CHAPTER-IV

GEOMETRY OF A WORLDLINE IN ROBERTSON-WALKER SPACE-TIME

1. Introduction

In this chapter we exploited the NP formalism to study the geometry of the World line of the time-like particle in the Robertson-Walker space-time. In Sec.2 using differential calculus the tetrad components relative to Robertson-Walker space-time are obtained. In Sec.3, we define the components of the null tetrad vector fields. In Sec.4 we find the values of the Christofell symbols. For the choice of the tetrad components in Sec.3, the Newman-Penrose Spin Coefficients are evaluated in the Sec.5. It is shown for Robertson-Walker space-time the spin coefficients $\kappa, \sigma, \tau, \lambda, \nu$ vanish. Consequently, on the basis of Goldberg-Sachs theorem we conclude that the Robertson-Walker space-time is of Petrov-type D. In Sec.6 the components of the Riemann Curvature tensor are evaluated. Later these are used to find the Riemann Curvature at a point of Robertson-Walker space-time in the Sec.7. In the Sec.-8, with the help of rheotetrad introduced in the first chapter the expression for the Curvature field K=0, the Torsion field T=0 and the Bitorsion field B=0 of the worldline of the time like particle are derived through NP spin coefficients. Hence the world line of the particle is a straight line.

2. ROBERTSON-WALKER SPACE-TIME AND TETRAD VECTORS

We start with the Robertson-Walker Space-time given by

$$ds^2 = dt^2 - s^2(t)\left[dr^2 + f^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$ds^2 = dt^2 - s^2(t)dr^2 - s^2(t)f^2(r)d\theta^2 - s^2(t)f^2(r)\sin^2\theta d\phi^2 \qquad (2.1)$$

where $f^2(r) = \sin^2 r$ where k=1

$= r^2$ where k=0

$= \sinh^2 r$ where k= -1

The covariant tensor components of the metric tensor are given by

$$g_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s^2(t) & 0 & 0 \\ 0 & 0 & -s^2(t)f^2(r) & 0 \\ 0 & 0 & 0 & -s^2(t)f^2(r)\sin^2\theta \end{bmatrix} \qquad (2.2)$$

Then $g = |g_{ab}| = -s^6(t)f^4(r)\sin^2\theta$ (2.3)

The contravariant components of the metric tensor are obtained using equation (2.4) from chapter-II.

$$g^{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{-1}{s^2(t)} & 0 & 0 \\ 0 & 0 & \dfrac{-1}{s^2(t)f^2(r)} & 0 \\ 0 & 0 & 0 & \dfrac{-1}{s^2(t)f^2(r)\sin^2\theta} \end{bmatrix} \qquad (2.4)$$

Thus by formula (2.6) from chapter-I, corresponding to 4-basis 1-forms $e_a^{(\alpha)}$, we have four basis 1-forms θ^α as $\theta^\alpha = e_a^{(\alpha)} dx^a$. We express the Robertson-Walker Space-time (2.1) in terms of the basis 1-forms θ^α as

$$ds^2 = \left(\theta^1\right)^2 - \left(\theta^2\right)^2 - \left(\theta^3\right)^2 - \left(\theta^4\right)^2 \tag{2.5}$$

where we define 1-forms θ^α as

$$\theta^1 = dt,$$

$$\theta^2 = s(t)dr,$$

$$\theta^3 = s(t)f(r)d\theta,$$

$$\theta^4 = s(t)f(r)\sin\theta d\phi. \qquad (2.6).$$

With respect to the metric (2.7) the tetrad components of the metric tensor $\eta_{\alpha\beta}$ are

$$\eta_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.7}$$

On using (2.4) and (2.6) we obtain

$$\theta^1 = e_a^1 dx^a = 1dt,$$

$$\theta^2 = e_a^2 dx^a = s(t)dr,$$

$$\theta^3 = e_a^3 dx^a = s(t)f(x)d\theta,$$

$$\theta^4 = e_a^4 dx^a = s(t)f(x)\sin\theta d\phi \tag{2.8}$$

Then the covariant components of the tetrad vector fields are obtained from equations (2.8) as

$$e_a^1 = (1, 0, 0, 0),$$

$$e_a^2 = (0, s(t), 0, 0),$$

$$e_a^3 = (0, 0, s(t) f(r), 0),$$

$$e_a^4 = (0, 0, 0, s(t) f(r) \sin\theta). \qquad (2.9)$$

The contravariant components of the tetrad vector fields are obtained by using the equation

