Chapter 1.

MULTIPLICATIVE LATTICES

1. MULTIPLICATIVE LATTICES

§ 1. INTRODUCTION.

The study of abstract commutative ideal theory is recently the principal incentive of new research in commutative algebra. Morgan Ward and R. P. Dilworth began a study of the ideal theory of commutative ring in abstract form in 1937. As their intension was to treat it as purely ideal theoretic, they chose the algebraic system viz., a multiplicative lattice with commutative multiplication.

Lattice theory has its own beauty with the extremely simplest concepts viz., partial order, infimum, supremum, which has been made more marvelous by adding a new binary operation called multiplication. During the study of abstract formulation of the ideal theory of commutative rings, the classical notion of a multiplicative lattice was first introduced by M. Ward and R. P. Dilworth [21] in 1938.

Now we recall the some basic concepts.

Definition 1.1 : Complete Lattice. [7]

A complete lattice L is defined as, for any nonempty subset S of L, l. u. b. of S = $\vee S \in L$, g. l. b. of S = $\wedge S \in L$.

As $L \subseteq L$, we have $0 = \wedge L$ is the least element and $1 = \vee L$ is the greatest element of the lattice L.

Example 1.2 :

(i) Let X be a non-empty set. Let the power set, i. e., the set of all subsets of X be $\wp(X)$. Then $< \wp(X), \cup, \cap >$ is a complete lattice with respect to the set inclusion relation $\subseteq$.

Note that, in this case, $0 = \varnothing$ and $1 = X$.

(ii) Every finite lattice (i. e., the lattice containing finite number of elements) is a complete lattice.

(iii) Define for real numbers a, b such that $a < b$, the open interval

$L = (a, b) = \{ x / x$ is a real number such that $a < x < b \}$.

Define, with respect to the partial order relation $\leq$, for a nonempty subset S of L,

$$\wedge S = \text{g. l. b. } S \quad \text{and} \quad \vee S = \text{l. u. b. } S.$$

Then L is a lattice, but it is not a complete lattice, since for example, the g. l. b. (a, b) = $a \notin L$.

Definition 1.3 : Multiplicative Lattice. [9]

Let $L = \langle L; \vee, \wedge \rangle$ be a complete lattice with the least element '0' and greatest element '1'. The lattice L is said to be a *multiplicative lattice*, if there is defined a binary operation " . " called multiplication on L which satisfies the following axioms:

(1) $ab = ba$,

(2) $a(bc) = (ab)c$,

(3) $a(\vee_\alpha b_\alpha) = \vee_\alpha (ab_\alpha)$,

(4) $a.1 = a$.

The element 1 is thus known as the multiplicative identity element.

Further, $a^0 = 1$ and for a positive integer n, $a^n = a.a.a....$ n times.

The classical notion of residuation was first introduced by Dedekind in the theory of modules. As an abstract version of residual of ideals of commutative ring, Ward and Dilworth [21] were able to introduce the concept of the residuation of element in the multiplicative lattices, which is very important basic concept in the development of the theory of the multiplicative lattices. Basically, Ward-Dilworth [21] implemented this concept as binary operation with certain axioms in a lattice and they call the lattice with this binary operation as residuated lattice. Here follows the beautiful concepts.

Definition 1.4 : Residuation of Elements. [21]

Suppose L is a multiplicative lattice and a, b $\in$L. The *residuation of* a *by* b is denoted by a:b and given by a:b = $\vee$ {x $\in$ L / xb $\leq$ a }.

If every a:b is in L, the lattice is known as residuated lattice.

Naturally, as every multiplicative lattice is a complete lattice, we have a:b $\in$ L, for all a, b $\in$L. Therefore, every multiplicative lattice is a residuated lattice.

Basically, the main intension behind the development of multiplicative lattice theory is to get abstract version of ideal theory of commutative rings. So naturally we have the following obvious but important example.

Example 1.5 : Let R be a commutative ring with unity. Let S $\subseteq$ R. Then (S) is the smallest ideal containing S, called the ideal generated by S.

For ideals I and J, we know that I$\cap$J and I + J = (I$\cup$J) = { i+j $\in$ R / i $\in$ I, j$\in$ J } are ideals of R.

For a $\in$ R, the ideal generated by a or the smallest ideal containing a is denoted by (a) and such ideal is known as **principal ideal** of R.

For a, b $\in$ R, we have (a, b) = (a) + (b).

