CHAPTER – III

On a subclass of univalent functions

ABSTRACT

In this third chapter of dissertation , we have introduced a new subfamily $D_n(\alpha, \beta, \gamma)$ of class S , of normalized univalent functions f in the unit disk $U = \{ z : |z| < 1 \}$, having Taylor's series expansion of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \ .$$

The main theme of the present chapter is to study various properties of functions in $D_n(\alpha, \beta, \gamma)$, having negative coefficients. We characterize the class and obtain distortion theorem, radius of convexity, closure properties and extreme points for the class $D_n(\alpha, \beta, \gamma)$.

Lastly by making use of known concept of neighborhood of analytic function introduced by Ruscheweyh [7], we give several inclusion relation involving $N_\delta(e)$. Also we define new classes $T_n^{(\lambda)}(\alpha, \beta, \gamma)$ and $P_n^{*(\lambda)}(\alpha, \beta, \gamma)$ and determine the neighborhood for these classes $T_n^{(\lambda)}(\alpha, \beta, \gamma)$ and $P_n^{*(\lambda)}(\alpha, \beta, \gamma)$.

1. INTRODUCTION

We introduce a new subfamily of S, of normalized univalent functions f that are holomorphic in the unit disk $U=\{z : |z|<1\}$.

Definition . Let $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ and let $n \in N_0$, we define, the class $D_n(\alpha,\beta,\gamma)$ of n-starlike function of order α, type β and γ by

$$D_n(\alpha,\beta,\gamma)=\{ f \in H(U): f(0)=f'(0)-1=0 \text{ and } |\ell_n(f,\alpha,\gamma;z)|<\beta \ , \ z \in U \}$$

where

$$l_n(f,\alpha,\gamma;z) = \frac{(D^n f(z))'-1}{2\gamma\left[(D^n f(z))'-\alpha \right]-\left[(D^n f(z))'-1 \right]} \quad , \quad z \in U .$$

We note that $D_0(\alpha,\beta,\gamma)$ is class introduced and studied by Kulkarni [5] .The class $D_0(0,\alpha,1)$ is the class studied by Caplinger [1]. The class $D_0(\alpha,1,\beta)$ is the class of holomorphic functions discussed by Juneja and Mogra [4].

In this section we are interesting in those members of $D_n(\alpha,\beta,\gamma)$ having negative coefficients.

Let T denote the subclass of S consisting of functions whose non-zero coefficients, from the second on, are negative; that is, an univalent function f is in T if and only if it can be expressed in the form

$$f(z) = z-\sum_{j=2}^{\infty} a_j z^j \quad a_j \geq 0, \ j = 2,3,\ldots \ . \tag{3.1.1}$$

We define the class $P_n^*(\alpha,\beta,\gamma)$ by

$$P_n^*(\alpha,\beta,\gamma) = D_n(\alpha,\beta,\gamma) \cap T \tag{3.1.2}$$

and obtain several interesting results for the class $P_n^*(\alpha,\beta,\gamma)$ and study basic properties , such as characterization, distortion theorems, radius of convexity and closure theorem in the line of Guta and Jain [2], Kulkarni [5] , Sarangi and Uralegaddii [8] .

2. CHARACTERIZATION OF CLASS $P_n^*(\alpha,\beta,\gamma)$

First we state the Characterization theorem, which completely characterizes the member of class $P_n^*(\alpha,\beta,\gamma)$.

Theorem 1. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and let* $n \in N_0$, *the function* f *of the form (1.1) is in* $P_n^*(\alpha,\beta,\gamma)$ *if and only if*

$$\sum_{j=2}^{\infty} j^{n+1}\left[1+\beta(2\gamma-1)\right] a_j \le 2\beta\gamma(1-\alpha) \quad . \tag{3.2.1}$$

The result is sharp.

Proof. We suppose that (3.2.1) holds. Then we have

$$|\,l_n(f,\alpha,\gamma;z)\,| = \left|\frac{\left(D^n f(z)\right)'-1}{2\gamma\left[\left(D^n f(z)\right)'-\alpha\right]-\left[\left(D^n f(z)\right)'-1\right]}\right|$$

$$= \left|\frac{\sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1}}{2\gamma(1-\alpha)-\sum_{j=2}^{\infty} j^{n+1} a_j (2\gamma-1) z^{j-1}}\right|$$

Let $|z|=1$, then

$$\left|\sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1}\right| - \beta\left|2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j^{n+1} a_j (2\gamma-1) z^{j-1}\right|$$

$$\leq \sum_{j=2}^{\infty} j^{n+1}\left[1+\beta(2\gamma-1)\right] a_j - 2\beta\gamma(1-\alpha) \leq 0 .$$

where we used (3.2.1).

From the last inequality we deduce

$$| l_n(f,\alpha,\gamma;z) | \leq \beta \qquad , |z|=1 .$$

Hence $\qquad | l_n(f,\alpha,\gamma;z) | < \beta \quad , z \in U$ and $f \in P_n^*(\alpha,\beta,\gamma)$.

