CHAPTER - I

INTRODUCTION

1.1 Integral Transform:-

The theory of Integral Transformation is widely used in both pure and applied mathematics. This theory is used in solving some boundary value problems and initial value problems.

The origin of Laplace transform is due to Oliver Heaviside (in 1850-1925) later investigation by Bromwich, Carson and Van der Pol placed the Heaviside calculus on foundation.

We can express many functions in analysis as improper Riemann integrals or Lebesgue integrals of the form,

$$F(s) = \int_{0 \text{ or } -\infty}^{\infty} K(s,t)\, f(t)\, dt \qquad .....(1.1.1)$$

where s is real or complex.

The function F defined by this type of equation is called integral transform of f. The function K(s,t) in the integrand is called kernel of the transformation. It is assumed that the infinite integral in the equation (1.1.1) is convergent.

Different forms of the kernel K(s,t) and the range of integration, gives different integral transformations, such as Laplace, Fourier, Mellin, Hankel, Convolution and K-transformations. For all these transformations inversion formulae are available in Erdelyi [7].

The problems involving several variables can be solved by applying integral transformations successively with regard to several variables. In physical problems Laplace transformation is generaly used first to remove the time variable and then other integral transformations on space variables are successively applied. Some examples of repeated applications of transforms are given by Sneddon [16] , Tranter [18] and Ditkin [5].

In the integral (1.1.1) when the kernel K(s,t) is e^{-st} and the range of integration is 0 to ∞ then we get the one sided Laplace transform. Thus, the conventional Laplace transformation of a function f(t) is defined by the integral,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad .....(1.1.2)$$

It has always been a subject of great interest because of its mathematical elegance and also because of its usefulness in solving certain types of boundary value problems. In certain cases it has already shown its distinct and superior mathematical character in comparison with the ordinary methods of solving such problems.

The inversion formula for Laplace transformation of (1.1.2) is obtained in the form,

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\mu t} F(s)\, ds \qquad .....(1.1.3)$$

under suitable conditions on F(s) or f(t).

The important aspect of the integral transformation is its inversion theorem.

The theory of integral transformation is developed by many mathematicians like Widder, Sneddon, Tranter, Boas, Brijmahon, Bhonsle, Saxena and others, who studied deeply in this field and made their contributions for integral transformations.

Many mathematicians have made deep study of the Laplace transformation in the various aspects and some others have introduced more generalized integral transformations of which the Laplace transform become a particular case.

We enumerate below some of these generalizations

[i] Meijer [9] has defined the transformation

$$F(s) = \left(\tfrac{2}{\pi}\right)^{1/2} \int_0^\infty (st)^{1/2} K_m(st) f(t)\, dt \qquad \text{.....(1.1.4)}$$

where $K_m(z)$ represents Bessels function of the second Kind with imaginary argument.

[ii] Meijer[10] has also defined the transformation

$$F(s) = \int_0^\infty (st)^{-k-\frac{1}{2}} e^{\frac{-st}{2}} W_{k-\frac{1}{2},m}(st)\, f(t)dt \qquad \text{.....(1.1.5)}$$

where $W_{k,m}(.)$ Represents the Whittaker function.

[iii] P. K. Banerji and Deepali Sinha [11] has taken the transformation

$$F(z) = 2^{\frac{-v}{2}} \int_0^\infty (zt)^{\lambda} e^{-\frac{1}{2zt}} D_v(\sqrt{2zt})\, f(t)\, dt \;\; \mathrm{Re}(\lambda) > 0 \qquad \text{.....(1.1.6)}$$

where D_v denotes Weber's parabolic cylinder function.

[iv] Bhise[2] has introduced the Meijer-Laplace transformation by the integral,

$$F(s) = \int_0^\infty G^{m+1,c}_{m,m+1}\left[st \,/\, {\{a_m+n_m\} \atop \{n,m\},\{e\}} \right] f(t)\, dt \qquad .....(1.1.7)$$

in which G(.) Represents Meijer's G - function.

[v] Choudhary M.S. [4] has taken the Laplace - Hankel transform of F(u,v) is

$$F(u,v) = L_+H_\lambda(f) = \int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\; J_\lambda(vy)\, f(x,y)\, dx\, dy \qquad .....(1.1.8)$$

where $J_\lambda(vy)$ is the Bessel function of the first kind with λ real.

[vi] Varma[19, 20] has introduced Verma transformation

$$F(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{st}{2}} W_{k,m}(st)\, f(t)\, dt \qquad .....(1.1.9)$$

where $W_{k,m}(.)$ represents the Whittaker function.

[vii] Varma[19, 20] has also introduced the Whittaker transformation

$$F(s) = \int_0^\infty (2st)^{-\frac{1}{4}}\; W_{k,m}(2st)\, f(t)\, dt \qquad .....(1.1.10)$$

where $W_{k,m}(.)$ represents the Whittaker function.

J.M.C.Joshi and P.C.Joshi [8] have defined the generalization of Laplace Transform, in the integral (1.1.1) when the kernel K(s,t) is $(st)^\lambda\, e^{-st}\, L_k^a(st)$ and the range of integration is 0 to ∞ then we get the generalized Laplace transform as

1.2 Generalized Laplace Transform:-

The generalized Laplace transform is defined as,

$$F(s) = \int_0^\infty (st)^\lambda\, e^{-st}\, L_k^a(st)\, f(t)\, dt \qquad .....(1.2.1)$$

where $L_k^a(st)$ is the Laguerre polynomial and $\lambda$is a non-negative integer. When $\lambda = 0$ and k = 0 this transform reduces to well known one sided Laplace transform.

