CHAPTER III

CHAPTER - III

DISTRIBUTIONAL GENEGALIZED LAPLACE TRANSFORMATION

3.1 Introduction:-

The conventional generalized Laplace transformation is defined by the integral,

$$F(s) = \int_0^\infty (st)^\lambda e^{-st} L_k^a(st) f(t) dt \qquad .....(3.1.1)$$

where $L_k^a(st)$ is the Laguerre polynomial.

The aim of this chapter is to extend the conventional generalized Laplace transformation defined in (3.1.1) to a certain class of generalized functions.

The generalized Laplace transform F(s) of certain generalized function f is defined directly as the application of f(t) to $(st)^\lambda e^{-st} L_k^a(st)$, that is

$$F(s) \triangleq < f(t),\ (st)^\lambda e^{-st} L_k^a(st) > \qquad .....(3.1.2)$$

For this we construct a certain space of testing functions on $0 < t < \infty$ which contains $(st)^\lambda e^{-st} L_k^a(st)$ for the various values of the complex parameter s.

We have obtained an inversion formula and Uniqueness theorem for the generalized Laplace transformation.

We use notations and terminology as those of Zemanian [24].

3.2 The Testing Function Spaces GL_a, GL(w) and Thier Dual Space:-

Let a be a fixed real number. We define the space GL_a as the set of all complex valued smooth functions $\phi(t)$ on $0 < t < \infty$ such that for each non-negative integer m,

$$\gamma_{a,m}(\phi) \triangleq \sup_{0<t<\infty} \left| e^{at}\, t^m\, D_t^m [\phi(t)] \right|, \quad m = 0, 1, 2, 3, \ldots \qquad \ldots..(3.2.1)$$

assumes finite values.

GL_a is a linear space under the pointwise addition of functions and their multiplication by complex numbers.

$\gamma_{a,m}$ is a seminorm on GL_a and $\gamma_{a,0}$ is a norm.

Therefore the collection $\{\gamma_{a,m}\}_{m=0}^{\infty}$ is a countable multinorm on GL_a. We assign to GL_a the topology generated by the countable multinorm $\{\gamma_{a,m}\}_{m=0}^{\infty}$ and this makes GL_a, a countably multinormed space.

We say that the sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in GL_a to ϕ if and only if for each non-negative integer m, $\gamma_{a,m}(\phi_v - \phi) \to 0$ *as* $v \to \infty$.

A sequence $\{\phi_v\}_{v=1}^{\infty}$ is said to be a Cauchy sequence in GL_a if and only if $\gamma_{a,m}(\phi_v - \phi_\mu) \to 0$ *as* $v \to \infty$ *and* $\mu \to \infty$ independently, for each non-negative integer m.

First we can prove that the kernel $(st)^{\lambda} e^{-st} L_k^a(st)$ is a member of GL_a.

Lemma 3.2.1:

The function $(st)^{\lambda} e^{-st} L_k^a(st)$ is a member of GLa , for Re. s > a, where λ be a non-negative integer.

Proof:-

Let $\phi(t) = (st)^{\lambda}\, e^{-st}\, L_k^{a}(st)$

The function $\phi(t)$ will be member of GL_a , if

(i) $\phi(t)$ is a smooth on $0 < t < \infty$.

(ii) $\sup_{0<t<\infty} \left|e^{at}\, t^m\, D_t^m[\phi(t)]\right| < \infty$ for each m = 0, 1, 2, 3, ...

The function $\phi(t)$ is a product of three smooth functions of t and threfore smooth on $0 < t < \infty$.

Consider,

$$D_t^m[\phi(t)] = D_t^m\left[(st)^{\lambda}\, e^{-st}\, L_k^{a}(st)\right]$$

$$= \sum_{j=0}^{m} \binom{m}{j} [(st)^{\lambda}\, e^{-st}]^{(m-j)} \left[L_k^{a}(st)\right]^{(j)} \qquad .....(3.2.2)$$

But,

$$[(st)^{\lambda}\, e^{-st}]^{(m-j)} = \sum_{l=0}^{m-j} \binom{m-j}{l} [(st)^{\lambda}]^{(m-j-l)}\, [e^{-st}]^{(l)}$$

$$= \sum_{l=0}^{m-j} \binom{m-j}{l} [\lambda\,(\lambda-1)\, ...\, (\lambda-m+j+l)\, (st)^{\lambda-m+j+l}\, (s)^{m-j-l}\, e^{-st}\, (-s)^{l}]$$

$$= \sum_{l=0}^{m-j} \binom{m-j}{l} [\lambda\,(\lambda-1)\, ...\, (\lambda-m+j+l)\, (st)^{\lambda-m+j+l}\, (-1)^{l}\, (s)^{m-j}\, e^{-st}]$$

Putting in (3.2.2) we get.