Conversely, we assume that $f \in P_n^*(\alpha,\beta,\gamma)$.Then

$$| l_n(f,\alpha,\gamma;z) | < \beta \qquad , z \in U . \tag{3.2.2}$$

For $z \in [0,1)$ the inequality (3.2.2) can be written

$$-\beta < \frac{\sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1}}{2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j^{n+1} (2\gamma-1) a_j z^{j-1}} < \beta . \tag{3.2.3}$$

We note that $\qquad E(z) = 2\gamma(1-\alpha) - \sum_{j=2}^{\infty} j^{n+1} (2\gamma-1) a_j z^{j-1} > 0 \quad z \in [0,1),$

because $E(z) \neq 0$ for $z \in [0,1)$ and $E(0) = 2\gamma(1-\alpha) > 0$.Upon clearing the denominator in (3.2.3) and letting $z \to 1$ through real values, we deduce

$$\sum_{j=2}^{\infty} j^{n+1}\, a_j \le 2\beta\gamma(1-\alpha) - \beta\sum_{j=2}^{\infty} j^{n+1}\,(2\gamma-1)\, a_j \quad .$$

Thus

$$\sum_{j=2}^{\infty} j^{n+1}\left[1+\beta(2\gamma-1)\right]\, a_j \le 2\beta\gamma(1-\alpha) \quad .$$

The extremal functions are

$$f_j(z) = z - \frac{2\beta\gamma\,(1-\alpha)}{j^{n+1}\,[\,1+\beta\,(\,2\gamma-1\,)\,]}\, z^j \qquad j = 2,3,\ldots \quad . \tag{3.2.4}$$

Corollary 1. *If* $f \in P_n^*(\alpha,\beta,\gamma)$ *then*

$$a_j \le \frac{2\,\beta\gamma\,(1-\alpha)}{j^{n+1}\;[\,1+\beta\,(2\gamma-1)\,]} \qquad j = 2,3,\ldots \quad .$$

The result is sharp and the extremal functions are given by (3.2.4).

We state following particular cases for Theorem 1.

Corollary 2. *A function of the form (3.1.1) is in* $P_0^*(0,\alpha,1)$, *if and only if*

$$\sum_{j=2}^{\infty} j\;(1+\alpha)\; a_j \le 2\alpha\gamma \quad .$$

This result is sharp. This result is due to Caplinger [1].

Next is the similar characterization for the class of univalent functions studied by Juneja and Mogra [4] having negative coefficients.

Corollary 3. *A function of the form (3.1.1) is in* $P_0^*(\alpha,1,\beta)$, *if and only if*

$$\sum_{j=2}^{\infty} j\ a_j \leq (1-\alpha)$$

This result is sharp.

In the same vein we also have a corresponding result for univalent function proved by Gupta and Jain [2].

Corollary 4. *A function of the form (3.1.1) is in* $P_0^*(\alpha, \beta, 1)$, *if and only if*

$$\sum_{j=2}^{\infty} j\ (1+\beta)\ a_j \leq 2\beta\gamma(1-\alpha)$$

This result is sharp.

Next we obtain a theorem which supplies the extreme point of the class $P_n^*(\alpha, \beta, \gamma)$.

Theorem 2. *Let*

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\gamma\beta(1-\alpha)}{j^{n+1}[1+\beta(2\gamma-1)]} z^j \quad (3.2.5)$$

Then $f \in P_n^*(\alpha, \beta, \gamma)$ *if it can be expressed in the form*

$$f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z) \quad (3.2.6)$$

where

$$\lambda_j \geq 0 \quad (j=1,2,3,\ldots) \quad and \quad \lambda_1 + \sum_{j=2}^{\infty} \lambda_j = 1. \tag{3.2.7}$$

Proof. Suppose that

$$f(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z)$$

$$= z - \sum_{j=2}^{\infty} \frac{2\gamma\beta(1-\alpha)\lambda_j}{j^{n+1}[1+\beta(2\gamma-1)]} z^j .$$

Since

$$\sum_{j=2}^{\infty} j^{n+1}[1+\beta(2\gamma-1)] \frac{2\gamma\beta(1-\alpha)}{j^{n+1}[1+\beta(2\gamma-1)]} \lambda_j$$

$$= 2\beta\gamma(1-\alpha)\sum_{j=2}^{\infty} \lambda_j$$

$$\leq 2\beta\gamma(1-\alpha) .$$

By Theorem 1, $f \in P_n^*(\alpha,\beta,\gamma)$.

Conversely, we suppose that $f \in P_n^*(\alpha,\beta,\gamma)$.
Since

$$a_j \leq \frac{2\gamma\beta(1-\alpha)}{j^{n+1}[1+\beta(2\gamma-1)]} \qquad j=2,3,\ldots,$$

setting

$$\lambda_j = \frac{j^{n+1}[\,1+\beta(2\gamma-1)\,]}{2\gamma\beta(1-\alpha)}\; a_j$$

and

$$\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j \,.$$

Then we have

$$f(z) = \lambda_1\, f_1(z) + \sum_{j=2}^{\infty} \lambda_j\, f_j(z)\,.$$

This completes the proof of Theorem 2.

