CHAPTER-0

DEFINITIONS AND RESULTS

Chapter 0

Definitions And Results

In this chapter we collect the definitions and results which will be used in the subsequent chapters.

§ 1 Definitions: -

For basic definitions in lattice theory we refer Gratzer [1].

0.1.1 Partially ordered set or Poset :

Let P be a non-void set. A partial order relation ' $\leq$ ' on P the relation satisfying the following conditions.

1) $a \leq a$ (Reflexivity)

2) If $a \leq b$ and $b \leq a$ then $a = b$ (Antisymmetry)

3) If $a \leq b$ and $b \leq c$ then $a \leq c$ (Transitivity)

for all $a, b, c \in P$

Then the ordered pair $< P, \leq >$ is called a partially ordered set or poset in short.

A poset $< P, \leq >$ is called chain or totally ordered set if $a \leq b$ or $b \leq a$, for all $a, b \in P$.

Example:- Set of all natural numbers together with usual ordering relation is a chain.

0.1.2 Bounded Poset

Let $< P, \leq >$ be a poset. If there exists an element 0 in P such that

$0 \leq x$, for all x in P, then the element 0 is called zero element in poset P.If there exists an element 1 in P such that $1 \geq x$, for all x in P, then the element 1 is called unit element in poset P.

A poset with zero element and the unit element is called a bounded poset.

Example: - The poset represented by the following diagram (Fig. 0.1) represents the bounded poset.

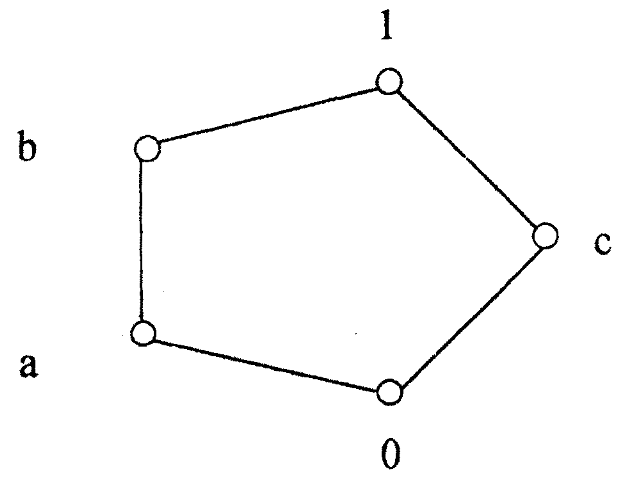

Fig.0.1

The poset represented by the following diagram (Fig.0.2) represents the non bounded poset.

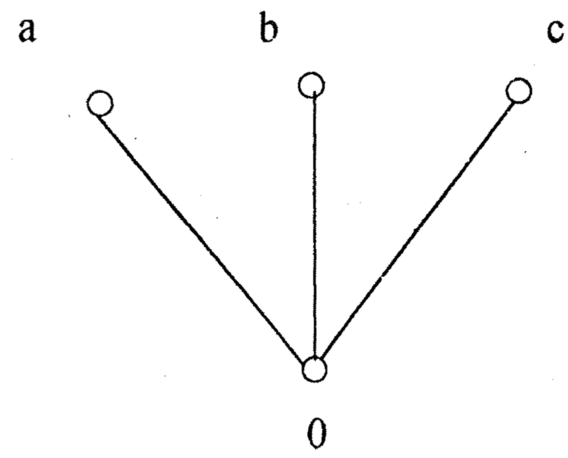

Fig.0.2

0.1.3 Supremum and Infimum in poset:

Let $< P, \leq >$ be a poset and $H (\neq \phi) \subseteq P$, $a \in P$ is an upper bound of H if $h \leq a$ for all $h \in H$. An upper bound a of H is the least upper bound of H or supremum of H if for any upper bound b of H, we have $a \leq b$. We write this $a = \text{Sup } H$ or $a = \vee H$.

Let $< P, \leq >$ be a poset and $H (\neq \phi) \subseteq P$, $a \in P$ is a lower bound of H if $a \leq h$ for all $h \in H$. A lower bound a of H is the greatest lower bound of H or infimum of H if for any lower bound b of H, we have $b \leq a$. We write this $a = \text{Inf } H$ or $a = \wedge H$.