$D_t^m[\phi(t)]$

$$= \sum_{j=0}^{m} \binom{m}{j} \left[\begin{array}{c} \left(\sum_{l=0}^{m-j} \binom{m-j}{l} \lambda (\lambda - 1) \ldots (\lambda - m + j + l)\, (st)^{\lambda - m + j + l} (-1)^l (s)^{m-j}\, e^{-st} \right) \\ \times \left(\sum_{n=1}^{k} \frac{(-1)^n (1+a)_k\, s^n\, [n(n-1) \ldots (n-j)]\, t^{n-j}}{n!\, (k-n)!\, (1+a)_n} \right) \end{array} \right]$$

Therefore,

$$\sup_{0<t<\infty} \left| e^{at}\, t^m\, D_t^m\, [\, \phi(t)\,] \right|$$

$$= \sup_{0<t<\infty} \left| e^{at}\, t^m \sum_{j=0}^{m} \binom{m}{j} \left[\begin{array}{c} \left(\sum_{l=0}^{m-j} \binom{m-j}{l} \lambda \ldots (\lambda - m + j + l)(st)^{\lambda - m + j + l} (-1)^l (s)^{m-j}\, e^{-st} \right) \\ \times \left(\sum_{n=1}^{k} \frac{(-1)^n (1+a)_k\, s^n\, [n(n-1) \ldots (n-j)]\, t^{n-j}}{n!\, (k-n)!\, (1+a)_n} \right) \end{array} \right] \right|$$

$$= \sup_{0<t<\infty} \left| e^{-(s-a)t}\, t^m \sum_{j=0}^{m} \binom{m}{j} \left[\begin{array}{c} \left(\sum_{l=0}^{m-j} \binom{m-j}{l} \lambda \ldots (\lambda - m + j + l)(st)^{\lambda - m + j + l} (-1)^l\, (s)^{m-j} \right) \\ \times \left(\sum_{n=1}^{k} \frac{(-1)^n (1+a)_k\, s^n\, [n(n-1) \ldots (n-j)]\, t^{n-j}}{n!\, (k-n)!\, (1+a)_n} \right) \end{array} \right] \right|$$

The expression under the supremum will be finite if and only if

Re.(s - a) > 0 i.e. Re. s > a and $\lambda \geq 0$.

i.e. $\sup_{0<t<\infty} \left| e^{at}\, t^m\, D_t^m [\phi(t)] \right| < \infty$ if and only if Re.s > a and $\lambda \geq 0$.

for each m = 0, 1, 2, 3, ...

Thus $(st)^{\lambda}\, e^{-st}\, L_k^{a}(st)$ is a member of GL_a if and only if Re.s > a and $\lambda \geq 0$.

Hence the proof.

Lemma 3.2.2:

GL_a is complete and therefore a Frechet space.

Proof:-

Let sequence $\{\phi_v\}_{v=1}^{\infty}$ be a Cauchy sequence in GL_a . Then by the equation (3.2.1), $e^{at}\, t^m\, D_t^m[\phi_v(t)]$ is a uniform Cauchy sequence on $0 < t < \infty$ *as* $v \to \infty$.

Hence by the standard theorem [1, P. 402] there exist a smooth function $\phi(t)$ such that for each m and t

$$D_t^m[\phi_v(t)] \to D_t^m[\phi(t)] \text{ as } v \to \infty.$$

Moreover, for each $\in\ > 0$ there exist an integer N_k such that, for every

$v, \mu \geq N_k$,

$$\left|e^{at}\, t^m\, D_t^m[\phi_v(t) - \phi_\mu(t)]\right| < \ \in$$

Taking the limit as $\mu \to \infty$. We obtain

$$\left|e^{at}\, t^m\, D_t^m[\phi_v(t) - \phi(t)]\right| < \ \in \quad v \geq N_k,\ 0 < t < \infty \qquad .....(3.2.3)$$

Thus as $v \to \infty$

$$\gamma_{a,m}(\phi_v - \phi) \to \infty \quad \text{for each m.}$$

Finally because of uniform convergence and the fact that each $e^{at}\, t^m\, D_t^m[\ \phi_v(t)]$ is bounded on $0 < t < \infty$, their exist a constant C_k not depending on v, such that

$$\left|e^{at}\, t^m\, D_t^m[\phi_v(t)]\right| < C_k \text{ for all t.}$$

Therefore from the equation (3.2.3) it implies that

$$\left|e^{at}\, t^m\, D_t^m[\phi(t)]\right| < C_k + \ \in$$

which shows that $e^{at}\, t^m\, D_t^m[\phi(t)]$ is bounded on $0 < t < \infty$.

Hence the limit function ϕ is a member of GL_a.

Thus, the Cauchy sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in GL_a to the unique limit ϕ.

Hence GL_a is complete.