Corollary 1. The extreme points of $P_n^*(\alpha,\beta,\gamma)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\gamma\beta(1-\alpha)}{j^{n+1}[1+\beta(2\gamma-1)]}\, z^j \qquad j = 2,3,\ldots\,.$$

We give the following particular cases for above theorem.

Corollary 2. The extreme points of $P_0^*(0,\alpha,1)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\alpha}{j\,(1+\alpha)}\, z^j \qquad (j = 1,2,3,\ldots)\,.$$

This result is due to Caplinger [1].

Corollary 3. The extreme points of $P_0^*(\alpha, \beta, \gamma)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\gamma\beta(1-\alpha)}{j\,[1+\beta(2\gamma-1)]}\, z^j \qquad (j = 1, 2, 3, \ldots) .$$

This is due to the class studied by Kulkarni [5].

Lastly we also state the corollary for the class of the functions introduced by Jain and Gupta [2].

Corollary 4. The extreme points of $P_0^*(\alpha, \beta, 1)$ are the functions

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{2\beta(1-\alpha)}{j^{n+1}(1+\beta)}\, z^j \quad (j = 1, 2, 3, \ldots) .$$

3. SOME PROPERTIES OF CLASS $P_n^*(\alpha,\beta,\gamma)$

Now we prove some properties of class $P_n^*(\alpha,\beta,\gamma)$, like distortion theorem , radius of convexity and closure theorems.

Theorem 3. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $\gamma \in (1/2,1]$ *and let* $n \in N_0$, *if* $f \in P_n^*(\alpha,\beta,\gamma)$, *then for* $0 < |z| = r < 1$, *we have*

$$r - \frac{\beta\gamma(1-\alpha)}{2^n[1+\beta(2\gamma-1)]}\, r^2 \le |f(z)| \le r + \frac{\beta\gamma(1-\alpha)}{2^n[1+\beta(2\gamma-1)]}\, r^2 \quad (3.3.1)$$

and

$$1-\frac{\beta\gamma(1-\alpha)}{2^{n-1}[1+\beta(2\gamma-1)]}\,r\le|f'(z)|\le 1+\frac{\beta\gamma(1-\alpha)}{2^{n-1}[1+\beta(2\gamma-1)]}\,r \quad (3.3.2)$$

The bounds in (3.3.1) and (3.3.2) are sharp.

Proof. From (3.2.1) we have

$$2^{n+1-k}[1+\beta(2\gamma-1)]\sum_{j=2}^{\infty}j^k a_j\le\sum_{j=2}^{\infty}j^{n+1}[1+\beta(2\gamma-1)]\,a_j\le 2\beta\gamma(1-\alpha)$$

and

$$\sum_{j=2}^{\infty}j^k a_j\le\frac{\beta\gamma(1-\alpha)}{2^{n-k}[1+\beta(2\gamma-1)]}\quad. \qquad (3.3.3)$$

Using (3.3.3) with k = 0, for $0<|z|=r<1$ we obtain

$$|f(z)|\le r+\sum_{j=2}^{\infty}a_j r^j\le r+r^2\sum_{j=2}^{\infty}a_j$$

$$\le r+\frac{\beta\gamma(1-\alpha)}{2^n[1+\beta(2\gamma-1)]}\,r^2,$$

and

$$|f(z)|\ge r-\frac{\beta\gamma(1-\alpha)}{2^n[1+\beta(2\gamma-1)]}\,r^2.$$

Similarly using (3.3.3) with k = 1, for $0<|z|=r<1$ we obtain

$$|f'(z)|\le 1+r\sum_{j=2}^{\infty}j\,a_j$$

$$\leq 1 + \frac{\beta\gamma(1-\alpha)}{2^{n-1}[1+\beta(2\gamma-1)]} r ,$$

and

$$|f'(z)| \geq 1 - \frac{\beta\gamma(1-\alpha)}{2^{n-1}[1 + \beta(2\gamma-1)]} r .$$

This completes the proof of Theorem 3. Sharpness are attained by the function

$$f(z) = z - \frac{\beta\gamma(1-\alpha)}{2^{n}[1+\beta(2\gamma-1)]} z^2 \qquad (z = \pm r) \qquad . \qquad (3.3.4)$$

Keeping our intension in view, we go to state some special cases of Theorem 3.

Corollary 1. *A function* $f \in P_0^*(\alpha,\beta,\gamma)$, *then for* $0 < |z| = r < 1$, *we have*

$$r - \frac{\beta\gamma(1-\alpha)}{[1+\beta(2\gamma-1)]} r^2 \leq |f(z)| \leq r + \frac{\beta\gamma(1-\alpha)}{[1+\beta(2\gamma-1)]} r^2$$

and

$$1 - \frac{2\beta\gamma(1-\alpha)}{[1+\beta(2\gamma-1)]} r \leq |f'(z)| \leq 1 + \frac{2\beta\gamma(1-\alpha)}{[1+\beta(2\gamma-1)]} r .$$

The result is sharp. This result is due to Kulkarni [5].