Let L = L(R) be the set of all ideals of R. Then L is a multiplicative lattice with

$I \wedge J = I \cap J, \qquad I \vee J = I + J$

$IJ = \{\sum_{finite} a_i b_i \in R / a_i \in I, b_i \in J\}$ and $I:J = \{x \in R / xJ \subseteq I\}$. [22]

§ 2. RADICAL ELEMENTS.

Now we have another concept which is abstraction of the concept of radical of ideal, as follows:

Definition 1.6 : Radical of an Element. [9]

Let L be the multiplicative lattice. The *radical* of $a \in L$ is the element, is given by

$\sqrt{a} = \vee \{x \in L / x^n \leq a, \text{for some } n \in Z^+\}$.

Now we recall following most important fundamental facts multiplicative lattices which are obviously useful constantly in the study of multiplicative lattice. Most of these basic facts are pointed out by R. P. Dilworth.

We can observe at a glance that, many of the following results are nothing but the abstract version of results of commutative ideal theory.

Property 1.7 : Let L be multiplicative lattice with a, b, c $\in$ L. Then [9]

i) $ab \leq a \wedge b \leq a, b$.

ii) $a \leq b$ implies $ac \leq bc$.

iii) $a \leq b$ implies $a{:}c \leq b{:}c$.

iv) $a \leq b$ implies $c{:}a \geq c{:}b$ or $c{:}b \leq c{:}a$.

v) $(a{:}b)b \leq a$.

vi) $c \leq a{:}b$ iff $cb \leq a$.

vii) $a{:}b = 1$ iff $b \leq a$.

viii) $(a \wedge b){:}c = a{:}c \wedge b{:}c$.

ix) $a{:}(bc) = (a{:}b){:}c$.

x) $a \leq a{:}b$.

xi) $a{:}1 = a$.

xii) $a \leq ab{:}b$.

xiii) $(a \vee b){:}c \geq a{:}c \vee b{:}c$.

xiv) $(a \wedge b)c \leq ac \wedge bc$.

xv) $(a \wedge b){:}b = a{:}b$.

xvi) $a{:}(a \vee b) = a{:}b$.

xvii) $a{:}(b \vee c) = a{:}b \wedge a{:}c$.

xviii) $a \vee c = b \vee c = 1$ implies $ab \vee c = 1$.

xix) $a \vee c = 1$ implies $(a \wedge b) \vee c = b \vee c$.

xx) $a \leq \sqrt{a}$.

xxi) $a \leq b$ implies $\sqrt{a} \leq \sqrt{b}$.

§ 3. COMPACTLY GENERATED LATTICES.

We now study a very important stronger concept. The development of multiplicative lattice theory is largely depending upon this strong concept.

Definition 1.8 : Compact Element. [7]

Let S be a lattice. An element a of S is *compact*, if $a \leq \vee_{\alpha} a_{\alpha}$ implies that $a \leq \vee_{i=1}^{n} a_{\alpha_i}$, where $a_{\alpha_i} \in \{a_{\alpha}\}$ and n is a positive integer.

Example 1.9 : We recall the very well known but important example 1.6. Note that, every principal ideal is compact in L = L(R), where R is a commutative ring with unity.

Since, if $a \in R$ and $\{I_{\alpha}\}$ is a family of ideals of R such that $(a) \subseteq \vee_{\alpha} I_{\alpha}$, then as $1 \in R$, we have $a \in (a)$ and so we have $a \in \cup_{j=1}^{n} Ij \subseteq \vee_{k=1}^{n} Ij$, for some finite number of ideals $Ij \in \{I_{\alpha}\}$ and $n \in Z^{+}$. Thus $(a) \subseteq \cup_{j=1}^{n} Ij$, as the ideal

(a) is the smallest ideal generated by x in R.

This shows that, each principal ideal (element) is compact in L(R).

Notation 1.10: The Set of Compact Elements (L_*). [2]

$L_* = \{ x \in L / x$ is a compact element in L $\}$.

Definition 1.11: Compactly Generated Lattice or Algebraic Lattice. [2]

A lattice L is said to be *compactly generated* or *algebraic*, if for every $a \in L$, there exists $x_\alpha \in L_*$ such that $a = \vee_\alpha x_\alpha$. [8]

That is a lattice is said to be compactly generated, if every element of it is a join of compact elements.

