CHAPTER 2

CHARACTERIZATIONS

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This chapter is devoted to obtain various characterizations of 0-distributive lattices.[9]

Theorem 2.1- Let L be a lattice with 0 . Then the following statements are equivalent.

1.L is 0-distributive.

2.If a, $a_1, \ldots, a_n$ are elements of a lattice L such that $a \wedge a_1 = \ldots = a \wedge a_n = 0$ then $a \wedge (a_1 \vee \ldots \vee a_n) = 0$.

3.If A is an ideal and $\{A_i \mid i \in I\}$ is a family of ideals of L such that $A \cap A_i = (0]$ for all i then $A \cap (\vee_{i \in I} A_i) = (0]$.

4.Every maximal filter of L is a prime.

5.If M is a maximal filter of L , L \ M is a minimal prime ideal.

6.If M is a maximal filter of L , L \ M is an ideal.

7.Every minimal prime semi ideal of L is a minimal prime ideal.

8.Every prime semi ideal of L contains a prime ideal.

9.Every proper filter of L is disjoint from a minimal prime ideal.

10. Every proper filter of L is disjoint from a prime ideal.

11. Every proper filter of L is contained in a prime filter.

12.For each non zero element a of L, there is a minimal prime ideal not containing a.

13.For each non zero element a of L, there is a prime ideal not containing a.

14.For each non zero element a of L is contained in a prime filter.

Proof:-

(1) $\Rightarrow$ (2)

Follows by Result 1.3.1 .

(2) $\Rightarrow$ (3)

Suppose (2) holds.

Let $A \in I(L)$ and $\{A_i, i \in I\} \subseteq I(L)$ such that $A \cap A_i = (0]$ for all i .

To prove $A \cap (\vee_{i \in I} A_i) = (0]$.

Let $x \in A \cap (\vee_{i \in I} A_i)$ implies $x \in A$ and $x \in (\vee_{i \in I} A_i)$.

Then $x \in A$ and $x \leq a_{i1} \vee \ldots \vee a_{in}$ where $a_{ij} \in A_i$

we get $x \wedge (a_{i1} \vee \ldots \vee a_{in}) = x$ ....(I)

Now $x \in A$ and $a_{ij} \in A_i$ implies $x \wedge a_{i1} \in A$ and $x \wedge a_{i1} \in A_1$.

Thus by data $x \wedge a_{i1} \in A \cap A_1 = (0]$.

Therefore $x \wedge a_{i1} = 0$, similarly $x \wedge a_{ij} = 0$ for all a_{ij} .

By (2) , $x \wedge (a_{i1} \vee \dots \vee a_{in}) = 0$ ....(II)

From (I) and (II) we get $x = 0$.

Thus $A \cap (\vee_{i \in I} A_i) = (0]$.

(3) ⇒ (1)

To prove that L is 0-distributive.

Let $x \wedge y = 0$ and $x \wedge z = 0$.

hen $(x \wedge y] = (0]$ and $(x \wedge z] = (0]$.

i.e. $(x] \cap (y] = (0]$ and $(x] \cap (z] = (0]$.

By (3) $(x] \cap [(y] \vee (z]] = (0]$.

i.e. $(x \wedge (y \vee z)] = (0]$. i.e. $x \wedge (y \vee z) = 0$.

Hence L is 0-distributive.

(2) ⇒ (4)

Let M be any maximal filter of L .

To prove M is a prime filter.

Let $x , y \notin M$. $x , y \in L \setminus M$.

By Result (0.2.5) , $a \wedge x = 0$ and $b \wedge y = 0$ for some $a , b \in M$.

Suppose $c = a \wedge b$ then $c \wedge x = 0$, $c \wedge y = 0$ and $c \in M$.

By (2) , $c \wedge (x \vee y) = 0$.

If $x \vee y \in M$, then $c \wedge (x \vee y) \in M$.

Thus we get $0 \in M$; a contradiction.

Thus $x \notin M$ and $y \notin M$ imply $x \vee y \notin M$.

Hence M is a prime filter.

(4) $\Rightarrow$ (5)

Let M be a maximal filter of L.

