CHAPTER - III

INTERPOLATING WAVELETS IN CLOSED FORM

Newton, when questioned about his method of work, could give no other answer but that he was wont to ponder again and again on subject.... Scientists and Artists both recommend persistent labour.

E. Mach

CHAPTER- 3

INTERPOLATING WAVELETS IN CLOSED FORM

3.1 Introduction:

In this chapter we show that if the orthogonality condition is replaced by the interpolating condition more examples of wavelets in closed form can be found.

These scaling functions and wavelets will be Riesz bases in addition to the property of being interpolating functions .The raised cosine wavelet can also be obtained as a special case of one of the examples (see example 3.2-6).

3.2 Lemma 3.2-1 :

Let $\phi(x)$ be defined by (2.1-1), then

$$\phi(t)=\frac{2\sin \pi t}{\pi t}\int_{0}^{\pi/3} h(x)\cos(tx)\,dx . \qquad (3.2\text{-}1)$$

Proof :

Taking inverse Fourier Transform and using integration by parts we get ,

$$\phi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\phi}(w)e^{-itw}dw$$

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$$= \frac{1}{2\pi it}\left\{ \int_{-\infty}^{\infty} e^{-itw}\left[h(w+\pi) - h(w-\pi)\right]dw \right\}$$

$$= \frac{1}{2\pi it}\left\{ \int_{-\infty}^{\infty} h(x)e^{-it(x-\pi)}dx - \int_{-\infty}^{\infty} h(x)e^{-it(x+\pi)}dx \right\}$$

$$= \frac{e^{it\pi} - e^{-it\pi}}{(\pi t)2i} \int_{-\infty}^{\infty} h(x)e^{-itx}dx$$

$$= \frac{2\sin \pi t}{\pi t} \int_{0}^{\pi/3} h(x)\cos(tx)\,dx \qquad \left(\because h \text{ is even and } Supp\, h \subset \left[\frac{-\pi}{3}\ \frac{\pi}{3}\right]\right)$$

If $\bar{h}(x)$ is a function satisfying conditions 1, 2, 4 and 5 of Lemma (2.1-1) then the function

$h(x) = \dfrac{\bar{h}(x)}{2g(0)}$ satisfies condition 1 to 5 and

Let $g(t) = \int_{0}^{\infty} \bar{h}(x)\cos(tx)\,dx$ then by Lemma (3.2-1) we have a scaling function

of MRA is given by $\phi(t) = \dfrac{\sin \pi t}{\pi t}\dfrac{g(t)}{g(0)}$ (3.2-2)

Now by Lemma (2.1-1) and Theorem (2.1-1) we can give examples of scaling function of MRA.

The associated Mother wavelet can be found by using

$$\psi\left(t + \frac{1}{2}\right) = 2\phi(2t) - \phi(t)$$

Example 3.2-1 :

Let $\bar{h}(x) = \left(a^2 - x^2\right)^{v-\frac{1}{2}}$ *where* $0 < x < a \le \frac{\pi}{3}$ *and* $\operatorname{Re} v > \frac{-1}{2}$

Then $g(t) = \int_0^\infty \bar{h}(x)\cos(tx)\,dx$

$$= \int_0^\infty \left(a^2 - x^2\right)^{\frac{v}{2}} \cos(tx)\,dx$$

$$= \frac{2^{v-1}\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}a^v J_v(at)}{t^v} \qquad (\because \text{ by [1] formula 8 P.11})$$

$$\therefore g(0) = \frac{\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}a^{2v}}{2\Gamma(v+1)}$$

then $h(x) = \dfrac{\bar{h}(x)}{2g(0)} X_{[-a\ \ a]}(x)$

$$= \frac{\left(a^2 - x^2\right)^{v-\frac{1}{2}}\Gamma(v+1)}{\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}a^{2v}} X_{[-a\ \ a]}(x)$$

Therefore h satisfies conditions 1 to 5 of Lemma (2.3-1).