0.1.4 Lattice (as a Poset):

A poset $< L, \leq >$ is called lattice if sup {a, b} and inf {a, b} exist for all a and b in L.

0.1.5 Lattice (as an algebra):

By a lattice $< L, \wedge, \vee >$ we mean a non empty set L together with binary operations $\wedge$ and $\vee$ defined on L, satisfying the following conditions,

for all a, b, c in L.

1) $a \wedge a = a$, $a \vee a = a$ (Idempotent).

2) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ (Commutative).

3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$ (Associative).

4) $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ (Absorption).

Example: - Consider the poset L = {0, a, b, c, 1} whose diagrammatic representation is given as in Fig. 0.3. Then L is a lattice.

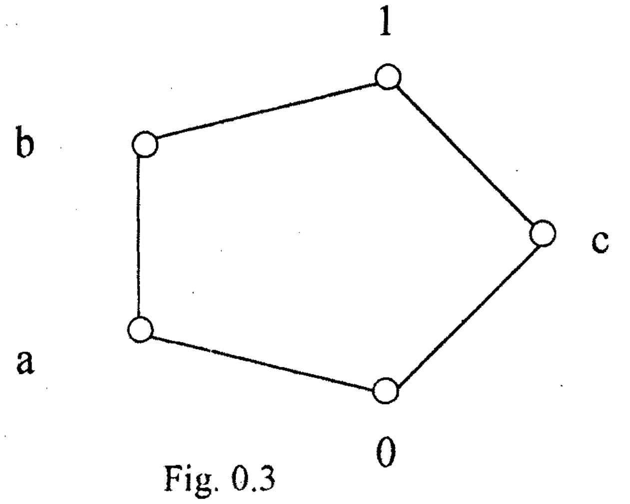

Fig. 0.3

Remark:- Every lattice is a poset but converse need not be true.

The poset represented by Fig.0.2 is an example of a poset, which is not lattice.

0.1.6 Semi Ideal in a Lattice:

A non empty subset I of L is called a semi ideal if $x \leq y$ and $y \in I$, then $x \in I$ for $x, y \in L$.

0.1.7 Ideal in a Lattice :

A non empty subset I of L is called an ideal if

i) $x \vee y \in I$, for all x and y in I.

ii) for $x, y \in L$, $x \leq y$ and $y \in I$ imply $x \in I$.

Remark:- Every ideal is a semi ideal but converse need not be true.

For this consider the lattice L = {0, a, b, c, 1} whose diagrammatic representation is given as in Fig. 0.4

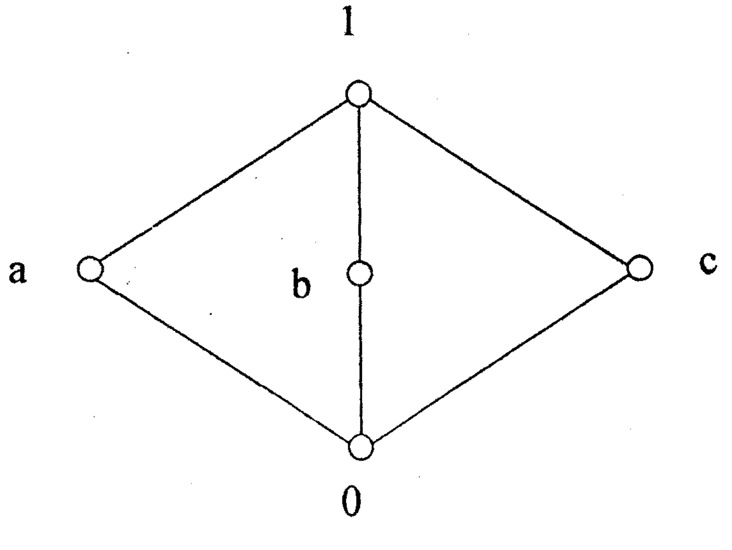

Fig. 0.4

Then S ={0, a, b} is a semi ideal in L but S is not ideal.

0.1.8 Lattice of ideals I(L) :

The set of all ideals I (L) in L is a poset under set inclusion and as a poset it is a lattice .For any two ideals I and J in L we have

$I \wedge J = I \cap J$ and $I \vee J = (I \cup J] = \{t \in L / t \leq i \vee j$ for some $i \in I$ and $j \in J\}$.

