CHAPTER 0

PRELIMINARIES

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This chapter is devoted to the summary of known concepts and results which will be used in subsequent chapters.

§ 0.1 DEFINITION :

0.1.1 Poset : ([1] , Page -1)

Let P be any nonvoid set. A binary relation ' $\leq$ ' on P is called partially ordering relation if it satisfies the following conditions for all a , b , c in L.

(1) $a \leq a$ (Reflexivity)

(2) $a \leq b$ and $b \leq a \Rightarrow a = b$ (Antisymmetricity)

(3) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (Transitivity)

A non-empty set equipped with the partially ordering relation is called partially ordered set or poset. It is denoted as $< P , \leq >$. A poset in which $a \leq b$ or $b \leq a$ for all a , b in P is called a chain.

0.1.2 Zero element in a poset : ([1], Page-2)

Let $< P , \leq >$ be a poset .If there exists 0 in P such that $0 \leq x$ for all x in P, then 0 is called zero element in poset P.

0.1.3 Unit element in a poset : ([1], Page-2)

Let $< P , \leq >$ be a poset .If there exists 1 in P such that $1 \geq x$ for all x in P, then 1 is called unit (or one) element in poset P.

0.1.4 **Bounded poset : ([1], Page-2)**

A poset with the zero element and the unit element is called bounded poset.

0.1.5 **Lattice (as a poset) : ([1], Page-2)**

A poset $< L , \leq >$ is called a lattice if $\sup\{a , b\}$ and $\inf\{a , b\}$ exist for all a and b in L.

0.1.6 **Lattice (as an algebra) : ([2], Page-3)**

Let L be any nonempty set. If ' $\wedge$ ' and ' $\vee$ ' are binary operation defined on L then $< L , \wedge , \vee >$ is called lattice if the following conditions hold for all a , b , c in L

1) $a \wedge b = b \wedge a$	1$'$) $a \vee b = b \vee a$	(commutativity)
2) $a\wedge(b\wedge c) = (a\wedge b)\wedge c$	2$'$) $a\vee(b\vee c) = (a\vee b)\vee c$	(associativity)
3) $a \wedge a = a$	3$'$) $a \vee a = a$	(idempotency)
4) $a \wedge (a \vee b) = a$	4$'$) $a \vee (a \wedge b) = a$	(absorption law)

0.1.7 **Filter in a lattice : ([2], Page-23)**

A nonempty subset F of a lattice L is called a filter if

1) $x \wedge y \in F$ for all x and y in F ,

2) $x \leq y$ and $x \in F$ imply that $y \in F$.

0.1.8 **Proper filter in a lattice : ([2], Page-23)**

A filter F which is different from lattice L is called proper filter.

0.1.9 **Prime filter in a lattice** : ([2], Page-23)

A proper filter in a lattice is called prime filter if for all x and y in L, $x \vee y \in F$ imply that $x \in F$ or $y \in F$.

0.1.10 **Ideal in a lattice** : ([2],Page-20)

A nonempty subset I of a lattice L is called an ideal if

i) $x \vee y \in I$ for all x and y in I

ii) $x \leq y$ and $y \in I$ imply $x \in I$.

0.1.11 **Proper ideal in a lattice** : ([2], Page-21)

An ideal I which is different from lattice L is called Proper ideal.

0.1.12 **Prime ideal in lattice** : ([2], Page-21)

A proper ideal I in a lattice L is called a prime ideal if for all x and y in L, $x \wedge y \in I$ imply that $x \in I$ or $y \in I$.

0.1.13 **Moore family in lattice** : ([1], Page-111)

Let x be any nonempty set and $\mathcal{F} \subseteq \mathcal{P}(x)$. $\mathcal{F}$ is said to form a Moore family of subsets of x if i) $x \in \mathcal{F}$, ii) $\bigcap_{F_\alpha \in \mathcal{F}} F_\alpha \in \mathcal{F}$.

0.1.14 **Principal filter in a lattice** : ([2], Page-23)

Given an element a in L, the filter generated by {a} denoted by [a) (= $\{x \in L / x \geq a\}$) is called principal filter of L.

0.1.15 **Principal ideal in a lattice** : ([2], Page-21)

Given an element a in L, the ideal generated by {a}, denoted by (a] (= $\{x \in L / x \leq a\}$) is called principal ideal of L.

0.1.16 Quasicomplemented lattice : ([2], Page- 184)

A lattice with 1 is called quasicomplemented if for all x in L there exists $x^{\perp}$, the smallest element in L such that $x \vee x^{\perp} = 1$, $x^{\perp}$ is called quasicomplement of x in L or dual pseudocomplement of x .