Since GL_a is countably multinormed space which is complete . Hence GL_a is a Frechet space.

Hence the proof.

Lemma 3.2.3:

GL_a is a testing function space.

Proof:-

Clearly, GL_a satisfies the first two conditions of testing function space. We shall prove the third.

Let sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in GL_a to zero.

Then $\gamma_{a,m}(\phi_v) \to 0$ for each m, as $v \to \infty$.

i.e. $\sup\limits_{0<t<\infty} \left| e^{at}\, t^m\, D_t^m[\phi_v(t)] \right| \to 0$ for each m, as $v \to \infty$.

i.e. $\left| e^{at}\, t^m\, D_t^m[\phi_v(t)] \right| \to 0$ for each m, as $v \to \infty$.

Since $\left| e^{at}\, t^m\, D_t^m[\phi_v(t)] \right| \to 0$ uniformly for each m and $|e^{at}\, t^m|$ has positive supremum on every compact subset of $I = (0, \infty)$.

Therefore,

$$\left| D_t^m[\phi_v(t)] \right| \to 0 \quad \text{for each m, as } v \to \infty.$$

Hence we must have the sequence $\{D_t^m[\phi_v(t)]\}_{v=0}^{\infty}$ converges to zero function uniformly on every compact subset of $I = (0, \infty)$.

Thus, GL_a satisfies all the three defining properties of a testing function space.

Hence GL_a is a testing function space.

Hence the proof.

The dual space of GL_a is GL'_a and GL'_a consists of all continuous linear functionals on GL_a. Thus, f is member of GL'_a, if f is a continuous linear functional on GL_a.

As we were already proved that GL_a is a testing function space and therefore GL'_a is a space of generalized functions.

Under the usual definitions of addition and multiplication by a complex numbers, GL'_a is a linear space. We assign to GL'_a its customary(weak) topology. It follows that GL'_a is also complete.

Now, we list some properties of the space GL_a, which can be easily established

(i) If $a < b$ then $GL_a \subset GL_b$. The topology of GL_a is stronger than the topology induced on GL_a by GL_b.

To see this first we note that $0 < e^{at}\, t^m < e^{bt}\, t^m$ on $0 < t < \infty$.

Therefore,

$$\left|e^{at}\, t^m\, D_t^m[\phi(t)]\right| \leq \left|e^{bt}\, t^m\, D_t^m[\phi(t)]\right|$$

So that

$\gamma_{a,m}(\phi) \leq \gamma_{b,m}(\phi)$

Our assertion follows by this inequality and [24, (lemma (1.6.3)].

(ii) $D(I) \subset GL_a$ and the topology of D(I) is stronger than that induced on it by GL_a.

Similarly, the other properties of the space GL'_a can be easily established as following,

(iii) If a < b, the restriction of $f \in GL'_b$ to GL_a is in GL'_a. Also the convergence in GL'_b implies convergence in GL'_a, it follows as a consequence of property (i).

We shall turn now to certain countable union space GL(w) that arises from the GL_a space.

Let w be either a real number or $-\infty$. Let $\{a_v\}_{v=1}^{\infty}$ be a monotic sequence of positive real numbers which converges to w^+ as $v \to \infty$.Then define GL(w) as a countable union space of GL_{a_v} space.

Thus,

$$\mathrm{GL(w)} = \bigcup_{v=1}^{\infty} GL_{a_v}.$$

Space of this type were introduced by Gelfand and Shilov.

The sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in GL(w) to ϕ if and only if

ϕ_v and ϕ belongs to some particular GL_{a_v}, for some fixed a_v and $\phi_v \to \phi$ in GL_{a_v}.

The sequence $\{\phi_v\}_{v=1}^{\infty}$ is said to be a Cauchy sequence in the countable union space GL(w) if it is a Cauchy in one of the spaces GL_{a_v}.

When all the the Cauchy sequences in GL(w) are convergent then GL(w) is complete.

Moreover, GL(w) does not depend on the choice of $\{a_\nu\}_{\nu=1}^{\infty}$.

The dual space of GL(w) is denoted by GL'(w). GL'(w) is linear under ususal definitions.

A sequence $\{f_\nu\}_{\nu=1}^{\infty}$ converges in GL'(w) if their exist a f in GL'(w) such that, for every $\psi \in GL_{a_\nu}$,

$$< f_\nu , \psi > \rightarrow < f, \psi > \text{ as } \nu \rightarrow \infty.$$

GL'(w) is also complete because GL(w) is complete. [24, theorem 1.8.2]

If w < u then GL(u) $\subset$ GL(w) and convergence in GL(u) implies convergence in GL(w). Thus the restriction of any f $\in$ GL'(w) to GL(u) is in GL'(u).