This lattice I (L) called lattice of ideals L.

0.1.9 Proper ideal in a Lattice :

An ideal I in a lattice which is different from L is called a proper ideal of L.

0.1.10 Prime ideal in a Lattice :

A proper ideal I in L is called prime if for all x and y in L.

$x \wedge y \in I$ imply that $x \in I$ or $y \in I$.

The set of all prime ideals of L denoted by P(L). It is a poset under set inclusion.

0.1.11Maximal ideal in a Lattice :

A proper ideal I in L is called maximal if I is not contained in any other proper ideal of L.

0.1.12 Minimal prime ideal in a Lattice :

A minimal prime ideal of lattice L is a prime ideal which does not contain any other prime ideal. i.e. minimal prime ideal is the minimal element in the set of all prime ideals in L.

0.1.13 Ideal generated by H ($\neq\phi$) in L :

For any non empty subset H of L , the ideal generated by H is denoted by (H].Further (H] = { $x\in L$ / $x \leq h_1 \vee \ldots\ldots \vee h_n$ for $h_i \in H$ and n is finite}

0.1.14 Principal ideal in a Lattice :

Given an element a in L the ideal generated by {a} denoted by

(a] = {$x\in L$ / $x \leq a$} and is called principal ideal of Lattice L generated by a.

0.1.15 Annihilator in a Lattice :

Let L be a lattice with 0. For any non empty subset A of L, the annihilator A* of A is defined as A*={$x\in L$ / $x \wedge a=0$ for each $a\in A$}.

In particular if A = {a}, we get {a}* ={$y \in L$ /$a \wedge y = 0$} (for $a\in L$).

0.1.16 Semi Filter in a Lattice / Dual semi ideal :

A non empty subset F of L is called an semi filter if $x \geq y$ and $y \in F$ imply $x \in F$ for $x, y \in L$.

0.1.17 Filter / Dual ideal in a Lattice :

A non empty subset F of a lattice L is called a filter / dual ideal if

i) $x \wedge y \in F$ for all x and y in F.

ii) for $x, y \in L$, $x \geq y$ and $y \in F$ imply $x\in F$.

Remark:- Every filter is semi filter but converse need not be true.

For this consider the lattice L = { 0 , a , b , c , 1 } whose diagrammatic representation is given as in Fig. 0.5

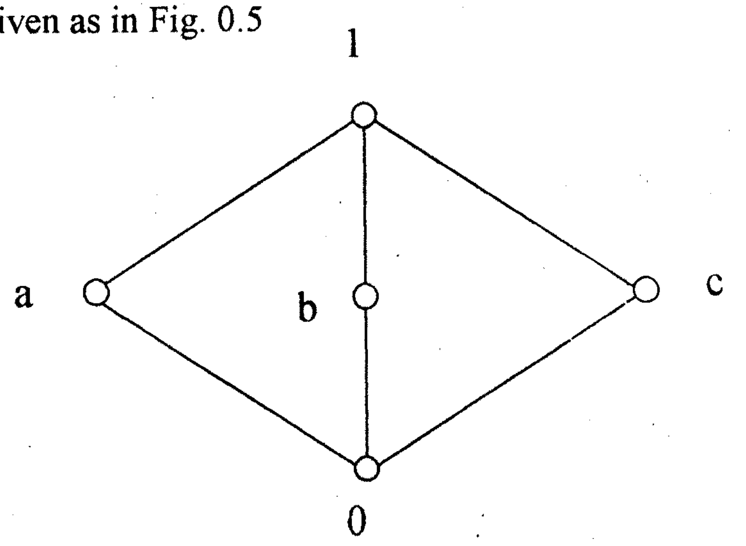

Fig. 0.5

Then S ={a, b, 1} is a semi filter in L but S is not a filter in L.

0.1.18 Proper Filter in a Lattice :

A filter F of L which different from L is called proper filter.

0.1.19 Prime filter in a Lattice :

A proper filter F of L is prime if $a \vee b \in F$ implies that $a \in F$ or $b \in F$

for a, b, $\in$ L.

0.1.20 Filter generated by H ($\neq \phi$) in L :

For any non empty subset H of L, the smallest filter in L containing H is denoted by [H). Further $[H) = \{x \in L \,/\, x \geq h_1 \wedge \ldots \wedge h_n$ for $h_i \in H$ and n is finite$\}$.