To prove that $L \setminus M$ is a minimal prime ideal.

By Result 0.2.11 , $L \setminus M$ is a minimal prime semi ideal.

To prove $L \setminus M$ is an ideal.

i.e. Prove that if $x, y \in L \setminus M$ then $x \vee y \in L \setminus M$.

By (4) , M is a prime filter.

If $x, y \notin M$, then $x \vee y \notin M$ implies $x \vee y \in L \setminus M$.

Hence $L \setminus M$ is a minimal prime ideal.

(5) $\Rightarrow$ (6)

Obviously true.

(6) $\Rightarrow$ (7)

Let N be any minimal prime semi ideal of L.

To prove N is a minimal prime ideal.

By Result 0.2.11 , $L \setminus N$ is a maximal filter.

By (6) , $L \setminus (L \setminus N)$ is an ideal .

Hence N is a minimal prime ideal.

(7) $\Rightarrow$ (8)

Let A be any prime semi ideal of L.

To prove prime semi ideal A of L contains a prime ideal

By Result 0.2.3 $N \subseteq A$ for some minimal prime semi ideal N .

By (7) , N is a prime ideal.

(8) $\Rightarrow$ (10)

Let A be any proper filter of L .

To prove A is disjoint from a prime ideal.

Then $L \setminus A$ is a prime semi ideal by Result 0.2.2.

By (8) , $B \subseteq (L \setminus A)$ for some prime ideal B.

Therefore $A \cap B = \phi$.

(11) $\Rightarrow$ (14)

Obviously true.

(5) $\Rightarrow$ (9)

Let A be any proper filter of L .

To prove A is disjoint from a minimal prime ideal.

By Result 0.2.1 , $A \subseteq M$ for some maximal filter M .

By (5) , L \ M is a minimal prime ideal.

Therefore $A \cap (L \setminus M) = \phi$.

(9) $\Rightarrow$ (12)

Let a be any non zero element of L .

Then [a) is proper filter.

To prove a minimal prime ideal not containing a.

By (9) , [a) is disjoint from the minimal prime ideal N.

Thus $a \notin N$.

(12) $\Rightarrow$ (13)

Obviously true.

(13) $\Rightarrow$ (14)

Let a be any non zero element of L.

To prove a is contained in a prime filter.

By (3) , there exists a prime ideal A such that $a \notin A$.

Then L \ A is a prime filter by Result (0.2.10) .

Therefore $a \in L \setminus A$.

(14) $\Rightarrow$ (1)

To prove that L is 0-distributive.

Let $a , b , c \in L$ and let $a \wedge b = 0$ and $a \wedge c = 0$.

Suppose $a \wedge (b \vee c) \neq 0$.

By (14), $a \wedge (b \vee c) \in B$ for some prime filter B.

Then $a \in B$ and $b \vee c \in B$. Thus $a \in B$ and $b \in B$ or $c \in B$.

Since B is a prime filter.

Thus ($a \in B$ and $b \in B$) or ($a \in B$ and $c \in B$).

i.e. $a \wedge b \in B$ or $a \wedge c \in B$.

Thus in either case $0 \in B$; a contradiction.

Hence $a \wedge (b \vee c) = 0$.

Therefore L is 0-distributive.

Thus we have proved

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

(2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (14) $\Rightarrow$ (1)

(5) $\Rightarrow$ (9) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14) $\Rightarrow$ (1)

Hence all the statements are equivalent.

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Theorem 2.2- Let L be a lattice with 0 . Then the following statements are equivalent.

L is 0-distributive.

2.If A is a non empty subset and B is a proper filter intersecting A in L , there is a minimal prime ideal containing A* and disjoint from B.

3. If A is a non empty subset of L and B is a proper filter intersecting A , there is a prime filter containing B and disjoint from A*.

4.If A is a nonempty subset of L and B is a prime semi ideal not containing A ,there is a minimal prime ideal containing A* and contained in B.

5. If A is a nonempty subset of L and B is a prime semi ideal not containing A ,there is a prime filter containing L \ B and disjoint from A*.

6. For each non zero element a of L and each proper filter B containing a , there is a prime ideal containing (a)* and disjoint from B.