3.3 The Distributional Generalized Laplace Transformation:-

We shall call, the generalized function f GL transformable if f $\in$ GL'(w) for some w, let σ_f be the infimum of all such w. Define

$$F(s) \triangleq GL(f)(s) \triangleq < f(t), (st)^{\lambda} e^{-st} L_k^{a}(st) > , \text{Re. } s > \sigma_f \quad .....(3.3.1)$$

Our aim is to obtain the very important aspect of the space GL_a is its Inversion theorem and Uniqueness theorem.

3.4 Inversion and Uniqueness theorems for the Distributional Generalized Laplace Transformation:-

In this section we shall derive the inversion formula for the distributional generalized Laplace transformation. From this we will obtain a Uniqueness theorem. We shall use the same technique as that used in proving the inversion formula for Convolution, K-transformation and Generalized Laplace transformation by Zemanian [24].

First we state and prove some lemmas which will be used for proving the inversion theorem.

Result 3.4.1:

$$\int_0^\infty [\tfrac{nt}{x}]^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k; 1+a;\, \tfrac{n}{x}\, t)\, dx \sim nt\,(\lambda+n+k-2)!$$

as $nt \to \infty$

Proof:-

By the definition of Lagurre polynomial, we have

$$L_k^a(\tfrac{n}{x}t) = \tfrac{(-1)^k}{k!}\, U(-k;\, 1+a;\, \tfrac{n}{x}\, t) \sim (\tfrac{n}{x}\, t)^k \text{ as } nt \to \infty$$

Therefore,

$$\int_0^\infty [\tfrac{nt}{x}]^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k; 1+a;\, \tfrac{n}{x}\, t)\, dx$$

$$\sim \int_0^\infty [\tfrac{nt}{x}]^{\lambda+n+k}\, e^{-\frac{n}{x}t}\, dx$$

$put\ \frac{n}{x} = u \ \Rightarrow \ dx = -\frac{n}{u^2}\, du.$

$when\ x = 0,\ u = \infty.$

$and\ x = \infty,\ u = 0$

$$= \int_0^{\infty} [ut]^{\lambda+n+k}\, e^{-ut} \frac{n}{u^2}\, du$$

$$= n\, t^{\lambda+n+k} \int_0^{\infty} u^{\lambda+n+k-2}\, e^{-ut}\, du$$

Integrating by parts, we get

$$= n\ \ t^{\lambda+n+k} \left\{ \left[u^{\lambda+n+k-2} \frac{e^{-ut}}{-t} \right]_0^{\infty} - \int_0^{\infty} (\lambda + n + k - 2)\ u^{\lambda+n+k-3} \frac{e^{-ut}}{-t}\, du \right\}$$

$$= n\, t^{\lambda+n+k-1}\, (\lambda + n + k - 2) \int_0^{\infty} u^{\lambda+n+k-3}\, e^{-ut}\, du$$

Continuing in this way, we get

$$= n\, t^{\lambda+n+k-(\lambda+n+k-2)}\, (\lambda + n + k - 2)! \int_0^{\infty} e^{-ut}\, du$$

$$= n\, t^2\, (\lambda + n + k - 2)! \left[\frac{e^{-ut}}{-t} \right]_0^{\infty}$$

$$= n\, t\, (\lambda + n + k - 2)!$$

Hence the proof.

<u>Lemma 3.4.1:</u>

Let a be a suitably fixed real number and $\psi \in GL'_a$ and $s = \frac{n}{x}$. Then

$$\int_0^{\infty} \psi(x) < f(t)\, ,\ \frac{s^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n}\, e^{-st}\, L_k^a(st) \right] > dx$$

$$= \left< f(t) , \int_0^\infty \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] \psi(x)\, dx \right> \qquad .....(3.4.1)$$

Proof:-

To prove this lemma, we shall use the technique of Riemann sums.

If $\psi(x) = 0$ then the lemma is obvious, hence we assume that $\psi(x) \neq 0$ in GL'_a.

First we shall show that,

$G(n,t) = \int_0^\infty \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] \psi(x)\, dx$ is a member of GL_a.

Consider,

$$e^{at}\, t^m\, D_t^m [G(n,t)] = e^{at}\, t^m D_t^m \int_0^\infty \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] \psi(x)\, dx \qquad .....(3.4.2)$$

Since the integrand is smooth function, we may carry the operator D_t^m under the integral sign in the equation (3.4.2), we get

$$e^{at}\, t^m\, D_t^m [G(n,t)] = e^{at}\, t^m \int_0^\infty \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} D_t^m \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] \psi(x)\, dx$$