$$e^{(\alpha)a} = g^{ab} e_b^{(\alpha)}$$

in the form

$$e^{(1)a} = (1, 0, 0, 0),$$

$$e^{(2)a} = (0, \frac{-1}{s(t)}, 0, 0)$$

$$e^{(3)a} = (0, 0, \frac{-1}{s(t) f(r)}, 0)$$

$$e^{(4)a} = (0, 0, 0, \frac{-1}{s(t) f(r) \sin\theta}). \qquad (2.10)$$

We notice from equations (2.9) and (2.10) that the tetrad vector field $e_a^{(1)}$ is a time-like vector field, while $e_a^{(2)}, e_a^{(3)}, e_a^{(4)}$ are space-like vector fields.

3. ROBERTSON-WALKER SPACE-TIME IN NEWMAN-PENROSE FORMALISM.

For the description of Robertson-Walker space-time in a Newman-Penrose formalism we choose four null vectors of the tetrad $e_{(\alpha)a} = (l_a, n_a, m_a, \bar{m}_a)$ using equations (3.1) from chapter-II with (2.9)

$$l_a = \frac{1}{\sqrt{2}}(1,\ -s(t),\ 0,\ 0\ )$$

$$n_a = \frac{1}{\sqrt{2}}(1,\ s(t),\ 0,\ 0\ )$$

$$m_a = \frac{1}{\sqrt{2}}(0,\ 0,\ s(t)f(r),\ is(t)f(r)\sin\theta\ )$$

$$\bar{m}_a = \frac{1}{\sqrt{2}}(0,\ 0,\ s(t)f(r),\ -is(t)f(r)\sin\theta\ ) \qquad (3.1)$$

while the contravariant components of tetrad vector fields can be obtained from $e^a_{(\alpha)} = g^{ab}e_{(\alpha)b}$ as

$$l^a = \frac{1}{\sqrt{2}}(1,\ \frac{1}{s(t)},\ 0,\ 0\ )$$

$$n^a = \frac{1}{\sqrt{2}}(1,\ \frac{-1}{s(t)},\ 0,\ 0\ )$$

$$m^a = \frac{1}{\sqrt{2}}(0,\ 0,\ \frac{-1}{s(t)f(r)},\ \frac{-i}{s(t)f(r)\sin\theta}\ )$$

$$\bar{m}^a = \frac{1}{\sqrt{2}}(0,\ 0,\ \frac{1}{s(t)f(r)},\ \frac{i}{s(t)f(r)\sin\theta}\ ) \qquad (3.2)$$

We observe from equation (3.1) and (3.3) that the null vector fields of the tetrad satisfy the orthogonality condition.

4. CHRISTOFFEL SYMBOLS FOR THE ROBERTSON-WALKER SPACE-TIME

Using the formula for the Christoffel symbols of first kind and second kinds given in equations (4.1) and (4.2) in chapter second, we find the Christoffel symbols for the Robertson-Walker space-time.

For example

$$\Gamma^1_{11} = g^{1b}\Gamma_{11,b} = g^{11}\Gamma_{11,1}$$

$$= \frac{g^{11}}{2}(g_{11,1} + g_{11,1} - g_{11,1}) = 1\{(1)_{,t}\}$$

$$\Gamma^1_{11} = 0$$

Similarly,

$$\Gamma^1_{22} = g^{11}\Gamma_{22,1}$$

$$= \frac{g^{11}}{2}\left(g_{21,2} + g_{21,2} - g_{22,1}\right)$$

$$= \frac{1}{2}\left(0 + 0 - (-s^2(t))_{,t}\right)$$

$$\Gamma^1_{22} = s(t)s'(t)$$

Thus we record below the non-vanishing Christoffel symbols for ready reference

$$\Gamma^1_{33} = s(t)s'(t)f^2(r)\,,$$

$$\Gamma^1_{44} = s(t)s'(t)f^2(r)\sin^2\theta\,,$$

$$\Gamma^2_{12} = \frac{s'(t)}{s(t)}\,,$$

$$\Gamma^2_{33} = -f(r)f'(r)\,,$$

$$\Gamma^2_{44} = -f(r)f'(r)\sin^2\theta\,,$$

$$\Gamma^3_{23} = \frac{f'(r)}{f(r)}\,,$$

$$\Gamma^3_{44} = 2\sin\theta\cos\theta\,,$$

$$\Gamma^4_{14} = \frac{s'(t)}{s(t)}\,,$$

$$\Gamma^4_{24} = \frac{f'(r)}{f(r)}\,,$$

$$\Gamma^4_{34} = \cot\theta\,. \qquad (4.1)$$

All other Christoffel symbols are zero.

