Chapter 4.

INVERTIBLE ELEMENTS

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§ 1. INTRODUCTION.

Throughout this chapter, L denotes a weak r-lattice.

In this chapter, we study the concept of invertible elements in multiplicative lattices and also study invertible prime elements. Using invertible prime elements, we see establishment of some equivalent conditions for a weak r-lattice L to be a finite direct product of Dedekind domains.

§ 2. INVERTIBLE ELEMENTS.

Definition 4.1 : Regular Element. [3]

An element a of a multiplicative lattice S is said to be regular if there is a principal element $b \in S$ such that $0{:}b = 0$ and $b \leq a$.

Definition 4.2 : Invertible Element. [3]

An element a of a multiplicative lattice S is said to be invertible if ac = d for some c$\in$S and for some principal regular element d of S.

Consequently, if an element is a principal regular, then it is regular.

We now turn to some results on invertible elements.

The following theorem gives characterization of invertible elements in terms of principal regular elements.

Lemma 4.3 : Let S be a multiplicative lattice. Then an element a$\in$ S is an invertible element if and only if a is a principal regular element of S. [3]

Proof. The proof of the lemma follows from Lemma 2.2. Since if a is invertible, then ac = d, for some c and d is principal regular. As d is principal regular, we have $0{:}e = 0$, where $e \leq d$. Then $e \leq d \leq a$. Thus, by property 1.7, we have $0{:}a \leq 0{:}d \leq 0{:}e = 0$. That is, $0{:}a = 0 = 0{:}d$. As d is principal and a is a factor of d, we have a is principal (see lemma 2.2).

The converse is obvious.

Now here follows an obvious result in virtue of the above lemma 4.3.

Theorem 4.4 : A domain L is a Dedekind domain if and only if every nonzero element is invertible. [3]

Proof is obvious due to lemma 4.3, since as Lis a domain, it has zero annihilator and conversely, if each element is regular it has zero annihilator and hence L is a domain. Consequently, result follows from theorem 3.24.

We note the following lemma that we need.

Lemma 4.5 : Let p be a proper invertible prime element of L. Then

(a) If $p = ab$, where $a, b \in L$, then either $a = 1$ or $b = 1$.

(b) If a is an invertible element of L and $a > p$, then $a = 1$.

(c) If $p' = \wedge_{n=1}^{\infty} p^n$ then p' is a prime element and $p'p = p'$ and if p'' is a prime element and $p'' < p$, then $p'' \leq p'$. If p' is compact and q is a primary element contained in p, then $p' \leq q$; in fact $p' = q$ or $\sqrt{q} = p$. In particular, if p' is compact, then p' is the only prime element properly contained in p.

(d) An element q is p-primary if and only if q is a power of p.

(e) The only invertible elements between p and p^n, where n is a positive integer are powers of p.

Now we characterize Dedekind domains in terms of invertible elements.

Theorem 4.6 : A domain L is a Dedekind domain if and only if every prime element is compact and every nonzero maximal element is invertible. [3]

Proof. If L is a Dedekind domain, then by Theorem 3.24, every element is principal and hence every prime element is compact and every nonzero element is invertible

Conversely, assume that every prime element is compact and every nonzero maximal element is invertible. By Lemma 4.5(c), every prime element is principal and hence every element is principal. Consequently L is a Dedekind domain.

Definition 4.7 : Proper Dedekind domains., [3]

A multiplicative lattice domain is said to be a proper domain if it is not a two element chain.

The following Theorem 4.8 establishes an equivalent condition for L to be a finite direct product of proper Dedekind domains.

First we recall the following result.

Theorem 4.8 : If L is lattice generated by join principal elements in which every semiprimary element is a primary element, then if p is a non-maximal prime element of L and q is p-primary, then p = q. [2]

Theorem 4.9 : Suppose L is not a two element chain. Then L is a finite direct product of proper Dedekind domains if and only if every prime element is compact and every maximal element is invertible. [3]

Proof. Suppose $L = L_1 \times \ldots \times L_n$, where each L_i is a proper Dedekind domain. Then each L_i is a principal element domain, by theorem 4.4 and so L is a principal element lattice, since principality property preserves in the direct product of lattices. If m is a maximal element of L, then $m = (1, 1, \ldots\ldots m_i \ldots\ldots 1)$. where m_i is a maximal element of L_i and so $0 : m = 0$. Thus, every maximal element is a principal regular element. Consequently, by lemma 4.3, every maximal element is invertible.

Conversely, assume that every prime element is compact and every maximal element is invertible. Then by theorem 4.6, L is a Dedekind domain consequently, L is a principal element lattice and so dim $L \leq 1$. As L is a Noether lattice, the zero element has a normal decomposition.

Let $0 = q_1 \wedge \ldots \wedge q_n$ be a normal decomposition and let $p_i = \sqrt{q_i}$. Suppose for i = 1,2, ..., k, the p_i's are non-maximal and for i = k + 1, ..., n, the p_i's are maximal. But by theorem 4.8, $q_i = p_i$ for $i = 1, 2, \ldots, k$. By Lemma 4.5 (c), $p_i' < q_i$ for $i = k + 1, \ldots, n$ where $p_i' =$ is a prime element. But this contradicts the hypothesis that a normal decomposition is redundant unless $k = n$. Hence $0 = p_1 \wedge \ldots \wedge p_n$. Further these prime elements are co-maximal and so $L \cong L/p_1 \times \ldots \times L/p_n$. Note that each factor is a proper principal element domain, and hence L is a finite direct product of proper Dedekind domains. **Q. E. D.**