0.1.21 Principal filter in a Lattice :

Given an element a in L, the filter generated by $\{a\}$ which denoted by $[a) = \{x \in L / x \geq a\}$ is called principal filter of L.

0.1.22 Maximal filter in a Lattice :

A filter A of lattice L is maximal filter if it is not contain in any other proper filter.

0.1.23 Minimal filter in a Lattice :

A minimal prime filter of lattice L is prime filter, which does not contain any other prime filter.

0.1.24 Lattice of filters F (L) :

The set of all filters in a poset under set inclusion and as a poset it is a lattice. For any two filters F and J in L we have $F \wedge J = F \cap J$ and

$F \vee J = [F \cup J) = \{x \in L / x \geq f \wedge j$ for some $f \in F$ and $j \in J\}$ which is called as the lattice of filter L.

0.1.25 Complemented Lattice :

A bounded lattice L is said to be complemented if for every x in L there exists y in L such that $x \wedge y = 0$ and $x \vee y = 1$.

0.1.26 Distributive Lattice :

A lattice L is said to be distributive if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for $x, y, z \in L$.

Example :- The lattice L = { 0 , a , b , c , 1 } whose diagrammatic representation is as shown in Fig. 0.6 is distributive.

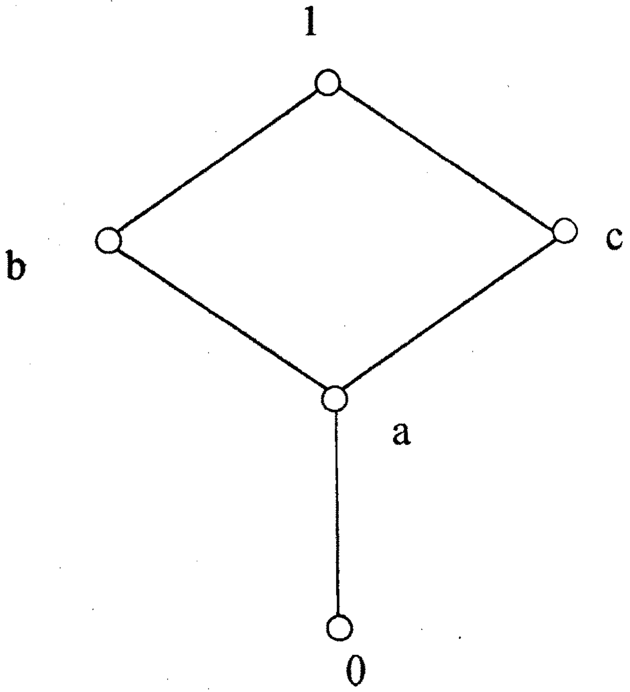

Fig. 0.6

0.1.27 0 -Distributive Lattice :

Let L be a lattice with 0. L is said to be 0-distributive if $(x \wedge y) = 0$, $(x \wedge z) = 0$ imply $x \wedge (y \vee z) = 0$ for $x, y, z \in L$.

0.1.28 1 -Distributive Lattice :

Let L be a lattice with 1 is said to be 1-distributive if $(x \vee y) = 1$, $(x \vee z) = 1$ imply $x \vee (y \wedge z) = 1$ for $x, y, z \in L$.

Remark :- A 0-1 distributive lattice is a bounded lattice which is both 0-distributive and 1-distributive lattice.

0.1.29 Pseudo-annihilator:

Let L be a lattice with 0. For any non empty subset A of L, the pesudo-annihilator A° of A is defined as $A^\circ = \{x \in L \,/\, x \wedge a = 0 \text{ for some } a \in A\}$.

In particular if $A = \{a\}$, we get $\{a\}^\circ = \{y \in L \,/ a \wedge y = 0\}$ (for $a \in L$).

If $a \in L$, $(a)^\circ$ for $\{a\}^\circ$.

0.1.30 Pseudo-complement of an element in L :

Let L be a lattice with 0 and $x \in L$. $x^* \in L$ is the pesudo-complement of x if $x \wedge y = 0 \Leftrightarrow y \leq x^*$.