7. For each non zero element a of L and each proper filter B containing a , there is a prime filter containing B and disjoint from (a)*.

8. For each non zero element a of L and each prime semiideal not containing a ,there is a prime ideal containing (a)* and contained in B.

9. For each non zero element a of L and each prime semiideal B

not containing a ,there is a prime filter containing L \ B and is disjoint from (a)*.

Proof:-

(1) $\Rightarrow$ (2)

Let A be a non empty subset of L and B be any proper filter such that

$A \cap B \neq \phi$.

To prove a minimal prime ideal containing A* and disjoint from B.

By Result 0.2.2 , L \ B is a prime semi ideal.

Then there exists a minimal prime semi ideal N such that $N \subseteq L \setminus B$

by Result 0.2.3.

Thus $N \cap B = \phi$ and $A \not\subset L \setminus B$.

Also as $A \not\subset N$ we get $A^* \subseteq N$ by Result 0.2.12.

Since L is 0-distributive, N is a minimal prime ideal [see Theorem 2.1(7)].

(2) $\Rightarrow$ (3)

Let A be a non empty subset of L and B is a proper filter such that

$A \cap B \neq \phi$.

To prove a prime filter containing B and disjoint from A*.

By (2), there exists a minimal prime ideal say Q such $A^* \subseteq Q$ and

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$Q \cap B = \phi$.

$L \setminus Q$ is a prime filter containing B and disjoint from A* by Result (0.2.10).

(3) ⇒ (5)

Let A be a non empty subset of L and B is a prime semi ideal not containing A.

As $B \not\subset A$ we get $A \cap L \setminus B \neq \phi$.

To prove a prime filter containing $L \setminus B$ and disjoint from A*.

By Result 0.2.2, $L \setminus B$ is a maximal filter.

By (3) ,there exists prime filter Q such that $L \setminus B \subseteq Q$ and

$Q \cap A^* = \phi$.

(5) ⇒ (9)

Let a be a nonzero element of L and B is a prime semi ideal not containing a.

i.e. $a \notin B$.

To prove a prime filter containing $L \setminus B$ and disjoint from (a)*.

By (3) , there exists prime filter F such that $L \setminus B \subseteq F$ and

$F \cap (a)^* = \phi$.

(9) ⇒ (1)

Let a be a non zero element of L .

To prove L is 0-distributive lattice.

Then [a) is proper filter of L.

By Result 0.2.2 , L \ [a) is a prime semi ideal and a $\notin$ L \ [a).

By (9) ,there exists prime filter F such that L \ (L \ [a)) $\subseteq$ F.

i .e. [a) $\subseteq$ F. Thus a $\in$ F.

Hence L is 0-distributive ,by Theorem 2.1 [14].

(8) $\Rightarrow$ (9) and (6) $\Rightarrow$ (7) by Result (0.2.10) .

(3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) and (7) $\Rightarrow$ (9) by Result (0.2.2) .

(4) $\Rightarrow$ (8) and (2) $\Rightarrow$ (6) by Result (0.2.6) .

Thus we have proved

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (9) $\Rightarrow$ (1) .

(1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1) .

(1) $\Rightarrow$ (2) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (9) $\Rightarrow$ (1) .

Hence all the statements are equivalents.

♦♦♦

Theorem 2.3- Let L be a lattice with 0 . Then the following statements are equivalent.

1. L is 0-distributive.

2. If A and B are filters of L such that A and B° are disjoint, there is a minimal prime ideal containing B° and disjoint from A.

3. If A and B are filters of L such that A and B° are disjoint, there is a minimal prime filter containing A and disjoint from B°.

4. If A is a filter of L and B is prime semi ideal containing A°, there is a minimal prime ideal containing A° and contained B.

5. If A is a filter of L and B is prime semi ideal containing A°, there is a prime filter containing L \ B and disjoint from A°.

6. For each nonzero element a in L and each filter A disjoint from (a)*, there is a prime ideal containing (a)* and disjoint from A.

7. For each nonzero element a in L and each filter A disjoint from (a)*, there is a prime filter containing A and disjoint from (a)*.