Therefore $\phi(t) = \dfrac{\sin \pi t}{\pi t} \dfrac{g(t)}{g(0)}$

$$= 2^v\Gamma(v+1)\frac{\sin \pi t}{\pi t}\frac{J_v(at)}{(at)^v}$$

Example 3.2-2:

Let $\bar{h}(x) = \left(a^2 - x^2\right)^{\frac{v}{2}} \; J_v\left[b\sqrt{a^2 - x^2}\right] \, where \;\; 0 < x < a \le \frac{\pi}{3} \; and \, b > 0$

Then $g(t) = \int\limits_0^{\infty} \bar{h}(x)\cos(tx)\,dx$

$$= \int\limits_0^{\pi/3} \left(a^2 - x^2\right)^{\frac{v}{2}} J_v\left[b\sqrt{a^2 - x^2}\right]\cos(tx)\,dx$$

$$= \sqrt{\frac{\pi}{2}} \; a^{v+\frac{1}{2}} b^v J_{v+\frac{1}{2}}\left[a\sqrt{b^2 + t^2}\right]\left(b^2 + t^2\right)^{-\frac{v}{2}-\frac{1}{4}}$$

($\because$ by [1] formula 50 P.57)

$$\therefore g(0) = \sqrt{\frac{\pi}{2}} a^{v+\frac{1}{2}} b^v J_{v+\frac{1}{2}}[ab] \; b^{-v-\frac{1}{2}}$$

$$= \sqrt{\frac{\pi}{2b}} a^{v+\frac{1}{2}} J_{v+\frac{1}{2}}[ab]$$

Now Let $h(x) = \dfrac{\bar{h}(x)}{2g(0)} X_{[-a \;\; a]}(x)$

$$= \sqrt{\frac{b}{2\pi}} \frac{\left(a^2 - x^2\right)^{\frac{v}{2}} J_v\left[b\sqrt{a^2 - x^2}\right]}{a^{v+\frac{1}{2}} J_{v+\frac{1}{2}}[ab]} X_{[-a \;\; a]}(x)$$

Then h satisfies conditions 1,3,4 and 5 of Lemma (2.3-1). To ensure condition 2 of Lemma (2.3-1) we must choose b so that $\frac{\alpha_{v,1}}{b} > a$ where $\alpha_{v,1}$ is the first nonzero zero of the Bessel function of order ν.

Therefore $\phi(t)=\dfrac{\sin\pi t\ g(t)}{\pi t\ g(0)}$

$$=\frac{\sin\pi t\ b^{v+\frac{1}{2}}\ J_{v+\frac{1}{2}}\left[a\sqrt{b^2+t^2}\right]}{\pi t\left(b^2+t^2\right)^{\frac{v}{2}+\frac{1}{4}}J_{v+\frac{1}{2}}\left[ab\right]}$$

Example (3.2-3) :

Let $\bar{h}(x)=\left(a-|x|\right)^{v-1}$ *where* $0<x<a\le\frac{\pi}{3}$ *and* $\operatorname{Re}v>0$

Using [2] (formula 6, P. 424)

Then $h(x)=\dfrac{\bar{h}(x)}{2g(0)}X_{[-a\ \ a]}(x)$

$$=\frac{v\left(a-|x|\right)^{v-1}}{2a^{v}}X_{[-a\ \ a]}(x)$$

then h satisfies conditions 1 to 5 of Lemma 2.3-1

Therefore $\phi(t)=\dfrac{\sin\pi t\ g(t)}{\pi t\ g(0)}$

$$=\frac{\overline{|v+1}\sin\pi t}{\pi t(at)^{v}}Y_{v}(at).$$

Example (3.2-4) :

Let $P_{v}^{\mu}(z)$ be the Legendre function defined by

$$P_{v}^{\mu}(z)=\frac{1}{\overline{|1-\mu}}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}}\ {}_2F_1\left(-v;v+1;1-\mu;\frac{2}{2}-\frac{z}{2}\right)$$

and Let $\bar{h}(x) = (\cos x - \cos a)^{v-\frac{1}{2}}$ *where* $0 < a < \pi$ *and* $\operatorname{Re} v > \frac{-1}{2}$

$$g(t) = \int_0^{\infty} \bar{h}(x)\cos(tx)\,dx$$

$$= \int_0^{\infty} (\cos x - \cos a)^{v-\frac{1}{2}} \cos(tx)\,dx$$

$$= \sqrt{\frac{\pi}{2}}\,(\sin a)^{v}\,\Gamma\!\left(v+\frac{1}{2}\right) P^{-v}_{t-\frac{1}{2}}(\cos a) \qquad (\because \text{by [1] formula 28 P.22})$$

$$\therefore g(0) = \sqrt{\frac{\pi}{2}}\,(\sin a)^{v}\,\Gamma\!\left(v+\frac{1}{2}\right) P^{-v}_{t-\frac{1}{2}}(\cos a)$$