0. 1.31 Pseudo-complemented Lattice :

A pesudo-complemented lattice L is a lattice with 0 in which every element has pesudo-complement in L.

A normal element of a pesudo-complemented lattice L is an element a such that $a = a^{**}$.

0.1.32 Dence element:

Let L be a lattice with 0. An element $a \in L$ is said to be dense element in L if $(a]^* = \{0\}$.

In a pesudo-complement lattice L. An element $a \in L$ is said to be dense if $a^* = \{0\}$.

0. 1.33 Quasi-complemented Lattice

A lattice L with 0 is a qausi-complemeted lattice if for any $x \in L$ there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is a dense element.

0. 1.34 Semi prime ideal

An ideal I of L is called semi prime ideal if for a, b, $c \in I$

$a \wedge b \in I$ and $a \wedge c \in I$ imply $a \wedge (b \vee c) \in I$.

Remark: Every prime ideal in L is semi prime ideal of L is semi prime but converse need not be true.

e.g:- Consider the lattice L = { 0 , a , b , c, d , 1 } whose diagrammatic representation is as shown in Fig. 0.7.

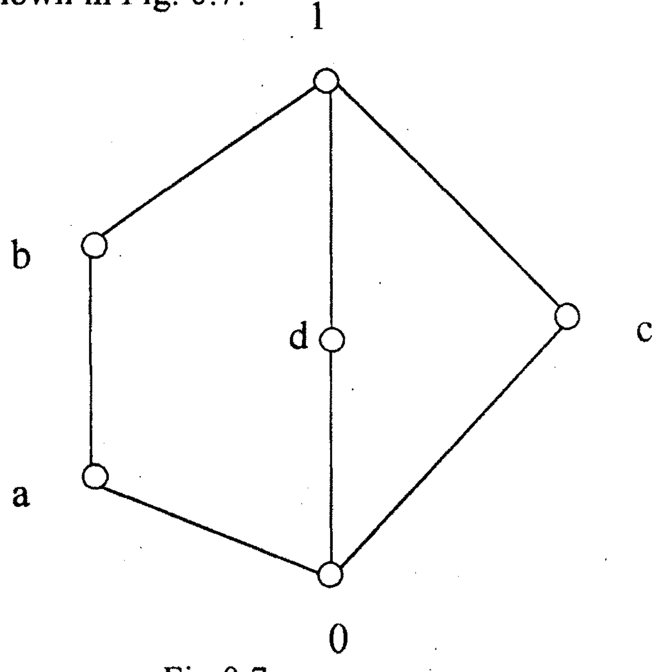

Fig.0.7

Then S ={0, a } is a prime ideal in L but S is not semi prime ideal.

§ 2 RESULTS

Throughout L denotes a bounded lattice.

Result 0.2.1 [7] - Every proper filter of a lattice with 0 is contained in a maximal filter.

Result 0.2.2 **[8]** - Let A be a non empty proper subset of a lattice L. Then A is a filter if and only if L\ A is a prime semi ideal.

Result 0.2.3 **[8]** - Any prime semi ideal of a lattice L with 0 contains a minimal prime semi ideal.

Result 0.2.4 [8] - Any normal semi ideal of a lattice L with 0 is the intersection of all minimal prime semi ideals containing it.

Result 0.2.5 **[3]** - Let A be proper filter of a lattice L with 0. Then A is maximal if and only if for each x in L\ A, there is some a in A such that $a \wedge x = 0$.

Result 0.2.6[9]Let A be a non empty subset of a lattice L with 0. Then A* and A° are semi ideals of L and [x]* = [x)°.

Result 0.2.7 - Let I (L) be the lattice of all ideals L with 0 .

Then for any $A_1, \ldots, A_n \in I(L)$

$A_1 \vee \ldots \vee A_n = \{ x \in L / x \leq a_1 \vee \ldots \vee a_n \ , a_j \in A_j \ , j = 1, \ldots, n\}$ and

$A_1 \wedge \ldots \wedge A_n = A_1 \cap \ldots \cap A_n = \{ a_1 \wedge \ldots \wedge a_n \ , a_j \in A_j \ , j = 1, \ldots, n\}$.