8. For each nonzero element a in L and each prime semi ideal B containing (a)*, there is a prime ideal containing (a)* and in B.

9. For each nonzero element a in L and each prime semi ideal B containing (a)*, there is a prime filter containing L \ B and disjoint (a)*.

Proof:-

(1) $\Rightarrow$ (2)

Suppose (1) holds.

Let A and B be filters of L such that $A \cap B^{\circ} = \phi$.

To prove a minimal prime ideal containing B° and disjoint from A.

By Result 0.2.13, there is minimal prime semi ideal N such that $B^{\circ} \subseteq N$

and $N \cap A = \phi$.

Since L is 0-distributive it follows that N is a minimal prime ideal.

[See Theorem 2.1 (7)].

(2) ⇒ (3)

It follows by Result 0.2.10.

(3) ⇒ (5)

Suppose (3) holds.

Let A be filter of L and let B be a prime semi ideal containing A°

such that $A^{\circ} \subseteq B$.

To prove a prime filter containing $L \setminus B$ and disjoint from A°.

By Result 0.2.2, $L \setminus B$ is a filter.

By (3), there exists prime filter Q containing $L \setminus B$ and disjoint from A°.

i.e. $L \setminus B \subseteq Q$ and $A^{\circ} \cap Q = \phi$.

(5) ⇒ (9)

Let a be non zero element of L and let B be a prime semi ideal containing $(a)^*$.

Then [a) is a proper filter of L.

To prove a prime filter containing L\ B and disjoint from (a)*.

By Result 0.2.2, L \ B is a filter.

Consider $A^\circ = [a)^\circ$.

As $x \in A^\circ$ then $x \in [a)^\circ$.

We get $x \wedge t = 0$ for some $t \in [a)$.

Then $t \geq a$ and $x \wedge t = 0$ implies $x \wedge a = 0$.

Hence $x \in (a)^*$.i.e. $x \in B$.

Therefore $A^\circ \subseteq B$.

By (5) , there exists a prime filter Q such that $L \setminus B \subseteq Q$ and

disjoint with A°.

Then $Q \cap A^\circ = \phi$ implies $Q \cap [a)^\circ = \phi$.

By result 0.2.6, $Q \cap (a)^* = \phi$

(9) ⇒ (1)

Suppose (9) holds.

We have to prove L is 0-distributive.

Let a be nonzero element of L.

Then [a) is a proper filter of L.

We get L \ [a) is prime semi ideal not containing a by Result 0.2.2.

Since $(a] \cap (a)^* = (0] \subset L \setminus [a)$, it follows $L \setminus [a)$ contains $(a)^*$.

By (9) , there is prime filter Q containing [a) and disjoint from $(a)^*$.

Clearly $a \in Q$. It follows that L is 0-distributive.[see Theorem 2.1(14)]

(2) ⇒ (4)

Suppose (2) holds.

Let A be filter of L and be B is prime semi ideal containing A°

such that $A^\circ \subseteq B$.

To prove a minimal prime ideal containing A° and contained in B.

By Result 0.2.2 , $L \setminus B$ is a filter .

By (2) , there exists a minimal prime ideal Q such that $A^\circ \subseteq Q$ and

$(L \setminus B) \cap Q = \phi$. i.e. $Q \subseteq B$.

(4) ⇒ (8)

Let a be non zero element of L and B is prime semi ideal containing $(a)^*$ such

that $(a)^* \subseteq B$.

since [a) is a proper filter of L.

To prove a prime ideal containing $(a)^*$ and contained in B.

By Result 0.2.6 , $([a))^\circ = (a)^*$. °

By (4), there exists a minimal prime ideal P such that $(a)^* \subseteq P \subseteq B$.

(8) ⇒ (9)

Let a be non zero element of L and B is prime semi ideal containing $(a)^*$ such that $(a)^* \subseteq B$.

To prove a prime filter containing $L\setminus B$ and disjoint from $(a)^*$.

By (8) , there is prime ideal P such that $(a)^* \subseteq P \subseteq B$.