i.e. $$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{13} = \Gamma^1_{14} = \Gamma^1_{23} = \Gamma^1_{24} = \Gamma^1_{34} = 0\,,$$

$$\Gamma^2_{11} = \Gamma^2_{13} = \Gamma^2_{14} = \Gamma^2_{22} = \Gamma^2_{23} = \Gamma^2_{24} = \Gamma^2_{34} = 0\,,$$

$$\Gamma^3_{11} = \Gamma^3_{12} = \Gamma^3_{13} = \Gamma^3_{14} = \Gamma^3_{22} = \Gamma^3_{24} = \Gamma^3_{33} = \Gamma^3_{34} = 0\,,$$

$$\Gamma^4_{11} = \Gamma^4_{12} = \Gamma^4_{13} = \Gamma^4_{22} = \Gamma^4_{23} = \Gamma^4_{33} = \Gamma^4_{44} = 0\,. \qquad (4.2)$$

5. NEWMAN-PENROSE SPIN COEFFICIENTS IN RBBERTSON-WALKER SPACE-TIME

The Newman-Penrose spin coefficients with respect to the chosen basis are obtained as follows. We have by definition,

$$\rho = l_{a;b} m^a \overline{m}^b$$

$$= m^a [l_{a,b} \overline{m}^b - \Gamma^c_{ab} l_c \overline{m}^b]$$

Expanding summation over b and c we write

$$\rho = m^a \left[l_{a,3} \overline{m}^3 + l_{a,4} \overline{m}^4 - \Gamma^1_{a3} l_1 \overline{m}^3 - \Gamma^2_{a3} l_2 \overline{m}^3 - \Gamma^1_{a4} l_1 \overline{m}^4 - \Gamma^2_{a4} l_2 \overline{m}^4 \right]$$

Now expanding the summation over a we get

$$\rho = m^3 \left[l_{3,3} \overline{m}^3 + l_{3,4} \overline{m}^4 - \Gamma^1_{33} l_1 \overline{m}^3 - \Gamma^2_{33} l_2 \overline{m}^3 - \Gamma^1_{34} l_1 \overline{m}^4 - \Gamma^2_{a4} l_2 \overline{m}^4 \right]$$

$$+ \; m^3 \left[l_{3,3} \overline{m}^3 - l_{3,4} \overline{m}^4 - \Gamma^1_{33} l_1 \overline{m}^3 - \Gamma^2_{33} l_2 \overline{m}^3 - \Gamma^1_{34} l_1 \overline{m}^4 - \Gamma^2_{a4} l_2 \overline{m}^4 \right]$$

Noting the non-vanishing components of the null vector fields from (3.2), (3.3) and using (4.1) we get,

$$\rho = m^3 \overline{m}^3 [\Gamma^1_{33} l_1 + \Gamma^2_{33} l_2] - m^4 \overline{m}^4 [\Gamma^1_{44} l_1 + \Gamma^2_{44} l_2]$$

$$\rho = -(\frac{-1}{\sqrt{2s(t)f(r)}})^2 [s(t) f^2(r) s'(t) \frac{1}{\sqrt{2}} - s(r) f'(r) \left(\frac{-s(t)}{\sqrt{2}} \right)]$$

$$-(\frac{-i}{\sqrt{2}s(t)f(r)\sin\theta})^2 [s(t)s'(t)f^2(r)\sin^2\theta \frac{1}{\sqrt{2}} - f(r)f'(rt)\sin^2\theta (\frac{-s(t)}{\sqrt{2}})]$$

$$\rho = \frac{-1}{\sqrt{2}}\left(\frac{f(r)s'(t) - f'(r)}{s(t)f(r)}\right) \tag{5.1}$$

We list here all other non-vanishing NP spin coefficients derived as

$$\mu = \frac{1}{\sqrt{2}}\left(\frac{f(r)s'(t) - f'(r)}{s(t)f(r)}\right),$$

$$\alpha = -\beta = \frac{-\cot\theta}{4s(t)f(r)},$$

$$\gamma = -\varepsilon = \frac{-s'(t)}{4\sqrt{2}s(t)}. \tag{5.2}$$

All other NP spin coefficients vanish

$$\text{i.e.} \quad \kappa = \sigma = \tau = \pi = \lambda = \nu = 0. \tag{5.3}$$