Dualizing Result 0.2.7 we have

Result 0.2.8 - Let F(L) be the lattice of all filters L with 1. Then for any

$A_1, \ldots, A_n \in F(L)$

$A_1 \vee \ldots \vee A_n = \{ y \in L / y \geq a_1 \wedge \ldots \wedge a_n , a_j \in A_j , j = 1, \ldots, n\}$ and

$A_1 \wedge \ldots \wedge A_n = A_1 \cap \ldots \cap A_n = \{ a_1 \vee \ldots \vee a_n , a_j \in A_j , j = 1, \ldots, n\}$.

Result 0.2.9 - If $\{ A_i / i \in I \}$ is a family of ideals of a lattice , then

$\vee A_i = \{ x / x \leq a_{i1} \vee \ldots \vee a_{in} , a_{i1} \in A_{i1}, \ldots, a_{in} \in A_{in} ; i_1, \ldots i_n \in I\}$.

Result 0.2.10[8] Let A be a nonempty proper subset of a lattice L. Then A is prime ideal if and only if L \ A is a prime filter.

Necessary and sufficient condition for a nonempty subset to be a maximal filter in any lattice is established in following result.

Result 0.2.11 [8] - Let A be a non empty proper subset of a lattice L. Then A is a maximal filter if and only if L \ A is a minimal prime semi ideal.

Proof:- Only if part:-

Let A be a maximal filter of L.

To prove that L \ A is a minimal prime semi ideal.

By Result 0.2.1, L\ A prime semi ideal.

Now prove L \ A is a minimal prime semi ideal.

Let T be any other prime semi ideal properly contained in L \ A.

As $T \subset L \setminus A$ implies $A \subset L \setminus T$.

Claim :- L\T is a filter .

i) $L \setminus T \neq \phi$

ii) Let $x \leq y, x \in L \setminus T$. To prove $y \in L \setminus T$

Suppose $x \notin L \setminus T$. Then $x \in T$ and $x \leq y$, T is a semi ideal imply $x \in T$; a contradiction. Hence $y \in L \setminus T$.

iii) $x , y \in L \setminus T$. Prove that $x \wedge y \in L \setminus T$.

As $x \in L \setminus T$ implies $x \notin T$ and $y \in L \setminus T$ implies $y \notin T$.

Then $x \wedge y \notin T$ since T is prime semi ideal.

Thus $x \wedge y \in L \setminus T$.

Hence from (i) , (ii) and (iii) , $L \setminus T$ is a filter.

As $A \subset L \setminus T$, A is a maximal filter and $L \setminus T$ is a filter; a ontradiction.

Therefore $L \setminus A$ is a minimal prime semi ideal.

If part:-

Let $L \setminus A$ be minimal prime semi ideal.

To prove A is a maximal filter.

By Result 0.2.1, A is a proper filter.

i.e. $L \setminus A \neq L$. Therefore $A \neq \phi$.

Since A is a filter.

To prove A is a maximal filter.

Let $a \notin A$. i.e $a \in L \setminus A$.

If there exits $b \in L \setminus A$ such that $a \wedge b = 0$.

i.e. given $a \notin A$ there exits $b \in A$ such that $a \wedge b = 0$.

Therefore A is a maximal filter by Result 0.2.5.

Hence A is a maximal filter.

♦♦♦

An important property of the annihilator A* of the nonempty subset A of L is proved in the following result.

Result 0.2.12 [8] - Let A be a nonempty proper subset of a lattice L with 0. Then A* is the intersection of all minimal prime semi ideals not containing A.

Proof:- To prove

$A^* = \cap \{ M / M$ is a minimal prime semi ideal such that $A \not\subset M\}$.

Let $x \in A^*$ we get $x \wedge a = 0$ for all $a \in A$.

Let M be a minimal prime semi ideal and let $a \in A$ such that $a \notin M$. Since M is a minimal prime semi ideal and $0 \in M$ we get

$x \wedge a \in M$. M being prime we get $x \in M$ as $a \notin M$.

Thus $x \in \cap \{ M / M$ is a minimal prime semi ideal such that $A \not\subset M\}$.

Hence

$A^* \subseteq \cap \{ M / M$ is a minimal prime semi ideal such that $A \not\subset M\}$—(I)

Now let $x \in \cap$ { M / M is a minimal prime semi ideal and $A \not\subset M$}

To prove $x \in A^*$.