Therefore $h(x) = \dfrac{\bar{h}(x)}{2g(0)} X_{[-a\ \ a]}(x)$

$$= \frac{(\cos x - \cos a)^{v-\frac{1}{2}}}{\sqrt{2\pi}\,(\sin a)^{v}\,\Gamma\!\left(v+\frac{1}{2}\right) P^{-v}_{-\frac{1}{2}}(\cos a)} X_{[-a\ \ a]}(x)$$

then h satisfies condition 1 to 5 of Lemma (2.3-1)

Therefore $\phi(t) = \dfrac{\sin \pi t\ g(t)}{\pi t\ g(0)}$

$$= \frac{\sin \pi t\ P^{-v}_{t-\frac{1}{2}}(\cos a)}{\pi t\ P^{-v}_{-\frac{1}{2}}(\cos a)}.$$

Example 3.2-5 :

Let $\bar{h}(x) = \left(\cos\left(\dfrac{\pi x}{2a}\right)\right)^{v-1}$ *where* $\operatorname{Re} v > 0$

Then $g(t)=\int_0^\infty \bar{h}(x)\cos(tx)\,dx$

$$=2^{1-\nu}\Gamma(\nu)\, a\left[\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{at}{\pi}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{2}-\frac{at}{\pi}\right)\right] \qquad (\because \text{ by [1] formula 27 P.22})$$

$$\therefore g(0)=\frac{2^{1-\nu}\Gamma(\nu)\, a}{\left(\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\right)^2}$$

Therefore $h(x)=\dfrac{\bar{h}(x)}{2g(0)}X_{[-a\;\,a]}(x)$

$$=\frac{\left(\cos\frac{\pi x}{2a}\right)^{\nu-1}}{2^{2-\nu}\Gamma(\nu)\, a}\left(\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\right)^2 X_{[-a\;\,a]}(x)$$

and h(x) satisfies condition 1 to 5 of Lemma (2.3-1)

Therefore $\phi(t)=\dfrac{\sin\pi t\;\, g(t)}{\pi t\;\, g(0)}$

$$=\frac{\sin\pi t\left(\Gamma\left(\nu+\frac{1}{2}\right)\right)^2}{\pi t\left[\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{at}{\pi}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{2}-\frac{at}{\pi}\right)\right]}$$

Now we consider two special cases

Case I : For $\nu = 1$

$h(x)=\dfrac{1}{2a}X_{[-a\;\,a]}(x)$

and $\phi(t) = \dfrac{\sin \pi t}{\pi t\left[\Gamma\left(1+\frac{at}{\pi}\right)\Gamma\left(1-\frac{at}{\pi}\right)\right]}$,but $\Gamma\left(1+\frac{at}{\pi}\right)\Gamma\left(1-\frac{at}{\pi}\right) = \dfrac{at}{\sin at}$

$$\therefore \phi(t) = \frac{\sin \pi t \, \sin at}{a\pi t^2}$$

Therefore by (5.3-3) Mother wavelet is $\psi(t) = \dfrac{\sin \pi t \, \sin at}{a\pi t^2}\left[2\cos \pi t \, \cos at - 1\right]$

Case II : ν = 2 then

$$h(x) = \frac{\pi \cos\left(\frac{\pi x}{2a}\right)}{4a} X_{[-a \;\; a]}(x) \quad \text{and} \quad \phi(t) = \frac{\pi \sin \pi t}{\pi t \, 4\,\Gamma\left(\frac{3}{2}+at\pi\right)\Gamma\left(\frac{3}{2}-at\pi\right)}$$

Using Legendre duplication formula

$\Gamma\left(x+\frac{1}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(2x)}{2^{2x-1}\Gamma(x)}$ and Let $\gamma = \dfrac{at}{\pi}$ we get $\Gamma\left(\frac{3}{2}+\gamma\right)\Gamma\left(\frac{3}{2}-\gamma\right) = \dfrac{\pi\,\Gamma(2+2\gamma)\,\Gamma(2-2\gamma)}{4\,\Gamma(1+\gamma)\,\Gamma(1-\gamma)}$