Corollary 2. *A function* $f \in P_0^*(\alpha,\beta,1)$, *then for* $0 < |z| = r < 1$, *we have*

$$r - \frac{\beta(1-\alpha)}{(1+\beta)} r^2 \leq |f(z)| \leq r + \frac{\beta(1-\alpha)}{(1+\beta)} r^2$$

and

$$1-\frac{2\beta(1-\alpha)}{(1+\beta)}\,r \le |f'(z)| \le 1+\frac{2\beta(1-\alpha)}{(1+\beta)}\,r\ .$$

The result is sharp. This result is due to Gupta and Jain [2].

Corollary 3. *A function* $f \in P_0^*(0,\alpha,1)$, *then for* $0 < |z| = r < 1$, *we have*

$$r-\frac{\alpha}{(1+\alpha)}\,r^2 \le |f(z)| \le r+\frac{\alpha}{(1+\alpha)}\,r^2$$

and

$$1-\frac{2\alpha}{(1+\alpha)}\,r \le |f'(z)| \le 1+\frac{2\alpha}{(1+\alpha)}\,r\ .$$

The result is sharp. This is due to the class studied by Caplinger [1].

We now state the theorem which gives the disk contained in the range set of functions in class $P_n^*(\alpha,\beta,\gamma)$.

Theorem 4. *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk*

$$|w| < 1-\frac{\gamma\beta\ (1-\alpha)}{2^n\ [1+\beta(2\gamma-1)]}$$

by any $f \in P_n^*(\alpha,\beta,\gamma)$.

Proof. The result follows upon by letting $r \to 1$ in (3.3.1).

In the next theorem we determine the radius of convexity for the functions in $P_n^*(\alpha,\beta,\gamma)$.

Theorem 5. *If the function* $f \in P_n^*(\alpha,\beta,\gamma)$ *, then* f *is convex in the disk*

$$|z| < r = r(\alpha,\beta,\gamma,n) = \inf_j \left(\frac{j^{n-1}[1+\beta(2\gamma-1)]}{2\beta\gamma(1-\alpha)} \right)^{\frac{1}{j-1}}, \quad (j=2,3,\ldots). \quad (3.3.5)$$

This result is sharp, with the extremal function as given in (3.2.4).

Proof. It suffices to show that

$$\left| \frac{z f''(z)}{f'(z)} \right| \le 1 \qquad \text{in} \quad |z| \le r(\alpha,\beta,\gamma,n). \qquad (3.3.6)$$

In view of (3.2.1), we have

$$\left| \frac{z f''(z)}{f'(z)} \right| \le \frac{\sum_{j=2}^{\infty} j(j-1) a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1}} .$$

Thus (3.3.6) follows if

$$\sum_{j=2}^{\infty} j(j-1) a_j |z|^{j-1} \le 1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1}$$

or

$$\sum_{j=2}^{\infty} j^2 a_j |z|^{j-1} \le 1 . \qquad (3.3.7)$$

Also by Theorem 1, we have

$$\sum_{j=2}^{\infty} \frac{j^{n+1}\,[\,1+\beta\,(2\gamma-1)\,]}{2\,\beta\gamma\,(1-\alpha\,)}\,a_j \le 1 \quad . \tag{3.3.8}$$

Hence f is convex if

$$j^2\,|\,z\,|^{j-1} \le \frac{j^{n+1}\,[1+\beta\,(2\gamma-1)\,]}{2\beta\gamma\,(1-\alpha\,)} \quad .$$

Solving for $|z|$, we obtain

$$|\,z\,| \le \left(\frac{j^{n-1}\,[\,1+\beta\,(2\gamma-1)\,]}{2\,\gamma\beta\,(1-\alpha\,)} \right)^{\frac{1}{j-1}}, \qquad (j=2,3,...)$$

setting $|z| = r(\alpha,\beta,\gamma,n)$, the result follows.

Now we state some particular case of above theorem.

Corollary 1. *If the function* $f \in P_0^*(\alpha,\beta,\gamma)$, *then* f *is convex in the disk*

$$|\,z\,| < r = r\,(\,\alpha,\beta,\gamma,0\,) = \inf_j \left(\frac{1+\beta\,(\,2\gamma-1\,)}{2\,j\,\beta\gamma\,(\,1-\alpha\,)} \right)^{\frac{1}{j-1}} \qquad (j=2,3,...).$$

This result is sharp.

Next corollary gives the radius of convexity for the class introduced and studied by Gupta and Jain [2].

Corollary 2. *If the function* $f \in P_0^*(\alpha,\beta,1)$, *then* f *is convex in the disk*

$$|\,z\,| < r = r\,(\,\alpha,\beta,1,0\,) = \inf_j \left(\frac{(\,1+\beta\,)}{2\,j\,\beta(\,1-\alpha\,)} \right)^{\frac{1}{j-1}} \qquad (j=2,3,...).$$

This result is sharp.