Then $L \setminus P$ is prime filter by Result 0.2.10 such that $L \setminus B \subseteq L \setminus P$ and $L \setminus P \cap (a)^* = \phi$

(2) ⇒ (6)

Let a be non zero element of L and A is a filter disjoint from $(a)^*$

then [a) is a proper filter of L .

By Result 0.2.6 $(a)^* =([a))^\circ$.

By (2), there exists a minimal prime ideal P such that $(a)^\circ \subseteq P$ and $P \cap A = \phi$.

i.e. $(a)^* \subseteq P$ and $P \cap A = \phi$.

(6) ⇒ (7)

Let a be non zero element of L and A is filter of L such that $(a)^* \cap A= \phi$.

To prove a prime filter containing A and disjoint from $(a)^*$.

By (6) , there is prime ideal P such that $(a)^\circ \subseteq P$ and $P \cap A = \phi$.

By Result 0.2.10 , $L \setminus P$ is prime filter such that $(a)^* \cap L\setminus P = \phi$ and $A \subseteq L \setminus P$.

(7) $\Rightarrow$ (9)

Let a be non zero element of L and B is prime semi ideal of L

such that $(a)^* \subseteq B$.

To prove a prime filter containing $L \setminus B$ and disjoint from $(a)^*$.

We get $L \setminus B$ is filter in L by Result 0.2.2 .

Such that $(a)^* \cap L \setminus B = \phi$

By (7), there is prime filter Q such that $L \setminus B \subseteq Q$ and $(a)^* \cap Q = \phi$.

Thus we have proved

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (9) $\Rightarrow$ (1) .

(1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1) .

(1) $\Rightarrow$ (2) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (9) $\Rightarrow$ (1) .

Hence all the statements are equivalents.

♦♦♦

Theorem 2.4- Let L be a lattice with 0 . Then the following statements are equivalent.

1. L is 0-distributive.

2. For any non empty subset A of L , A^* is the intersection of all the minimal prime ideal not containing A.

3. For any filter A of L , A° is a the intersection of all minimal prime ideals disjoint from A.

4. For each a of L , (a)* is an ideal .

5. Every normal semi ideal of L is an intersection of minimal prime ideals.

6. For any ideal A of L and any family of ideals $\{ A_i \ / i \in I \}$ of L ,

$(A \cap [\vee_{i\in I} A_i])^* = \cap_{i\in I} (A \cap A_i)^*$.

7. For any three ideals A , B ,C of $\in$L

$(A \cap [B \vee C])^* = (A \cap B)^* \cap (A \cap C)^*$.

8 . For any finite numbers of filters A, $A_1 \ldots A_n$ of L

$(A \vee (A_1 \cap \ldots \cap A_n))^\circ = (A \vee A_1)^\circ \cap \ldots \cap (A \vee A_n)^\circ$.

9. For any filters A , B ,C of L

$(A \vee (B \cap C))^\circ = (A \vee B)^\circ \cap (A \vee C)^\circ$.

10. For all a , b, c in L ,

$(a \wedge (b \vee c))^* = (a \wedge b)^* \cap (a \wedge c)^*$.

Proof:-

(1) $\Rightarrow$ (2)

Follows by Result 0.2.12 and theorem 2.1 (7)

(1) $\Rightarrow$ (3)

Follows by Result 0.2.14 and theorem 2.1 (7).

(3) ⇒ (4)

Follows by Result 0.2.6 and $(a)^* = [a)^\circ$. Hence result.

(4) ⇒ (1)

We have to prove L is 0-distributive.

Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$.

Then $b, c \in (a)^*$.

By (4), $b \vee c \in (a)^*$.

Thus we get $a \wedge (b \vee c) = 0$.

Hence L is 0-distributive.

(3) ⇒ (5)

Since a semi ideal A of L is normal if and only if $A = B^*$ for somesemi ideal B.

(5) ⇒ (4)

By Result 0.2.6, $(a)^* = (a]^*$ for all $a \in L$. Hence the result.

(2) ⇒ (6)

Suppose (2) holds.

To prove $(A \cap [\vee A_i])^* = \cap_{i \in I} (A \cap A_i)^*$.