But we have relations

$\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1+x)\Gamma(1-x) = \pi x \operatorname{cosec} \pi x$

$$\therefore \Gamma\left(\frac{3}{2}+\gamma\right)\Gamma\left(\frac{3}{2}-\gamma\right) = \frac{\pi(1+2\gamma)(1-2\gamma)\Gamma(1+2\gamma)\Gamma(1-2\gamma)}{4\pi\gamma \operatorname{cosec} \pi\gamma}$$

$$= \frac{\pi\left(1-4\gamma^2\right)\pi 2\gamma \operatorname{cosec}^2 \pi\gamma}{2\operatorname{cosec} at}$$

$$= \frac{\pi\left(1-\frac{4a^2t^2}{\pi^2}\right)}{4\cos at}$$

$$\therefore \phi(t) = \frac{\sin \pi t \cos at}{\pi t\left(1 - \frac{4a^2t^2}{\pi^2}\right)}$$

Now we show that the raisedcosine wavelet (2.1-5) is a special case of Example (3.2-5) Case II

Example (3.2-6) :

Let a = $\pi\beta$ in example (3.2-5, case II)

$$\therefore \phi(t) = \frac{\sin \pi t \cos \pi\beta t}{\pi t\left(1 - 4\beta^2 t^2\right)}$$

$$\therefore \hat{\phi}(w) = \begin{cases} 0 & \text{if } w \le -\pi(1+\beta) \\ \frac{1}{2}\left[1 + \sin\frac{w+\pi}{2\beta}\right] & \text{if } -\pi(1+\beta) \le w \le -\pi(1-\beta) \\ 1 & \text{if } -\pi(1-\beta) \le w \le -\pi(1+\beta) \\ \frac{1}{2}\left[1 - \sin\frac{w-2\pi}{2\beta}\right] & \text{if } \pi(1-\beta) \le w \le \pi(1+\beta) \\ 0 & \text{if } \pi(1+\beta) \le w \end{cases}$$

Let $\eta = \frac{w}{4\pi}$ *and* $\delta = \frac{\pi(1-\beta)}{4\beta}$

$$\therefore \hat{\phi}(w) = \begin{cases} 0 & \text{if } w \le -\pi(1+\beta) \\ \cos^2(\eta+\delta) & \text{if } -\pi(1+\beta) \le w \le -\pi(1-\beta) \\ 1 & \text{if } -\pi(1-\beta) \le w \le -\pi(1+\beta) \\ \cos^2(\eta-\delta) & \text{if } \pi(1-\beta) \le w \le \pi(1+\beta) \\ 0 & \text{if } \pi(1+\beta) \le w \end{cases}$$

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As ϕ is not orthogonal scaling function by Lemma (1.3-1) we have another function $\hat{\phi}_1(w) = \sqrt{\hat{\phi}(w)}$ which is an orthogonal scaling function of a MRA .

Taking inverse Fourier Transform

$$\phi_1(t) = \frac{1}{\pi} \int_0^{(1+\beta)\pi} \hat{\phi}_1(w) \cos(tw)\, dw$$

$$= \frac{1}{\pi} \left\{ \int_0^{(1-\beta)\pi} \cos(tw)\, dw + \int_{(1-\beta)\pi}^{(1+\beta)\pi} \cos^2\left(\frac{w}{4\pi} - \delta\right) \cos(tw)\, dw \right\}$$

$$\therefore \phi_1(t) = \frac{\sin \alpha t}{\pi t} + \frac{1}{2\pi} \left\{ \frac{\cos \gamma t - \cos \alpha t}{\left[t + \frac{1}{4\beta}\right]} - \frac{\cos \gamma t + \sin \alpha t}{\left[t - \frac{1}{4\beta}\right]} \right\}$$

where $\alpha = (1-\beta)\pi$ and $\gamma = (1+\beta)\pi$

Thus $\phi_1(t) = \dfrac{\sin \alpha t + 4\beta t \cos \gamma t}{\pi t\left[1 - (4\beta t)^2\right]}$

Which is same as the raised cosine wavelet (2.1-5).