Corollary 3. *If the function* $f \in P_0^*(0,\alpha,1)$, *then* f *is convex in the disk*

$$|z| < r = r(0,\alpha,1,0) = \inf_j \left(\frac{(1+\alpha)}{2j\alpha} \right)^{\frac{1}{j-1}} \qquad (j=2,3,\ldots).$$

This result is sharp. This is due to Caplinger [1].

In [8] Sarangi and Uralegaddi obtained the radius of univalence of holomorphic functions with negative coefficients under the different conditions, on the same line we also obtain results for the class $P_n^*(\alpha,\beta,\gamma)$.

Theorem 6. *If the function* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geq 0, j=2,3,\ldots$ *is in* $P_n^*(\alpha,\beta,\gamma)$ *and* $f(z) = \frac{1}{2}[\,zF(z)\,]'$ *then* $f(z)$ *is* n*-starlike function of order* λ, *type* δ *in*

$$|z| < r = r(\alpha,\beta,\gamma,\delta,\lambda,n) = \inf_j \left(\frac{j^{n+1}[1+\beta(2\gamma-1)]\,(2\delta-\lambda-1)}{j^n\,\beta\gamma\,(1-\alpha)(j+1)(j-\lambda)} \right)^{\frac{1}{j-1}}$$

$$(j=2,3,\ldots). \qquad (3.3.9)$$

Proof. It suffices to show that $\operatorname{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \lambda$ for $|z| < r = r(\alpha,\beta,\gamma,\delta,\lambda,n)$, by definition of $f(z)$ we have

$$f(z) = \frac{1}{2}[\,zF(z)\,]' = z - \sum_{j=2}^{\infty} \left(\frac{j+1}{2} \right) a_j z^j \quad ,$$

now

$$\left| \frac{D^{n+1} f(z)}{D^n f(z)} - \delta \right| = \left| \frac{(1-\delta) - \sum_{j=2}^{\infty} j^{n+1} \left(\frac{j+1}{2} \right) a_j \, z^{j-1} (j-\delta)}{1 - \sum_{j=2}^{\infty} j^n \left(\frac{j+1}{2} \right) a_j \, z^{j-1}} \right|$$

$$\leq \frac{(1-\delta) - \sum_{j=2}^{\infty} j^{n+1} \left(\frac{j+1}{2} \right) a_j \, |z|^{j-1} (j-\delta)}{1 - \sum_{j=2}^{\infty} j^n \left(\frac{j+1}{2} \right) a_j \, |z|^{j-1}} \quad .$$

Hence , $\left| \frac{D^{n+1} f(z)}{D^n f(z)} - \delta \right| \leq (\delta - \lambda)$ if

$$(1-\delta) + \sum_{j=2}^{\infty} j^{n+1} \left(\frac{j+1}{2} \right) a_j \, (j-\delta) \, |z|^{j-1} \leq (\delta-\lambda) \left[1 - \sum_{j=2}^{\infty} j^n \left(\frac{j+1}{2} \right) a_j \, |z|^{j-1} \right]$$

$$\sum_{j=2}^{\infty} \frac{j^n \left(\frac{j+1}{2} \right) (j-\lambda) \, |z|^{j-1} \, a_j}{(2\delta - \lambda - 1)} \leq 1 \quad .$$

On account of coefficient inequality , we have

$$\sum_{j=2}^{\infty} \frac{j^{n+1} \left(\frac{j+1}{2} \right) (j-\lambda) \, |z|^{j-1}}{(2\delta - \lambda - 1)} \, a_j \leq \sum_{j=2}^{\infty} \frac{j^{n+1} \left[1 + \beta (2\gamma - 1) \right]}{2 \, \beta\gamma \, (1-\alpha)} \, a_j \ ,$$

solving for $|z|$, we get

$$|z| \leq \left(\frac{j^{n+1} \left[1 + \beta \, (2\gamma - 1) \right] \, (2\delta - \lambda - 1)}{j^n \, \beta\gamma \, (1-\alpha) \, (j+1) \, (j-\lambda)} \right)^{\frac{1}{j-1}}$$

.

We state some particular cases for above theorem.

Corollary 1. *If the function* $F \in P_0^*(\alpha,\beta,\gamma)$ *and* $f(z) = \frac{1}{2}[\, z\, F(z)\,]'$, *then* $f(z)$ *is starlike function of order* λ, *type* δ *in*

$$|z| < r = r(\alpha,\beta,\gamma,\delta,\lambda,0) = \inf_j \left(\frac{j\ [1+\beta\,(2\gamma-1)]\ (2\delta-\lambda-1)}{\beta\gamma\,(1-\alpha)\,(j+1)\,(j-\lambda)} \right)^{\frac{1}{j-1}}.$$

$(j = 2,3,\ldots)$.

This result is due to Joshi [3].