It should be noted that, in a compactly generated multiplicative lattice L, the *radical* of $a \in L$ is given as,

$\sqrt{a} = \vee \{ x \in L_* / x^n \leq a$, for some $n \in Z^+ \}$. [2]

Example 1.12 : Recalling the above example 1.6. The lattice L = L(R) is compactly generated. For, if I is an ideal, then we have $I = \cup_{x \in I} (x) \subseteq \Sigma_{x \in I} (x) = \vee_{x \in I} (x)$. This implies that $I = \vee_{x \in I} (x)$. This shows that, I is join of compact elements, as each (x) is a compact.

Property 1.13 : If L is a compactly generated multiplicative lattice, then

i) $\sqrt{\sqrt{a}} = \sqrt{a}$, ii) $\sqrt{a \wedge b} = \sqrt{a} \wedge \sqrt{b} = \sqrt{ab}$. [16]

§ 4. PRINCIPAL ELEMENTS.

Historically, the basic intension behind the development of multiplicative lattice theory was to formulate the beautiful results of ideal theory of a commutative ring in abstract form. Consequently, many concepts of ring theory were abstracted in multiplicative lattice theory. For example, residuation of ideals of a commutative ring R as residuation of elements in L(R), the lattice of ideals of R, radical of ideal of R as radical of element in L(R) and so many.

But always it was/is not possible very easily to get abstract version of concepts of commutative ring theory in multiplicative lattice theory. The main but very strong problem was arose with the abstract formulation of the concept of principal ideal as principal element in multiplicative lattice theory.

For suitably defined multiplicative lattices, Ward and Dilworth [21] extended the Noether decomposition theory of commutative ideal theory. But further development was however almost impossible; the essential difficulty was a proper concept of principal elements as an abstraction of principal ideals. The old concept of principal elements was:

Definition 1.14 : *" An element q of a multiplicative lattice is principal, if for any b with q > b implies there exists an element c such that qc = b".* [24]

This concept was sufficient for the development of multiplicative lattice theory up to very limited extend. For, this concept of principal element sufficed for the abstract version of the some results like the decomposition theorems in primaries.

But abstract version of many of the natural and deeper results of ring theory, like Krull Intersection Theorem, Nakayama Lemma etc., was almost a difficult task for the mathematician.

Naturally, the essential difficulty was to get proper abstract version of the notion of principal element. After realizing this fact, almost all the mathematicians, working in this theory, tried hard to get proper abstract version of principal element.

Eventually, after nearly 23 years of continuous research work on this problem, R. P. Dilworth [9] was able to introduce a new stronger notion of such a principal element, in 1961 and extended Krull intersection theorem, principal ideal theorem to Noether lattice. He divided this notion into following two identities. Till now this concept of principal elements seems to be very effective.

Definition 1. 15 : Meet Principal Element. [9]

An element $a \in L$ in a multiplicative lattice L, is called *meet principal,* if

$x \wedge ay = (x:a \wedge y)a, \quad \forall\, x, y \in L.$

Definition 1.16 : Join Principal Element. [9]

An element $a \in L$ in a multiplicative lattice L, is called *join principal*, if

$x \vee y:a = (xa \vee y):a, \quad \forall\, x, y \in L.$

It is clear that, each of these two identities is obtained from the other by interchanging the roles of "$\vee$" and "$\wedge$" and of "•" and ":".

Definition 1.17 : Principal Element. [9]

An element which is both meet and join principal element is called a *principal element.*

Definition 1.18 : Join Principally Generated Lattice. [2]

A multiplicative lattice L is *join principally generated*, if every element of it is a join of join principal elements.

Definition 1.19 : Principally Generated Lattice. [2]

A multiplicative lattice L is *principally generated*, if every element of it is a join of principal elements.

Example 1.20 : Obviously, we recall the example 1.5.

After defining the abstract version of principal ideal, Dilworth [9] naturally has shown that, for every element x of a commutative ring R with unity, the principal ideal (x) is a principal element of L = L(R), the lattice of ideals of R.

Also this lattice L is principally generated. Since, for any ideal I of R, we have $I = \vee_{x \in I} (x)$.

Example 1.21 : Recall the example (i) of 1.2. Define $AB = A \wedge B$. Then the lattice will be a multiplicative lattice such that every element of it is a principal element.

Thus, the lattice is principally generated and hence is join principally generated.

To show that the concepts of meet principal elements and join principal elements are independent, Johnson and Anderson [13] have given following examples.