Let $A \in I(L)$ and $\{ A_i \ / \ i \in I \} \subseteq I(L)$.

If Q is any minimal prime ideal of L such that $A \cap [\vee A_i] \not\subset Q$, then $(A \cap A_j) \not\subset Q$ for some $j \in I$.

By (2), it follows that

$(A \cap [\vee A_i])^* \supseteq \cap_i (A \cap A_i)^*$.

The reverse inclusion being obvious.

Hence $(A \cap [\vee A_i])^* = \cap_i (A \cap A_i)^*$.

(6) ⇒ (7)

Obviously true.

(7) ⇒ (10)

Follows by Result 0.2.6.

(10) ⇒ (1)

We have to prove L is 0-distributive.

Suppose (10) holds.

Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$.

Then $(a \wedge b)^* = L = (a \wedge c)^*$.

By (10), $(a \wedge (b \vee c))^* = L$.

It follows that $a \wedge (b \vee c) = 0$.

Thus L is 0-distributive.

(3) ⇒ (8)

Suppose (3) holds and let $A, A_1 \ldots A_n$ be the filters of L.

To prove $(A \vee (A_1 \cap \ldots \cap A_n))^\circ = (A \vee A_1)^\circ \cap \ldots \cap (A \vee A_n)^\circ$.

If Q is any minimal prime ideal of L such that

$Q \cap (A \vee (A_1 \cap \ldots \cap A_n)) = \phi$, then $Q \cap A = \phi$ and $Q \cap (A_1 \cap \cdots \cap A_m) = \phi$

So $Q \cap (A \vee A_i) = \phi$ for some $i \in I$.

By (3), it follows that

$(A \vee (A_1 \cap \ldots \cap A_n))^\circ \supseteq (A \vee A_1)^\circ \cap \ldots \cap (A \vee A_n)^\circ$.

The reverse inclusion being obvious.

(8) ⇒ (9)

Obviously true.

(9) ⇒ (10)

Suppose (9) holds and let $a, b, c, \in L$.

To prove $(a \wedge (b \vee c))^* = (a \wedge b)^* \cap (a \wedge c)^*$.

Then $([a) \vee ([b) \cap [c)))^\circ = ([a) \vee [b))^\circ \cap ([a) \vee ([c))^\circ$.

Now $([a) \vee ([b) \cap [c)))^\circ = ([a) \vee [b \vee c))^\circ = ([a \wedge (b \vee c)))^\circ =$

$(a \wedge (b \vee c))^*$ by result (0.2.6).

And $([a) \vee [b))^\circ \cap ([a) \vee ([c))^\circ = ([a \wedge b))^\circ \cap ([a \wedge c))^\circ =$

$([a \wedge b))^* \cap ([a \wedge c))^*$ by Result 0.2.6.

Thus $(a \wedge (b \vee c))^* = (a \wedge b)^* \cap (a \wedge c)^*$.

Thus we have proved

(1)$\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (1) .

(1)$\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) .

(1) $\Rightarrow$ (2) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (10)$\Rightarrow$ (1) .

(1)$\Rightarrow$ (3) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (1) .

Hence all the statements are equivalents.

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Theorem 2.5 - Let L be a lattice with 0 . Then the following statements are equivalent.

1.L is 0-distributive.

2. For any finite numbers of ideals A, $A_1, \ldots A_n$ of L

$$((A \vee A_1) \cap \ldots \cap (A \vee A_n))^* = A^* \cap (A_1 \cap \ldots \cap A_n)^*.$$

3. For any three ideals A , B , C of L

$$((A \vee B) \cap (A \vee C))^* = A^* \cap (B \cap C)^*.$$

4.For any filter A of L and any family of filters $\{A_i \ / i \in I\}$ of L ,

$$[\vee_{i \in I}(A \cap A_i)]^\circ = A^\circ \cap [\vee_{i \in I} A_i]^\circ.$$

5.For any three filters A , B , C of L

$[(A \cap B) \vee (A \cap C)]^{\circ} = A^{\circ} \cap [B \vee C]^{\circ}$.