Corollary 2. *If the function* $F \in P_0^*(\alpha,\beta,1)$ *and* $f(z) = \frac{1}{2}[\, z\, F(z)\,]'$, *then* $f(z)$ *is starlike function of order* λ, *type* δ *in*

$$|z| < r = r(\alpha,\beta,1,\delta,\lambda,0) = \inf_j \left(\frac{j\ (\,1+\beta\,)\ (2\delta-\lambda-1)}{\beta\,(1-\alpha)\,(j+1)\,(j-\lambda)} \right)^{\frac{1}{j-1}}.$$

$(j = 2,3,\ldots)$.

This is the result for the class studied by Gupta and Jain [2] .

Corollary 3. *If the function* $F \in P_0^*(\,0,\alpha,1\,)$ *and* $f(z) = \frac{1}{2}[\, z\, F(z)\,]'$ *then* $f(z)$ *is Starlike function of order* λ, *type* δ *in*

$$|\, z\, | < r = r(0,\alpha,1,\delta,\lambda,0) = \inf_j \left(\frac{2\,j\ (\,1+\alpha\,)\ (\delta-1)}{\alpha\ (j+1)\,(j-\lambda)} \right)^{\frac{1}{j-1}}.$$

$(j = 2,3,\ldots)$.

This is due to Caplinger [1] .

Theorem 7. *If the function* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$ $\quad a_j \geq 0, j = 2,3,\ldots$ *is in* $P_n^*(\alpha,\beta,\gamma)$ *and* $f(z) = \frac{1}{2}[\, z\, F(z)\,]'$, *then* $\operatorname{Re} f'(z) > \lambda$ *for* $0 \leq \lambda < 1$ *of order* λ, *type* δ *in*

$$|z| < r = r(\alpha,\beta,\gamma,\lambda,n) = \inf_j \left(\frac{j^n \, [1+\beta\,(2\gamma-1)\,] \,\, (1-\lambda)}{(j+1)\,\beta\gamma\,(1-\alpha)} \right)^{\frac{1}{j-1}} \qquad (j=2,3,\ldots).$$

(3.3.10)

Proof. We show that $|\, f'(z)-1 \,| \leq 1-\lambda$ for $|z| < r = r(\alpha,\beta,\gamma,\lambda,n)$.

We have

$$|\, f'(z)-1 \,| \leq \sum_{j=2}^{\infty} j \left(\frac{j+1}{2}\right) |z|^{j-1} \, a_j \quad .$$

Hence $|\, f'(z)-1 \,| \leq 1-\lambda$ if

$$\sum_{j=2}^{\infty} j \left(\frac{j+1}{2}\right) |z|^{j-1} \, a_j \leq 1-\lambda \, .$$

On account of coefficient inequality, we have

$$\sum_{j=2}^{\infty} \frac{j \left(\frac{j+1}{2}\right) |z|^{j-1}}{(1-\lambda)} \, a_j \leq \sum_{j=2}^{\infty} \frac{j^{n+1} \left[1+\beta(2\gamma-1)\right]}{2\,\beta\gamma\,(1-\alpha)} \, a_j \ ,$$

solving for $|z|$, we get

$$|z| \leq \left(\frac{j^n \, [1+\beta\,(2\gamma-1)\,] \,\, (1-\lambda)}{(j+1)\,\beta\gamma\,(1-\alpha)} \right)^{\frac{1}{j-1}}$$

Hence the Theorem 7.

Now we put some particular cases .

Corollary 1. *If the function* $F \in P_0^*(\alpha,\beta,\gamma)$ *and* $f(z) = \frac{1}{2}[\, z\, F(z)\,]'$ *then* Re $f'(z) > \lambda$ *for* $0 \le \lambda < 1$ *of order* λ, *type* δ *in*

$$|z| < r = r(\alpha,\beta,\gamma,\lambda,0) = \inf_j \left(\frac{[1+\beta\,(2\gamma-1)\,]\,(1-\lambda)}{(j+1)\,\beta\gamma\,(1-\alpha)} \right)^{\frac{1}{j-1}} \qquad (j = 2,3,\ldots).$$

This result is due Joshi [3].

Corollary 2. *If the function* $F \in P_0^*(\alpha,\beta,1)$ *and* $f(z) = \frac{1}{2}[\, z\, F(z)\,]'$ *then* Re $f'(z) > \lambda$ *for* $0 \le \lambda < 1$ *of order* λ, *type* δ *in*

$$|\, z\, | < r = r(\alpha,\beta,1,\lambda,0) = \inf_j \left(\frac{(\,1+\beta\,)\,(1-\lambda)}{(j+1)\,\beta\,(1-\alpha)} \right)^{\frac{1}{j-1}} \qquad (j = 2,3,\ldots).$$

This is new result obtained for class defined by Gupta and Jain [2].