Example 1.22 : Meet principal element need not be join principal element.

Let Q^+ be the semi-ring of nonnegative rational numbers, x be an indeterminate and let $R = Q^+[x]$. The ideal (1 + x) of R is a meet principal element but it is not join principal element of L = L(R). [15]

Example 1.23 : A join principal element need not be a meet principal element.

Let $(L_n, M_n) = RL_n$, where n = 1, 2.

Let $L = \{ (a, b) \in L_1 \oplus L_2 / a = m_1 \text{ iff } b = m_2 \}$. Then $(L, (m_1, m_2))$ is a multiplicative lattice in which every element is a join principal element but not a meet principal element. Hence it is join principally generated lattice, but not a principally generated lattice. [15]

§ 5. NOETHER LATTICES.

In the poset theory, 'a poset satisfying the ascending chain condition is known as a Notherian poset'.

The great mathematician Emmy Noether is the first to recognize the power of ascending chain condition and descending chain condition. These conditions are largely but effectively applied by Hilbert as a tool in the theory of ideals (see [7]). Ideal theory is concerned with the ascending chain condition at large, whereas the descending chain condition is too restrictive for many purpose. Consequently, such things happen in the theory of multiplicative lattices.

The weak concept of a Noether lattice as an abstraction of Noether ring was first introduced by M. Ward and R. P. Dilworth [21] as a multiplicative lattice satisfying the ascending chain condition in which every meet irreducible element is primary, by defining well two binary operations viz., multiplication and residuation. Further they proved successfully that, not only these two operations correspond to each other but also they have the properties of the like-named operations in particular instances of polynomial ideal theory.

After defining the stronger concept of principal elements, Dilworth then introduced stronger formulation of Noether lattices [9].

Principal elements in multiplicative lattice are the analogue of principal ideals in commutative rings with unity. The theory of multiplicative lattices and the abstract ideal theory are largely based on principal elements. Thus Principal elements are the cornerstones.

The Krull intersection theorem and principal ideal theorem in a Noether lattice are extended to explain the concept of principle element by Dilworth [9].

To understand the concept of Noether lattices, first we recall the following concepts.

Definition 1.24 : Chain.

In a lattice L, a family $\{a_i\} \subseteq$ L is an ascending (or descending) *chain*, if $a_1 \leq a_2 \leq a_3 \leq$ --- (or $a_1 \geq a_2 \geq a_3 \geq$ ---). [6]

Definition 1.25 : Ascending Chain Condition (ACC).

A lattice L is said to satisfy *ACC*, if for every ascending chain $a_1 \leq a_2 \leq a_3 \leq$ ---, $\exists N \in Z^+$ such that $a_N = a_n$, $\forall n \geq N$. [6]

Example 1.26 : (i) Let Z be the ring of integers. The lattice L = L(Z) of ideals of Z satisfies the ACC (see [22]).

Since, every ideal of Z is a principal ideal and every integer has finite number of prime factors. Thus, if I is an ideal of Z, then we have I = (n), for some integer n. Then there exists finite number of prime factors of n, p_1, p_2, --- , p_m in the ring Z such that $n = p_1 p_2 ... p_m$ and the ACC holds as follows:

$$I = (n) = (p_1 p_2 ... p_m) \subset (p_1 p_2 ... p_{m-1}) \subset (p_1 p_2 ... p_{m-2}) \subset \text{----} \subset (p_1).$$

This implies that, L satisfies ACC.

(ii) Obviously, every finite lattice satisfies ACC.

(iii) Let X be an infinite set. Then the lattice L = $\wp$(X) of subsets of X does not satisfy the ACC. Since, for instance, the chain of infinite subsets $\{a_n\}$

of X has no any upper bound as $\{ a_1 \} \subset \{ a_1, a_2 \} \subset \{ a_1, a_2, a_3 \} \subset$---.

Definition 1.27 : Modular Lattice.