6.For all a ,b ,c in L

$((a \vee b) \wedge (a \vee c))^{*} = (a)^{*} \cap (b \wedge c)^{*}$

7. For any family of ideals $\{A_i \,/\, i \in I\}$ of L ,

$[\vee_{i \in I} A_i]^{*} = \cap_{i \in I} A_i^{*}$.

8. For any two ideals A , B of L ,

$[A \vee B]^{*} = A^{*} \cap B^{*}$.

9. For any finite numbers of filters $A_1, \ldots A_n$ of L

$(A_1 \cap \ldots \cap A_n)^{\circ} = A_1^{\circ} \cap \ldots \cap A_n^{\circ}$.

10. For any two of filters A, B of L

$(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$.

11. For all a ,b in L

$(a \vee b)^{*} = (a)^{*} \cap (b)^{*}$.

12 . I(L) is pseudocomplemented.

13 . I(L) is 0-distributive.

proof:-

(1) $\Rightarrow$ (2)

Suppose (1) holds and $A, A_1 \ldots, A_n$ be ideals of L .

To prove $((A \vee A_1) \cap \ldots \cap (A \vee A_n))^* = A^* \cap (A_1 \cap \ldots \cap A_n)^*$.

If Q is any minimal prime ideal of L such that $A_1 \cap \ldots \cap A_n \not\subset Q$ be ideals of L.

It follows that $((A \vee A_1) \cap \ldots \cap (A \vee A_n))^* \supseteq A^* \cap (A_1 \cap \ldots \cap A_n)^*$.

The reverse inclusion is obvious.

(2) ⇒ (3)

Obviously true.

(3) ⇒ (6)

Suppose (3) holds and let $a, b, c \in L$.

To prove $((a \vee b) \wedge (a \vee c))^* = (a)^* \cap ((b \wedge c)^*$.

Then $(((a] \vee (b]) \cap ((a] \vee (c]))^* = (a]^* \cap ((b] \cap (c]))^*$.

Now $(((a] \vee (b]) \cap ((a] \vee (c]))^* = ((a \vee b] \cap ((a \vee c])^* =$

$(((a \vee b) \wedge ((a \vee c)])^* = ((a \vee b) \wedge (a \vee c))^*$.

And $(a]^* \cap ((b] \cap (c]))^* = (a)^* \cap ((b \wedge c])^* = (a)^* \cap ((b \wedge c)^*$ by Result 0.2.6.

Thus $((a \vee b) \wedge (a \vee c))^* = (a)^* \cap ((b \wedge c)^*$.

(6) ⇒ (11)

By (6), we have $((a \vee b) \wedge (a \vee c))^* = (a)^* \cap ((b \wedge c)^*$.

By taking c = b, we get $(a \vee b)^* = (a)^* \cap (b)^*$.

(11) ⇒ (1)

Suppose (11) holds.

To prove L is 0-distributive.

Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$.

Then $a \in (b)^*$ and $a \in (c)^*$ implies $a \in (b)^* \cap (c)^* = (b \vee c)^*$

by Result 0.2.6.

It follows that $a \wedge (b \vee c) = 0$.

Hence L is 0-distributive.

(1) ⇒ (4)

Suppose (1) holds.

To prove $(\vee_{i \in I}(A \cap A_i))^\circ = A^\circ \cap (\vee_{i \in I} A_i)^\circ$.

Then for any filter A of L, A° is the intersection of all minimal prime ideals

disjoint from A [see theorem 2.4 (3)].

Let $A \in F(L)$ and $\{A_i / i \in I\} \subseteq F(L)$.

If Q is any minimal prime ideal of L such that

$Q \cap [\vee_{i \in I} (A \cap A_i)] = \phi$ then $Q \cap A = \phi$ or $Q \cap [\vee_{i \in I} A_i)] = \phi$

It follows that

$(\vee_{i \in I}(A \cap A_i))^{\circ} \subseteq A^{\circ} \cap (\vee_{i \in I} A_i)^{\circ}$.

The reverse inclusion is obviously.

Hence $(\vee_{i \in I}(A \cap A_i))^{\circ} = A^{\circ} \cap (\vee_{i \in I} A_i)^{\circ}$.