Corollary 3. *If the function* $F \in P_0^*(0,\alpha,1)$ *and* $f(z) = \frac{1}{2}[\, z\, F(z)\,]'$ *then* Re $f'(z) > \lambda$ *for* $0 \le \lambda < 1$ *of order* λ, *type* δ *in*

$$|\, z\, | < r = r(0,\alpha,1,\lambda,0) = \inf_j \left(\frac{(\,1+\alpha\,)\,(1-\lambda)}{(j+1)\,\alpha} \right)^{\frac{1}{j-1}} \qquad (j = 2,3,\ldots).$$

This is due to Caplinger [1] .

Theorem 8. *The class* $P_n^*(\,\alpha,\beta,\gamma\,)$ *is convex.*

Proof. Let $f_1(\,z\,) = z - \sum_{j=2}^{\infty} a_j\, z^j$ and $f_2(\,z\,) = z - \sum_{j=2}^{\infty} b_j\, z^j$ be in $P_n^*(\,\alpha,\beta,\gamma\,)$.

For $0 \le \lambda \le 1$, we shall prove that $F(\,z\,) = \lambda\, f_1(\,z\,) + (\,1-\lambda\,)\, f_2(\,z\,)$ is also in class $P_n^*(\,\alpha,\beta,\gamma\,)$.

Since for $0 \le \lambda \le 1$,

$$F(z) = z - \sum_{j=2}^{\infty} \left[\lambda a_j + (1-\lambda) b_j\right] z^j \ , \qquad (3.3.11)$$

we observe that

$$\sum_{j=2}^{\infty} j^{n+1} \left[1+\beta(2\gamma-1)\right] \left\{ \lambda a_j + (1-\lambda) b_j \right\}$$

$$= \lambda \sum_{j=2}^{\infty} j^{n+1} \left[1+\beta(2\gamma-1)\right] a_j + (1-\lambda) \sum_{j=2}^{\infty} j^{n+1} \left[1+\beta(2\gamma-1)\right] b_j$$

$$\le 2\beta\gamma(1-\alpha).$$

Hence $F(z) \in P_n^*(\alpha, \beta, \gamma)$. This completes the proof of Theorem 8.

4. NEIGHBORHOODS OF UNIVALENT FUNCTIONS

The main object of present section is to investigate the δ-neighborhoods of the classes $T_n(\alpha, \beta, \gamma)$ and $P_n^*(\alpha, \beta, \gamma)$ subclasses of the class T of normalized analytic and univalent functions in unit disk U with negative coefficients.

We define the δ-neighborhood of a function $f \in T$ by

$$N_\delta(f) = \left\{ g \in T : g(z) = z - \sum_{j=2}^{\infty} b_j z^j \quad and \quad \sum_{j=2}^{\infty} j \left| a_j - b_j \right| \le \delta \right\}. \qquad (3.4.1)$$

In particular, for the identity function

$$e(z) = z, \tag{3.4.2}$$

we immediately have

$$N_\delta(e) = \{ g \in T : g(z) = z - \sum_{j=2}^{\infty} b_j z^j \quad and \quad \sum_{j=2}^{\infty} j | b_j | \le \delta \}. \tag{3.4.3}$$

Motivated by Ruscheweyh [7], Orhan and Kamali [6], we now prove some inclusion relations involving $N_\delta(e)$

Theorem 9. Let

$$\delta = \frac{2\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} \tag{3.4.4}$$

then

$$T_n(\alpha,\beta,\gamma) \subset N_\delta(e). \tag{3.4.5}$$

Proof. Let $f \in T_n(\alpha,\beta,\gamma)$, then using characterization theorem of $P_n^*(\alpha,\beta,\gamma)$, it follows that

$$2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]\sum_{j=2}^{\infty} j a_j \le 2\gamma\beta(1-\alpha). \tag{3.4.6}$$

Thus

$$\sum_{j=2}^{\infty} j a_j \le \frac{2\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]}$$

that is

$$\sum_{j=2}^{\infty} j a_j \le \frac{2\gamma\beta(1-\alpha)}{2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]} = \delta \tag{3.4.7}$$

which in view of definition (3.4.3), proves the Theorem 9.

Theorem 10. Let

$$\delta = \frac{2\gamma\beta(1-\alpha)}{2^n[1+\beta(2\gamma-1)]} \tag{3.4.8}$$

then

$$P_n^*(\alpha,\beta,\gamma) \subset N_\delta(e) . \tag{3.4.9}$$

Proof. Let $f \in P_n^*(\alpha,\beta,\gamma)$, then using chacterization theorem of $P_n^*(\alpha,\beta,\gamma)$, it follows that

$$2^n[1+\beta(2\gamma-1)]\sum_{j=2}^{\infty} j a_j \le 2\gamma\beta(1-\alpha). \tag{3.4.10}$$

Thus

$$\sum_{j=2}^{\infty} j a_j \le \frac{2\gamma\beta(1-\alpha)}{2^n[1+\beta(2\gamma-1)]}$$

that is

$$\sum_{j=2}^{\infty} j a_j \le \frac{2\gamma\beta(1-\alpha)}{2^n[1+\beta(2\gamma-1)]} = \delta \tag{3.4.11}$$

which in view of definition (3.4.3), proves the Theorem 10.

