weak join principal element and $x \leq ap:p = a \vee 0:p = a \vee 0 = a$, as L is a domain. This implies that $a^2:p \leq a$.

Consequently, $1 = p \vee a \vee a^2:p = p \vee a$. **Q. E. D.**

Theorem 3.23 : Suppose L is principally generated. If L is a Dedekind domain then every non-trivial prime element of L is a maximal element.

Proof: Assume that, L is a Dedekind domain.

First we show that, every principal nontrivial prime element of L is maximal.

Let p be a non-trivial principal prime element.

Suppose, if possible, p is not maximal. Then $p < m$, for some maximal element $m \in L$. As L is principally generated, there exists a principal element

$a \not\leq p$ such that $a \leq m$. As L is a Dedekind domain, we have $p \vee a = p_1 \dots p_k$ and $p \vee a^2 = q_1 \dots q_n$, where p_i's and q_i's are non-trivial prime elements of L.

Let $\tilde{a} = p \vee a = p_1 p_2 \dots p_k \vee p$, $\tilde{a}^2 = p \vee a^2 = q_1 q_2 \dots q_n \vee p$.

Note that, for $x \vee p, y \vee p$ L/p, we have $(x \vee p) \circ (y \vee p) = (x \vee p)(y \vee p) \vee p = xy \vee xp \vee yp \vee p^2 \vee p = xy \vee p$, as $xp, yp, p^2 \leq p$.

Thus, we have $\tilde{a} = \bar{p}_1 \circ \bar{p}_2 \circ \dots\dots \circ \bar{p}_k$ and $\tilde{a}^2 = \bar{q}_1 \circ \bar{q}_2 \circ \dots\dots \circ \bar{q}_n$, where $\bar{p}_i = p_i \vee p$ and $\bar{q}_i = q_i \vee p$.

Obviously, as $a \not\leq p$, $\tilde{a} \neq \tilde{o} = p$. Also by Proposition 2.22 (4), we have $\tilde{a}$, $\tilde{a}^2$ are weak join principal elements in L/p.

We show L/p is domain. Let $x \vee p, y \vee p \in L/p$. Such that $(x \vee p) \circ (y \vee p) = \tilde{o} = p$. Then $p = (x \vee p) \circ (y \vee p) \vee p = xy \vee xp \vee yp \vee p^2 \vee p = xy \vee p$. Thus $xy \leq p$. As p is prime either $x \leq p$ or $y \leq p$. Consequently $x \vee p = p$ or $y \vee p = p$. Thus, L/p is a domain.

Claim : If $a \neq 0$ is a weak join principal element in a domain L, then its factor is weak join principal element.

Proof : Let b be factor of a. Then $a = bk$ for some $k \in L$.

Let $x \in L$. We have to show $x \vee 0 : b = xb : b$. As L is a domain, we have only to show $x = xb:b$.

Let $z \leq xb:b$. Then $zb \leq xb$. Consequently, $zkb \leq xkb$. Hence $za \leq xa$. This gives, $z \leq xa:a = x \vee 0:a$, as a is join principal. Thus as $0:a = 0$, we have $z \leq x$. This implies that, $xb:b \leq x$. But by property 1.7 , $x \leq xb : b$. Hene $x = xb : b$. Hence the claim.

Therefore by afore-proved claim, each p_i and each q_i are weak join principal elements. Now note that, $\tilde{a}^2 = \bar{p}_1{}^2 \circ \bar{p}_2{}^2 \circ \ldots\ldots \circ \bar{p}_k{}^2 = \bar{q}_1 \circ \bar{q}_2 \circ \ldots\ldots \circ \bar{q}_n$. Hence by Lemma 3.21, $n = 2k$ and we may number q_i so that for $i = 1, \ldots, k$, $q_{2i-1} = q_{2i} = p_i$. Thus $(p \vee a)^2 = p \vee a^2$.

This gives, $p \leq (p \vee a)^2$. But note that, $a \nleq p$. Hence by lemma 3.22, we have $p \vee a = 1$. Consequently, $1 = p \vee a \leq m$, i.e., $m = 1$, which is a contradiction to the fact that m is a maximal element.

This shows that, every principal nontrivial prime element of L is a maximal element.

Now we show that, every nontrivial prime element is maximal.

Let p be a non-trivial prime element. Then there exists a nonzero principal element $a \leq p$. As L is a Dedekind domain $a = p_1 \ldots . p_n$, where p_i's are nontrivial prime elements. Since L is a domain and a is a principal element, by Lemma 2.2, each p_i is principal and hence a maximal element.

As $a = p_1 \ldots p_n \leq p$, we have some $p_i \leq p$. Consequently, due to maximality of p_i, p is maximal. Thus completes the proof. **Q. E. D.**

Theorem 3.24 : Suppose L is principally generated. If L is a Dedekind domain, then every principal element is principal. [3]

Proof : Assume that, L is a Dedekind domain. By theorem 3.23, we have every nontrivial prime element is a maximal element. As L is a Dedekind domain, every element of L is a finite product of prime elements. Hence, by theorem 2.23, every element of L is a finite product of maximal element.

Consequently, by theorem 3.22, every element of L is a principal element. **Q. E. D.**