A lattice L is *modular*, if for a, b, c $\in$ L, $a \vee (b \wedge c) = (a \vee b) \wedge c$, $\forall a \leq c$.[6]

Example 1.28 : (i) Consider the very well known example, the lattice structure $M_5 = \{0, a, b, c, 1\}$. This lattice is a modular lattice, with its lattice structure as follows. [6]

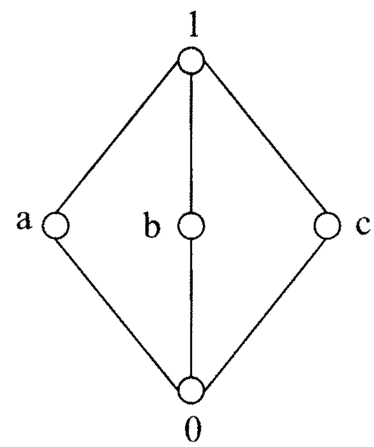

Figure 1.1: the lattice M_5.

(ii) Not every lattice is modular. For instance, the lattice N_5 is non-modular. Since, we have

$a < b$ and $a \vee (c \wedge b) = a \vee 0 = a$, while $(a \vee c) \wedge b = 1 \wedge b = b$.

(iii) Recall the example 1.5. According to [7], the lattice L = L(R) is a modular lattice.

Definition 1.29 : Noether Lattice.

If a multiplicative lattice L is modular, satisfying ACC and principally generated then it is called a *Noetherian lattice.*

As noted before, the concept of Noether lattice is an abstraction of the concept of the lattice of ideals of a Noetherian ring. Thus, every lattice of ideals of a Noetherian ring is a Noetherian lattice. This fact is cleared well with the illustrations 1.14 and (iii) of example 1.22.

Now we see some more basic concepts which are generalizations of concept of principal element.

Definition 1.30 : Weak Join Principal Element.

In a multiplicative lattice L, an element $a \in L$ is called as *weak join principal,* if $x \vee 0{:}a = xa{:}a, \ \forall \ x \in L$. [1]

Definition 1.31 : Weak Meet Principal Element.

In a multiplicative lattice L, an element $a \in L$ is *weak meet principal,* if $x \wedge a = (x{:}a)a, \quad \forall x \in L$. [1]

Definition 1.32 : Weak Principal Element.

An element is called *weak principal*, if it is both weak meet and weak join principal element. [1]

Definition 1.33 : Weak Principally Generated Lattice.

A multiplicative lattice L is called *weak principally generated*, if every element of L is a join of weak principal elements. [1]

We can observe that, by substituting 0 and 1 respectively in the definitions of join principal and meet principal elements we get the aforementioned definitions.

Thus, every join principal element is a weak join principal element, every meet principal element is a weak meet principal element and every principal element is a weak principal element.

Johnson and Anderson [13] provided the following examples to indicate that the above concepts are independent. Just we note them here.

Example 1.34 : A weak meet principal element which is not meet principal.

Consider L = { 0, a, b, c, d, e, 1 } is a lattice with the lattice structure:

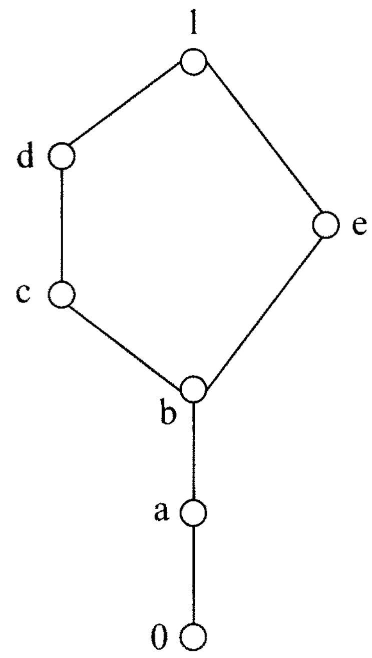

Figure 1.2

The multiplication table is as follows.

•	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	0	0	0	0	b
c	0	0	0	b	b	a	c
d	0	0	0	b	b	b	d
e	0	0	0	a	b	b	e
1	0	a	b	c	d	e	1

Then c is a weak principal element, but it is not a meet principal element. [13]

Example 1.35 : Weak join principal element does not imply join principal.

If F is a field and $X_i = (x_i)$, i = 1, 2, ----, n, then RL_n is the sublattice, which is generated by the power products of $X_1, X_2, ---, X_n$, of the ideal lattice of $F[x_1, x_2, ---, x_n]$.

In $RL_2 / (X_1X_2^2 \vee X_1^2X_2)$, the maximal element is a weak join principal element, but not join principal. [13]

Example 1.36 : Recall the example 1.23. In this lattice the only elements of the form (a, 0) or (b, 0) are weak meet principal elements and every element is a weak join principal element. Thus the lattice is a weak join principally generated lattice.