By definition of Laguerre polynomial and $s = \frac{n}{x}$ we get,

$e^{at}\, t^m\, D_t^m [G(n,t)]$

$$= e^{at} t^m \int_0^\infty \frac{n^{\lambda+n+1}}{|\lambda+n+k+1} D_t^m \left[\frac{t^{\lambda+n}}{x^{\lambda+n+1}} e^{-\frac{n}{x}t} \frac{(-1)^k}{k!} U(-k;\ 1+a;\ \tfrac{n}{x}t) \right] \psi(x)\, dx$$

$$= e^{at}\, t^m \frac{n^{\lambda+n+1}}{|\lambda+n+k+1} \int_0^\infty (-1)^m D_x^m \left[\frac{t^{\lambda+n-m}}{x^{\lambda+n+1-m}} e^{-\frac{n}{x}t} \frac{(-1)^k}{k!} U(-k; 1+a;\ \tfrac{n}{x}t) \right] \psi(x)\, dx$$

$$= e^{at}\, (-1)^m \frac{n}{|\lambda+n+k+1} \int_0^\infty \psi^m(x)\, x^{m-1} \left[(\tfrac{nt}{x})^{\lambda+n} e^{-\frac{n}{x}t} \frac{(-1)^k}{k!} U(-k;\ 1+a;\ \tfrac{n}{x}t) \right] dx$$

Consider,

$\left| e^{at} \, t^m \, D_t^m [G(n,t)] \right|$

$$\leq e^{at} \frac{n}{|\lambda + n + k + 1} \int_0^{\infty} \left| \psi^m(x) \; x^{m-1} \left[\left(\tfrac{nt}{x}\right)^{\lambda + n} e^{-\frac{n}{x}t} \frac{(-1)^k}{k!} U(-k;\; 1+a;\; \tfrac{n}{x}\, t) \right] \right| dx$$

$< \infty.$

Since the expression is finite because

By the result (3.4.1), we have

$$\int_0^{\infty} \left[\left(\tfrac{nt}{x}\right)^{\lambda + n} e^{-\frac{n}{x}t} \frac{(-1)^k}{k!} U(-k;\; 1+a;\; \tfrac{n}{x}t) \right] \, dx \sim nt(\lambda + n + k - 2)!$$

$$\text{as } nt \to \infty$$

and ψ has smooth and bounded support.

This shows that G(n,t) is a member of GL_a.

This will insures that the right hand side of (3.4.1) has a sense.

Now we prove that two sides of (3.4.1) are equal.

Let $\Phi(x) = \; < f(t), \frac{s^{\lambda+n+1}}{|\lambda + n + k + 1} \left[t^{\lambda + n} e^{-st} L_k^a(st) \right] >$

Then the left hand side of (3.4.1) is $\int_0^{\infty} \psi(x) \, \Phi(x) \, dx$

Then its Riemann sum is $\sum_{i=0}^{n} \psi(x_i) \, \Phi(x_i) \Delta_{x_i}$

Now conside the Riemann sum of the left hand side of (3.4.1),

$$\sum_{i=0}^{n} \psi(x_i) < f(t), \frac{s_i^{\lambda+n+1}}{|\lambda + n + k + 1} \left[t^{\lambda+n} e^{-s_i t} L_k^a(s_i t) \right] > \Delta_{x_i} \qquad .....(3.4.3)$$

Since $< f(t), \frac{s_i^{\lambda+n+1}}{|\lambda + n + k + 1} \left[t^{\lambda+n} e^{-s_i t} L_k^a(s_i t) \right] >$ is a continuous function on

$0 < x < \infty$.

Hence (3.4.3) equals to

$$< f(t)\,,\ \sum_{i=0}^{n} \psi(x_i) \frac{s_i^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n} e^{-s_i t} L_k^a(s_i t)\right] \Delta_{x_i} > \qquad .....(3.4.4)$$

Since, $\sum_{i=0}^{n} \psi(x_i) \frac{s_i^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n} e^{-s_i t} L_k^a(s_i t)\right] \Delta_{x_i}$ is the Riemann sum of the integral

$$\int_0^\infty \frac{s^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st)\right] \psi(x)\, dx$$

Therefore (3.4.4) equals

$$< f(t)\,,\ \int_0^\infty \frac{s^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st)\right] \psi(x)\, dx >$$

Hence the proof.

Lemma: 3.4.2:

Let $\psi \in D(I)$ then,

$$\rho(n,t) = \int_0^\infty \frac{s^{\lambda+n+1}}{\lfloor \lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st)\right] \psi(x)\, dx \quad s = \tfrac{n}{x}$$ converges GL_a

to $\psi(t)$ as n $\to \infty$, for every real number a.

Proof:-

We have to show that $\rho(n,t)$ converges uniformly to $\psi(t)$ in GL_a , for every real number a, as n $\to \infty$, means, we have to show that

$e^{at}\, t^m D_t^m[\rho(n,t) - \psi(t)]$ converges uniformly to zero function in

$0 < t < \infty$ as $n \to \infty$.