Here as $\kappa, \sigma, \lambda, \nu$ vanish, it confirms the type-D character of the space-time. The vanishing of the spin coefficients κ, σ shows that the null geodesic l_a is shear-free, while the null geodesic n_a is also shear-free as λ, ν vanish. Then by Goldberg-Sachs theorem, the shear-free character of the null geodesic congruences l_a and n_a shows that the Robertson-Walker space-time is of Petrov-type D.

6. COMPONENTS OF RIEMANN CURVATURE TENSOR

In V4, the 20 independent components of the Riemann-Curvature tensor are obtain by using the formula (6.1) from chapter-II. For example

$$R_{1212} = \frac{1}{2}(g_{12,21} + g_{21,12} - g_{11,22} - g_{22,11}) + g_{22}\Gamma^2_{12}\Gamma^2_{21}$$

On using (2.2), (4.1) and (4.2) we obtain

$$= \frac{-1}{2}\left[(-s'(t)_{,t})_{,t}\right] + (-s'(t))\left(\frac{s'(t)}{s(t)}\right)$$

$$R_{1212} = s(t)s''(t)$$

We site below the non-vanishing components of the Riemann-Curvature tensor as

$$R_{1323} = \frac{s(t)s'(t)f'(r)}{f(r)},$$

$$R_{1424} = -2s(t)f(r)s'(t)f'(r)\sin^2\theta,$$

$$R_{1434} = -s(t)f^2(r)s'(t)\sin\theta\cos\theta,$$

$$R_{2323} = s^2(t)f(r)\left[f''(r) - f(r)s'^2(t)\right],$$

$$R_{2434} = \frac{s^2(t)f^2(r)f'(r)\sin\theta\cos\theta}{f(r)},$$

$$R_{3434} = s^2(t)f^2(r)\sin^2\theta\left[f'^2(r) - f^2(r)s'^2(t) - 1\right]. \qquad (6.1)$$

We record below the vanishing components of Riemann-Curvature tensor

$$R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1234} = 0,$$

$$R_{1313} = R_{1314} = R_{1324} = R_{1334} = 0,$$

$$R_{1414} = R_{1423} = R_{1424} = R_{3423} = 0. \tag{6.2}$$

7. CURVATURE OF ROBERTSON-WALKER SPACE-TIME

The Riemann curvature at a point of a given space –time V_4 for the orientation determined by the real null vector fields l^a and n^a is

$$\mathrm{K} = \frac{R_{abcd} l^a n^b l^c n^d}{(g_{ac} g_d - g_{ad} g_{bc}) l^a n^b l^c n^d} \tag{7.1}$$

Using equatios (2.2), (3.3), (6.1) and (6.2) we obtain

$$\mathrm{K} = \frac{-s''(t)}{4s(t)} \tag{7.2}$$

Similarly the Riemann curvature at a point of the Robertson-Walker space-time for the orientation determined by the complex null vector fields m^a and $\overline{m}^a$ is given by

$$\mathrm{K} = \frac{\left[f'^2(r) - f^2(r)s'^2(t) - 1\right]}{\left(s^2(t) f^2(r)\right)} \tag{7.3}$$

8. CURVATURE, TORSION, BITORSION FIELDS OF THE WORLD LINE OF THE TME-LIKE VECTOR FIELD IN ROBERTSON-WALKER SPACE-TIME

Here we exploit the Rheotetrad introduced in Chapter-0, to study the geometry of the world line of the time-like particle in the Robertson-Walker space-time. In this section we explicitly find the expression for the Curvature, Torsion and Bitiorsion of the worldline of the particle by using the expressions (3.7),(3.10) and (3.16) of chapter-0. Using the equations (5.2) and (5.3) we get,

$$K = 0 \tag{8.1}$$

$$T = 0 \tag{8.2}$$

$$B = 0 \tag{8.3}$$

Thus for the present choice of tetrad the curvature, torsion and bitorsion of the time-like particle in the Robertson-Walker space-time are 0. Then by conclusion from Chapter-0, as K=0, the world line of the particle in Robertson-Walker space-time is a straight line.