0.1.17 Lattice homomorphism : ([2], Page-19)

Let L and L' be any two lattices. A function $f : L \dashrightarrow L'$ is called homomorphism if for all x and y in L

$$f(x \wedge y) = f(x) \wedge f(y) \quad \text{and}$$

$$f(x \vee y) = f(x) \vee f(y)$$

0.1.18 Pseudocomplemented lattice : ([2], Page-58)

A lattice with 0 is called pseudocomplemented if for all x in L there exists x^{*}, the largest element in L such that $x \wedge x^{*} = 0$, x^{*} is called the pseudocomplement of x in L.

0.1.19 Distributive lattice : ([2], Page-36)

A lattice L is said to be distributive lattice if for all x , y , z in L

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

0.1.20 Equivalence relation in a lattice : ([2], Page-24)

A relation ' θ ' defined on a lattice L is called an equivalence relation if the following conditions hold for all x , y , z in L

1) $x \equiv x(\theta)$ [Reflexivity]

2) $x \equiv y(\theta) \Rightarrow y \equiv x(\theta)$ [symmetrivity]

3) $x \equiv y(\theta)$ and $y \equiv z(\theta) \Rightarrow x \equiv z(\theta)$ [Transitivity]

0.1.21 <u>Congruence relation in a lattice</u> : ([2], Page-24)

An equivalence relation ' θ ' defined on a lattice L is called a congruence relation on L if for all x_1 , x_2 , y_1 , y_2 in L ,

$$x_1 \equiv y_1 (\theta) \quad \text{and} \quad x_2 \equiv y_2 (\theta) \quad \text{imply that}$$

$$x_1 \wedge x_2 \equiv y_1 \wedge y_2 (\theta) \text{ and } x_1 \vee x_2 \equiv y_1 \vee y_2 (\theta)$$

0.1.22 <u>Congruence class in a lattice</u> : ([2], Page-24)

Let ' θ ' be a congruence relation on a lattice L . For any x in L we define congruence class containing x as

$$[x]^\theta = \{ y \in L / x \equiv y(\theta) \}$$

0.1.23 <u>Quotient lattice</u> : ([2], Page-26)

Let ' θ ' be a congruence relation on a lattice L . Define

$$L/\theta = \{ [x]^\theta / x \in L \}$$

Define $\barwedge$ and $\veebar$ on L/θ as $[x]^\theta \barwedge [y]^\theta = [x \wedge y]^\theta$ and $[x]^\theta \veebar [y]^\theta = [x \vee y]^\theta$ for all x , y in L. The lattice $< L/\theta , \barwedge , \veebar >$ is called quotient lattice of a lattice $< L , \wedge , \vee >$.

0.1.24 <u>Cokernel of homomorphism of L</u> :

Let $f : L \to L'$ be a homomorphism of lattice L onto a lattice L′. We define co-kernel of f as coker f $= \{ x \in L / f(x) = 1' \} = \{ 1 \}$.

0.1.25 <u>Maximal ideal in a lattice</u> : ([1], Page-28)

A proper ideal M in a lattice L is called a maximal ideal in L if there does not exist any proper ideal J in L such that $M \subset J \subset L$.

0.1.26 **<u>Maximal filter in a lattice</u>** **: ([1], Page-28)**

A proper filter F in a lattice L is called maximal filter in L if there does not exist any proper filter J in L such that $M \supset J \supset L$.

□

§ 0.2 ***<u>Results</u>*** *:*

0.2.1 **<u>Result</u>** **:([2], Page-30)** Let $h : L \to L'$ be an onto homomorphism. Let P be prime ideal in L'. Inverse image of a prime ideal P is a prime ideal in L.

0.2.2 **<u>Zorn's lemma</u>** **: ([2], Page-74)**

Let A be any nonvoid set . Let $\phi \neq K \subseteq \mathcal{P}(A)$. Let $\mathcal{C}$ be any chain in K . If $\bigcup_{x \in \mathcal{C}} x \in K$ then K contains a maximal element .

0.2.3 **<u>Stone's (Separation) theorem</u>** **: ([2], Page-74)**

Let L be distributive lattice . If I is an ideal in L and F is a filter in L such that $I \cap F = \phi$, then there exists a prime ideal P in L such that $I \subseteq P$ and $P \cap F = \phi$.

0.2.4 **<u>Fundamental theorem of homomorphism</u>** **:([2], P-26 theorem 11)**

Every homomorphic image of a lattice L is isomorphic with some suitable quotient lattice L / θ .

0.2.5 **<u>Result</u>** **: ([2], Page-21)** Intersection of all filters in L is a filter.

0.2.6 **<u>Result</u>** : **([ ], Page-)** Intersection of all prime filters in a lattice need not be prime.