Consider

$$e^{at}\, t^m\, D_t^m \left\{ \int_0^\infty \left(\frac{s^{\lambda+n+1}}{|\overline{\lambda+n+k+1}} \left[t^{\lambda+n}\, e^{-st}\, L_k^a(st) \right] \psi(x) \right) dx \; - \psi(t) \right\}$$

Since ψ is smooth and is of bounded support, we may repeatedly differentiate under the integral sign as,

$$e^{at}\, t^m \left\{ \int_0^\infty \left(\frac{s^{\lambda+n+1}}{|\overline{\lambda+n+k+1}}\, D_t^m \left[t^{\lambda+n}\, e^{-st}\, L_k^a(st) \right] \psi(x) \right) dx \; - \psi^m(t) \right\}$$

since $s = \frac{n}{x}$ and by definition of Laguree polynomial, we get

$$e^{at}\, t^m \left\{ \int_0^\infty \left(\frac{n^{\lambda+n+1}}{|\overline{\lambda+n+k+1}}\, D_t^m \left[\frac{t^{\lambda+n}}{x^{\lambda+n+1}}\, e^{-\frac{n}{x} t}\, \frac{(-1)^k}{k!}\, U(-k\,; 1+a\,; \tfrac{n}{x}\, t) \right] \psi(x) \right) dx \; - \psi^m(t) \right\}$$

$$e^{at} t^m \left\{ \int_0^\infty \left(\frac{n^{\lambda+n+1}}{|\overline{\lambda+n+k+1}} (-1)^m D_x^m \left[\frac{t^{\lambda+n-m}}{x^{\lambda+n+1-m}}\, e^{-\frac{n}{x} t}\, \frac{(-1)^k}{k!}\, U(-k\,; 1+a\,; \tfrac{n}{x}\, t) \right] \psi(x) \right) dx \; - \psi^m(t) \right\}$$

integrating by parts m times, we get

$$e^{at}\, t^m \left\{ \int_0^\infty \left(\frac{n^{\lambda+n+1}}{|\overline{\lambda+n+k+1}}\, (-1)^m \psi^m(x) \left[\frac{t^{\lambda+n-m}}{x^{\lambda+n+1-m}}\, e^{-\frac{n}{x} t}\, \frac{(-1)^k}{k!}\, U(-k\,; 1+a;\, \tfrac{n}{x}\, t) \right] \right) dx \; - \psi^m(t) \right\}$$

$$e^{at} \left\{ \int_0^\infty \left(\frac{n}{|\overline{\lambda+n+k+1}} (-1)^m [\psi^m(x)\, x^{m-1}] \left[[\tfrac{nt}{x}]^{\lambda+n} e^{-\frac{n}{x} t} \frac{(-1)^k}{k!} U(-k; 1+a;\, \tfrac{n}{x}\, t) \right] \right) dx - \; t^m\, \psi^m(t) \right\}$$

but, by the result (3.4.1), we have

$$\int_0^\infty [\tfrac{nt}{x}]^{\lambda+n}\, e^{-\frac{n}{x} t} \frac{(-1)^k}{k!}\, U(-k; 1+a;\, \tfrac{n}{x}\, t)\, dx \sim nt\, (\lambda+n+k-2)! \quad \text{.....(3.4.5)}$$

$$e^{at} \int_0^\infty \left(\begin{array}{c} \left[\frac{n}{|\overline{\lambda+n+k+1}}\, (-1)^m\, [\psi^m(x)\, x^{m-1}\,] - \frac{t^{m-1}\, \psi^m(t)}{n\,(\lambda+n+k-2)\,!} \right] \\ \times \left[\, (\tfrac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x} t}\, \frac{(-1)^k}{k\,!}\, U(-k; 1+a;\, \tfrac{n}{x}\, t) \right] \end{array} \right) dx$$

$$= e^{at} \frac{n}{\lfloor \overline{\lambda + n + k + 1}} (-1)^{m} \left\{ \int_{0}^{\infty} \left([\psi^{m}(x)\, x^{m-1}] - \frac{\psi^{m}(t)\, t^{m-1} (\lambda+n+k)(\lambda+n+k-1)}{n} \right) . \times \left[[\tfrac{nt}{x}]^{\lambda+n} e^{-\frac{n}{x} t} \frac{(-1)^{k}}{k!} U(-k; 1+a; \tfrac{n}{x} t) \right] dx \right\}$$

$$= I_1(t) + I_2(t) + I_3(t) \qquad \text{.....(3.4.6)}$$

where $I_1(t)$, $I_2(t)$ *and* $I_3(t)$ denotes the terms obtained by integrating over the intervals $0 < x < t - \delta$, $t - \delta < x < t + \delta$ and $t + \delta < x < \infty$ respectively. δ being a positive number.