Now , we determine the neighborhood for each classes

$$T_n^{(\lambda)}(\alpha,\beta,\gamma) \text{ and } P_n^{*(\lambda)}(\alpha,\beta,\gamma)$$

which we define as follows. A function $f \in T$ is said to be in class $T_n^{(\lambda)}(\alpha,\beta,\gamma)$, if there exists a function $g \in T_n(\alpha,\beta,\gamma)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1-\lambda \qquad (0 \le \lambda < 1). \tag{3.4.12}$$

Analogously, a function $f \in T$ is said to be in class $P_n^{*(\lambda)}(\alpha, \beta, \gamma)$, if there exists a function $g \in P_n^*(\alpha, \beta, \gamma)$ such that inequality (3.4.12) holds.

Theorem 11. *If* $g \in T_n(\alpha, \beta, \gamma)$ *and*

$$\lambda = 1 - \frac{\delta\, 2^{n-1}\left[\,1+\beta\,(4\gamma-2\gamma\alpha-1)\,\right]}{2^n\left[\,1+\beta\,(4\gamma-2\gamma\alpha-1)-2\,\gamma\beta\,(1-\alpha)\,\right]} \tag{3.4.13}$$

then

$$N_\delta(g) \subset T_n^{(\lambda)}(\alpha, \beta, \gamma)\,. \tag{3.4.14}$$

Proof. Suppose that $f \in N_\delta(g)$, then

$$\sum_{j=2}^{\infty} j\,\left| a_j - b_j \right| \le \delta\,, \tag{3.4.15}$$

which implies

$$\sum_{j=2}^{\infty} \left| a_j - b_j \right| \le \frac{\delta}{2}\,. \tag{3.4.16}$$

Since $g \in T_n(\alpha, \beta, \gamma)$ from characterization theorem of $T_n(\alpha, \beta, \gamma)$, we have

$$\sum_{j=2}^{\infty} b_j \le \frac{2\,\gamma\beta\,(1-\alpha)}{2^n\left[\,1+\beta\,(4\gamma-2\gamma\alpha-1)\,\right]}, \tag{3.4.17}$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{j=2}^{\infty} |a_j - b_j|}{1 - \sum_{j=2}^{\infty} b_j}$$

$$\leq \frac{\delta}{2} \; \frac{1}{1 - \dfrac{2\gamma\beta(1-\alpha)}{2^n[1+\beta(4\gamma-2\gamma\alpha-1)]}}$$

$$= \frac{\delta\, 2^{n-1}[1+\beta(4\gamma-2\gamma\alpha-1)]}{2^n[1+\beta(4\gamma-2\gamma\alpha-1)-2\gamma\beta(1-\alpha)]}$$

$$= 1 - \lambda,$$

provided that λ is given precisely by (3.4.13). Thus by definition , $f \in T_n^{(\lambda)}(\alpha,\beta,\gamma)$ for λ given by (3.4.13), which completes the proof of Theorem 11.

Theorem 12. *If* $g \in P_n^*(\alpha,\beta,\gamma)$ *and*

$$\lambda = 1 - \frac{\delta\, 2^n[1+\beta(2\gamma-1)]}{2^{n+1}[1+\beta(2\gamma-1)-2\gamma\beta(1-\alpha)]} \qquad (3.4.18)$$

then

$$N_\delta(g) \subset P_n^{*(\lambda)}(\alpha,\beta,\gamma) . \qquad (3.4.19)$$

Proof. Suppose that $f \in N_\delta(g)$, then

$$\sum_{j=2}^{\infty} j\, |a_j - b_j| \leq \delta \;, \qquad (3.4.20)$$

which implies

$$\sum_{j=2}^{\infty} |a_j - b_j| \leq \frac{\delta}{2} \;. \qquad (3.4.21)$$

Since $g \in P_n^*(\alpha,\beta,\gamma)$ from characterization theorem of $P_n^*(\alpha,\beta,\gamma)$, we have

$$\sum_{j=2}^{\infty} b_j \leq \frac{2\gamma\beta(1-\alpha)}{2^{n+1}[1+\beta(2\gamma-1)]}, \tag{3.4.22}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{j=2}^{\infty} |a_j - b_j|}{1 - \sum_{j=2}^{\infty} b_j}$$

$$\leq \frac{\delta}{2} \frac{1}{1 - \frac{2\gamma\beta(1-\alpha)}{2^{n+1}[1+\beta(2\gamma-1)]}}$$

$$= \frac{\delta 2^n [1+\beta(4\gamma-2\gamma\alpha-1)]}{2^{n+1}[1+\beta(2\gamma-1)-2\gamma\beta(1-\alpha)]}$$

$$= 1 - \lambda,$$

provided that λ is given precisely by (3.4.18). Thus by definition, $f \in P_n^{*(\lambda)}(\alpha, \beta, \gamma)$ for λ given by (3.4.18), which completes the proof of Theorem 12.

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