(3) $\Rightarrow$ (5)

Obviously true.

(5) $\Rightarrow$ (6) similar to (3) $\Rightarrow$ (6) with obvious modification.

(1) $\Rightarrow$ (7)

Suppose (1) holds.

To prove $[\vee_{i \in I} A_i]^* = \cap_{i \in I} A_i^*$.

Then for any ideal A and for any family of ideals $\{ A_i \,/\, i \in I \}$ of L ,

$(A \cap [\vee_{i \in I} A_i])^* = \cap_{i \in I} (A \wedge A_i)^*$. [see Theorem 2.4 (6)].

Taking $A = \vee_{i \in I} A_i$ we get

$[\vee_{i \in I} A_i]^* = \cap_{i \in I} A_i^*$.

(7) $\Rightarrow$ (8)

Obviously true.

(8) $\Rightarrow$ (11)

Suppose (8) holds and let a , b $\in$ L .

To prove $(a \vee b)^* = (a)^* \cap (b)^*$.

Then $((a] \vee (b])^* = (a]^* \cap (b]^*$.

Now $((a] \vee (b])^* = ((a \vee b])^* = (a \vee b)^*$.

And $(a]^* \cap (b]^* = (a)^* \cap (b)^*$ by Result 0.2.6.

Thus $(a \vee b)^* = (a)^* \cap (b)^*$.

(1) $\Rightarrow$ (9)

Suppose (1) holds .

To prove $(A_1 \cap \ldots \cap A_n)^\circ = A_1{}^\circ \cap \ldots \cap A_n{}^\circ$.

Then for any finite numbers of filters $A_1, \ldots A_n$ of L,

$(A \vee (A_1 \cap \ldots \cap A_n))^\circ = (A \vee A_1)^\circ \cap \ldots \cap (A \vee A_n)^\circ$.

[see theorem 2.4 (8)]

Taking $A = (A_1 \cap \ldots \cap A_n)$ we get

$(A_1 \cap \ldots \cap A_n)^\circ = A_1{}^\circ \cap \ldots \cap A_n{}^\circ$.

(9) $\Rightarrow$ (10)

Obviously true.

(10) $\Rightarrow$ (11)

suppose (10) holds and let a , b $\in$ L .

To prove $(a \vee b)^* = (a)^* \cap (b)^*$.

Then $([a) \vee [b))^\circ = [a)^\circ \cap [b)^\circ$.

Now $([a) \vee [b))^\circ = ([a \vee b))^\circ = (a \vee b)^*$

and $[a)^\circ \cap [b)^\circ = (a)^* \cap (b)^*$ by Result 0.2.6.

Thus $(a \vee b)^* = (a)^* \cap (b)^*$.

(1) ⇒ (12).

Suppose (10) holds.

To prove I(L) is pesudocomplemented.

Let $A \in I(L)$. Then A^* is an ideal [see Theorem 2.4 (2)].

If $B \in I(L)$ such that $A \cap B = (0]$ and $x \in B$, then $a \wedge x = 0$ for all

$a \in A$ and so $x \in A^*$.

Thus $B \subseteq A^*$.

It follows that A^* is pseudocomplement of A.

(12) ⇒ (13)

Follows by Result 1.3.2.

(13) ⇒ (1)

Suppose (13) holds.

To prove L is 0-distributive.

Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$.

Then $(a] \cap (b] = (0] = (a] \cap (c]$.

By (13) $(a] \cap ((b] \vee (c]) = (0]$.

i.e. $(a \wedge (b \vee c]) = (0]$

It follows that $a \wedge (b \vee c) = 0$.

Hence L is 0-distributive.

Thus we have proved

(1) ⇒ (2) ⇒ (3) ⇒ (6) ⇒ (11) ⇒ (1).

(1) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (11) ⇒ (1).

(1) ⇒ (7) ⇒ (8) ⇒ (11) ⇒ (1) .

(1) ⇒ (9) ⇒ (10) ⇒ (11) ⇒ (1) .

(1) ⇒ (12) ⇒ (13) ⇒ (1) .

Hence all the statements are equivalents.

♦♦♦

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