Consider,

$$|\, I_2(t) \,| \leq e^{at} \int_{t-\delta}^{t+\delta} |\psi^{m}(x)\, x^{m-1} - \psi^{m}(t)\, t^{m-1}| dx$$

Let $\phi(x) = \psi^{m}(x)\, x^{m-1}$

Now ϕ is bounded as ψ is bounded.

Therefore,

$$|\, I_2(t) \,| \leq \delta\, e^{at} \sup_{t-\delta < \Upsilon < t+\delta} |\phi'(\Upsilon)|$$

Restrict δ by $0 < \delta < 1$. Then, since ϕ is smooth and of bounded support, the last expression is bounded by δB, where B is a constant with respect to t and δ.

Thus, given an $\in > 0$, we have that

$$|\, I_2(t) \,| \leq \in . \text{ For } \delta = \min\ (1 ; \tfrac{\in}{B})$$

and for all n. Fix δ in this way.

Now consider,

$$|I_1(t)| \le e^{at} \int_0^{t-\delta} \left| \psi^m(x)\, x^{m-1} \left[(\tfrac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k;\ 1+a;\ \tfrac{n}{x}\, t\,) \right] \right| dx$$

$$+ \frac{|\psi^m(t)\, t^{m-1}|}{n} \int_0^{t-\delta} \left[(\tfrac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k;\ 1+a;\ \tfrac{n}{x}\, t) \right] dx$$

but, as n $\to \infty$ the second term of the above expression tends to zero uniformly.

Therefore,

$$|I_1(t)| \le e^{at} \int_0^{t-\delta} \left| \psi^m(x)\, x^{m-1} \left[(\tfrac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k;\ 1+a;\ \tfrac{n}{x}\, t) \right] \right| dx$$

Again,

Let $\phi(x) = \psi^m(x)\, x^{m-1}$ and ϕ is bounded.

$$|I_1(t)| \le e^{at}\, C_k \int_0^{t-\delta} \left[(\frac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x}t}\, \frac{(-1)^k}{k!}\, U(-k;\ 1+a;\ \frac{n}{x}\, t\,) \right] dx$$

$$\le e^{at}\, C_k \int_0^{t-\delta} (\frac{nt}{x})^{\lambda+n+k}\, e^{-\frac{n}{x}t}\, dx$$

$$put\ \frac{n}{x} = u \implies dx = -\frac{n}{u^2}\, du.$$

$$when\ x = 0,\ u = \infty.$$

$$and\ x = t - \delta,\ u = \frac{n}{t-\delta}$$

$$|I_1(t)| \le\ e^{at}\, C_k \int_{\frac{n}{t-\delta}}^{\infty} (ut)^{\lambda+n+k}\, e^{-ut}\, \frac{n\, du}{u^2}$$

$$\le n\, e^{at}\, C_k\, t^{\lambda+n+k} \int_{\frac{n}{t-\delta}}^{\infty} u^{\lambda+n+k-2}\, e^{-ut}\, du$$

Integrating by parts repeatedly, since the integrated part is vanish as n $\to \infty$.

$$|I_1(t)| \le n\, e^{at}\, C_k(\lambda + n + k - 2)!\, t^{\lambda+n+k-(\lambda+n+k-2)} \int_{\frac{n}{t-\delta}}^{\infty} e^{-ut}\, du$$

$$\le n\, e^{at}\, C_k\,(\lambda + n + k - 2)!\, t^2 \left[\frac{e^{-ut}}{-t}\right]_{\frac{n}{t-\delta}}^{\infty}$$

$$\le n\, e^{at}\, C_k\,(\lambda + n + k - 2)!\, t\, e^{-\frac{n}{t-\delta}t}$$

$$\to 0 \qquad as\ n \to \infty.$$

Thus, $I_1(t)$ converges to zero uniformly on $0 < x < t - \delta$ as $n \to \infty$.

Finally,

$$|I_3(t)| \le e^{at} \int_{t+\delta}^{\infty} \left|\psi^m(x)\, x^{m-1}\left[(\tfrac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k;\ 1+a;\ \tfrac{n}{x}t)\right]\right| dx$$

$$+\frac{|\psi^m(t)\, t^{m-1}|}{n} \int_{t+\delta}^{\infty} \left[(\tfrac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k;\ 1+a;\ \tfrac{n}{x}t)\right] dx$$

but, as $n \to \infty$ the second term of the above expression tends to zero uniformly.

$$|I_3(t)| \le e^{at} \int_{t+\delta}^{\infty} \left|\psi^m(x)\, x^{m-1}\left[(\tfrac{nt}{x})^{\lambda+n}\, e^{-\frac{n}{x}t}\, \tfrac{(-1)^k}{k!}\, U(-k;\ 1+a;\ \tfrac{n}{x}t)\right]\right| dx$$

Again, *Let* $\phi(x) = \psi^m(x)\, x^{m-1}$ and ϕ is bounded, and let $A \le x \le B$ be the finite interval containing support of $\phi(x)$. For $A < t+\delta \le B$; $I_3(t) \equiv 0$. and on the other hand $t+\delta < B < \infty$, $|\phi(x)| \le C_k$, C_k is sufficiently large constant.