The Laguerre polynomial $L_k^a(st)$is defined as,

$$L_k^a(st) = \sum_{n=0}^{k} \frac{(-1)^n (1+a)_k (st)^n}{n! (k-n)! (1+a)_n}$$

which is the simple set of polynomials. The coefficient of $(st)^k$ is $\frac{(-1)^k}{k!}$,

and $(a)_n = a (a+1)(a+2)(a+3) \ldots\ldots (a+n-1), \; n \geq 1.$

$(a)_0 = 1 \;, \quad a \neq 0.$

1.3 Generalized Functions:-

The generalized functions were first introduced in 1927 by P. Dirac in his research in quantum Mechanics. The generalized functions is a generalization of the classical concept of mathematical functions.

The first impact of such a generalization of function have laid by Bochner [3] and Sobolev [17]. But the Laurent Schwartz's [14] deep study gives the construction of the theory of generalized functions on firm foundation (1950-51).

Let I be an open subset of R^n or C^n, where C^n is the complex n-dimensional Euclidean space. The set V(I) is said to be testing function space if the following conditions are satisfied,

(i) V(I) consists entirely of smooth functions defined on I.

(ii) V(I) is either a complete countablly multinormed space or complete countable union space.

(iii) If the sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in V(I) to zero, then for every non-negative integer k $\in R^n$, $\{ D^k \phi_v\}_{v=1}^{\infty}$ converges to the zero function uniformly on every compact subset of I.

A generalized function on I is any continuous linear functional on any testing function space V(I) on I.

In other word, f is called a generalized function, if it is a member of the dual space V'(I) of some testing function space V(I).

1.4 <u>Distributions:</u>-

Let I be a non-empty open set in R^n and K be a compact subset of I. $D_k(I)$ is the set of all complex valued smooth functions defined on I which vanish outside K. $D_k(I)$ is linear space under the usual definition of addition of functions and their multiplication by complex numbers. The zero element in $D_k(I)$ is the identically zero function on I. For each non negative integer $k \in R^n$ define γ_k by,

$$\gamma_k(\phi) = \underset{t \in I}{Sup} |D^k \phi(t)| \quad ; \quad \phi \in D_k(I) \ , k = 0, 1, 2, ... \qquad .....(1.4.1)$$

Then $\{\gamma_k\}$ is countable multinorm on $D_k(I)$. The topology of $D_k(I)$ is generated by the multinorm $\{\gamma_k\}_{k=0}^{\infty}$ where γ_k is a seminorm on $D_k(I)$ defined in (1.4.1).

The space $D_k(I)$ is a testing function space on I.

Let $\{K_m\}_{m=1}^{\infty}$ be the sequence of compact subsets of I with the following two properties,

(i) $K_1 \subset K_2 \subset K_3 \subset ....$

(ii) Each compact subset of I is contained in one of the K_m.

consequently, $I = \bigcup_{m=1}^{\infty} K_m$

and $D_{k_m}(I) \subset D_{k,m+1}(I)$ and the topology of $D_{k_m}(I)$ is stronger the topology induced on it by $D_{k,m+1}(I)$.

Therefore, the countable union space D(I) is the space

$$D(I) = \bigcup_{m=1}^{\infty} D_{k_m}(I)$$

The dual of D(I) is denoted by D'(I) and members of D'(I) are called distributions on I .

In other words, a continuous linear functional on the space D(I) is called a distribution on I.

Thus, every distribution is a generalized function but not conversely. Because of this convention, the members of D'_k (I) will be called generalized functions but not distributions because D_k (I) does not contain D(I).

1.5 <u>Generalized Integral Transformations:-</u>

The Generalized Integral Transformation is evolved from the theory of integral transforms and the theory of generalized functions. By using this theory Fourier transform is extended to the generalized functions.

Schwartz in 1952 extended Laplace transform to generalized function. In 1966 Zemanian [24] has extended it to generalized functions as follows,

$$F(s) = \langle f(t), e^{-st} \rangle \qquad .....(1.5.1)$$

Zemanian [24] has also extended Weierstrass, Convolution, Hankel and K- transformations to generalized functions.

In the present work, we extend the generalized Laplace transform defined by (1.2.1) to the certain class of generalized functions using the technique used in the extension of Convolution and K- transformations by Zemanian [24] as

$$F(s) = < f(t)\,,\ (st)^{\lambda}\, e^{-st}\, L_k^{a}(st)\ >,\ f \in GL'_a \qquad .....(1.5.2)$$

where GL_a is a suitably chosen space.

1.6 Notations and Terminology:-

Here we mention the most common notations and terminology's used in the development of the present work follows from that of Zemanian [24].

R^n and C^n denotes the real and complex n-dimensional Euclidean spaces respectively. Throughout this work x, y and t are real variables and s is complex variable.

By the compact set in R^n we mean a closed and bounded set in R^n. If I is an open set in R^n and K is compact set in R^n such that $K \subset I$, then K is called compact subset of I.

A conventional function is a function whose domain is contained in R^n or C^n and whose range is either in R^1 or C^1.

A conventional function is said to be smooth if all its derivatives of all orders are continuous at all points of its domain.

If k is non-negative integer in R^1 the partial differential with respect to t is denoted by $D_t^k = \frac{\partial^k}{\partial t^k}$.

The support of a continuous function f(t) defined on open set Ω in R^n is the closure with respect to Ω of the set of points t where $f(t) \neq 0$.

Whenever a certain equation is a definition, the symbol $\triangleq$ is used for equality.

If the f is a generalized function on R^1, the notation $f(x)$, $x \in R^1$ is used merely to indicate that the testing function, on which f is defined, x as their independent variable, it does not mean that f is a function of x.

$< f, \phi >$ denotes the number assigned to the element ϕ in a testing function space by a member of the dual space.

D(I) denotes the space of smooth functions that have compact support. D'(I) is its dual space.

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