Suppose $x \notin A^*$ .Then there exists $y \in A$ such that $x \wedge y \neq 0$.

consider $[x \wedge y)$ .This is a proper filter as $x \wedge y \neq 0$.

By Result 0.2.4 , $[x \wedge y)$ contained in a maximal filter say Q .

But $x \wedge y \in Q$ implies $x \in Q$ and $y \in Q$.

By Result 0.2.11 , L\Q is a minimal prime semi ideal.

Since $y \in A$ and $y \notin L \setminus Q$ we get $A \not\subset L \setminus Q$.

Hence by assumption $x \in L \setminus Q$, a contradiction as $x \in Q$.

Since $x \in A^*$

Therefore

$\cap$ { M / M is a minimal prime semi ideal such that $A \not\subset M$} $\subseteq A^*$—(II)

From (I) and (II) we get

$A^* = \cap$ { M / M is a minimal prime semi ideal such that $A \not\subset M$}.

♦♦♦

A Stone type separation theorem for a lattice L is proved in following result[9].

Result 0.2.13 - Let A and B be two filters of a lattice L with 0 such that A and B° are disjoint. Then there exists a minimal prime semi ideal containing B° and disjoint with A.

Proof: - Let A and B be two filters of a lattice L with 0.

If $0 \in A \vee B$ then $0 \geq a \wedge b$ for $a \in A$ and $b \in B$.

Then $a \in A \cap B^\circ = \phi$;

A contradiction. Hence $A \vee B$ is a proper filter. Then by Result 0.2.4

$A \vee B$ is contained in some maximal filter say M in L.

But then $L \setminus M$ is minimal prime semi ideal by Result 0.2.11.

Take $L \setminus M = P$.

Then $A \vee B \subset M$ implies $B \subseteq M$. Hence $M \cap B^\circ = \phi$.

Now $B^\circ \subseteq P$. As $t \in B^\circ$. Then $t \wedge b = 0$, for some $b \in B$.

Thus $0 \in P$ implies $t \wedge b \in P$. As P is a prime semi ideal, $t \in P$.

Thus $B^\circ \subseteq P$.

Hence we get a minimal prime semi ideal P such that $B \subseteq P$ and disjoint with B°.

♦♦♦

A property of the pesudo-annihlator A° of a non empty subset A is proved in following result[9]

Result 0. 2.14 - Let A be a filter of a lattice L with 0. Then A° is the intersection of all minimal prime semi ideals disjoint from A.

Proof :-

To prove

$A^\circ = \cap \{ M / M$ is a minimal prime semi ideal and $A \cap M = \phi \}$.

Let $x \in A^\circ$. we get $x \wedge a = 0$ for some $a \in A$.

Let M be a minimal prime semi ideal and $A \cap M = \phi$.

As $0 \in M$ we get $x \wedge a \in M$.

As $x \in M$. M is prime, we get either $x \in M$ or $a \in M$.

As $a \in M$ is not possible. It follows that

$x \in \cap \{M / M$ is a minimal prime semi ideal and $A \cap M = \phi \}$.

Hence

$A^\circ \subseteq \cap \{ M / M$ is a minimal prime semi ideal and $A \cap M = \phi \}$----(I)

Now let $x \in \cap \{ M / M$ is a minimal prime semi ideal and $A \cap M = \phi \}$

To prove $x \in A^\circ$.

Suppose $x \notin A^\circ$. Then there exists $y \in A$ such that $x \wedge y \neq 0$.

consider $[x \wedge y)$ is a proper filter .

By Result 0.2.1 , $[x \wedge y)$ contained in a maximal filter say Q .

i.e. $[x \wedge y) \subseteq Q$. By Result 0.2.6 , $L \setminus Q$ is minimal prime semi ideal.

As $A \cap L \setminus Q = \phi$, by assumption $x \in L \setminus Q$.

Thus $x \in Q \cap L \setminus Q = \phi$; a contradiction .

Therefore

$x \in \cap\{ M / M$ is a minimal prime semi ideal and $A \cap M = \phi \}$.

$\{ M / M$ is a minimal prime semi ideal and $A \cap M = \phi) \subseteq A^{\circ}$—(II)

From (I) and (II) we get

$A^{\circ} = \cap\{ M / M$ is a minimal prime semi ideal and $A \cap M = \phi \}$

♦ ♦ ♦