Thus,

$$|I_3(t)| \le e^{at} C_k \int_{t+\delta}^{B} (\tfrac{nt}{x})^{\lambda+n+k}\, e^{-\frac{n}{x}t}\, dx$$

$put\ \frac{n}{x} = u \Rightarrow dx = -\frac{n}{u^2}\,du.$

$when\ x = t+\delta,\ u = \frac{n}{t+\delta}.$

$and\ x = B,\ u = \frac{n}{B}.$

$$|I_3(t)| \le n\, e^{at}\, C_k\, t^{\lambda+n+k} \int_{\frac{n}{t+\delta}}^{\frac{n}{B}} u^{\lambda+n+k-2}\, e^{-ut}\, du$$

$$\le n\, e^{at}\, C_k\, t^2 (\lambda+n+k-2)!\left[\frac{e^{-ut}}{-t}\right]_{\frac{n}{t+\delta}}^{\frac{n}{B}}$$

$$\le n\, e^{at}\, C_k\, t\,(\lambda+n+k-2)!\,[e^{-\frac{n}{t+\delta}t} - e^{-\frac{n}{B}t}]$$

$$\to 0 \qquad as\ n \to \infty.$$

Thus, $I_3(t)$ converges to zero uniformly on $t+\delta < x < \infty$ as $n \to \infty$.

Altogether this proves that as limit $n \to \infty$ (3.4.6) is uniformly bounded by $\in$ on $0 < x < \infty$. Since $\in > 0$ is arbitrary we get (3.4.6) tends to zero uniformly as $n \to \infty$.

Hence the proof.

By using this lemmas we prove the inversion theorem.

Theorem 3.4.1: (Inversion theorem)

For a given kernel $(st)^{\lambda}\, e^{-st}\, L_k^a(st)$, let $M_{n,x}$ denotes some shifting and differentiation operator as defined in the section 2.3 of second chapter.

If $F(s) \triangleq < f(t), (st)^{\lambda}\, e^{-st}\, L_k^a(st) >$, $f \in GL'_a$, $s = \frac{n}{x}$, then

$$\lim_{n\to\infty} M_{n,x}\,[F(s)] = f \quad \text{in } GL'_a \qquad .....(3.4.7)$$

Proof:-

Let $\psi \in D$, then in the sense of convergence in D', we shall show that

$$\lim_{n \to \infty} < M_{n,x} [F(s)] , \psi(x) > = < f(t) , \psi(t) > \quad .....(3.4.8)$$

Since, F(s) is a smooth function and

$$M_{n,x} [F(s)] = M_{n,x} < f(t) , (st)^{\lambda} e^{-st} L_k^a(st) >$$

$$= < f(t) , M_{n,x} \left[(st)^{\lambda} e^{-st} L_k^a(st) \right] >$$

$$= < f(t) , \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] >$$

Consider,

$$< M_{n,x} [F(s)] , \psi(x) >$$

$$= < < f(t) , \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] > , \psi(x) >$$

Since $M_{n,x}$ [F(s)] is a smooth function, therefore $< M_{n,x} [F(s)] , \psi(x) >$ is an integral and for $\psi \in D$ we write,

$$< M_{n,x} [F(s)] , \psi(x) > = \int_0^{\infty} \psi(x) < f(t) , \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] > dx$$

By the lemma (3.4.1) this equals,

$$< f(t) , \int_0^{\infty} \frac{s^{\lambda+n+1}}{|\lambda+n+k+1} \left[t^{\lambda+n} e^{-st} L_k^a(st) \right] \psi(x) \, dx > \quad .....(3.4.9)$$

This expression tends to $< f(t) , \psi(t) >$ as $n \to \infty$ because $f \in GL'_a$ and according to the lemma (3.4.8), the testing function in the expression (3.4.9) converges to $\psi(t)$ in GL_a.

Hence the proof.

Theorem 3.4.2: (Uniquness theorem)

Let f, g $\in GL'_a$ and GL(f)(s) = F(s), Re. $s > \sigma_f$, GL(g) = G(s) Re. $s > \sigma_g$ and GL(f)(s) = GL(g)(s) for s > max(σ_f, σ_g) . Then f = g in the sense of equality in D'.

Proof:-

Let $M_{n,x}$ [F(s)] be as specified in the theorem (3.4.1), then in the sence of convergence in D', we have

$$f = \lim_{n\to\infty} M_{n,x}[F(s)] = \lim_{n\to\infty} M_{n,x}[G(s)] = g$$

Hence the proof.