CHAPTER - II.

CONSERVATION LAWS AND ISOMETRIES.

"Energy-momontum conservation has been a cornerstone of physics for more than a century. Nowhere does its essence shine forth so clearly as in [redacted] Einstein's geometric formulation of it" -

MISNER, THORNE, WHEELER (1973)

The local conservation laws of energy and momentum for gravitating systems as invisaged by well-known contracted Biachi identities are given through the divergence of stress-energy tensor in the form,

$$T^{ab}_{;b} = 0. \qquad \ldots (1.1)$$

1. Equation of continuity :

For the F-magnetofluid described by (I, 2.14) This provides,

$$\left[(\varrho + p + 2m)\ U^a U^b\right]_{;b} - \left[(p + 2m - m\mu)\ g^{ab}\right]_{;b} - - \left[\mu\ H^a H^b\right]_{;b} = 0\ . \qquad \ldots (1.2)$$

The equation (1.2) when contracted with U, we get,

$$(\varrho + m\mu)_{;b}\ U^b + (\varrho + p + 2m)\ U^b_{;b} + \mu U_{a;b} H^a H^b = 0, \quad \ldots (1.3)$$

i.e., $(\varrho + m\mu)_{;b}\ U^b + (\varrho + p + 2m)\ \theta + \mu\ (U_{a;b} H^a H^b) = 0, \ \ldots$

i.e., $(\varrho + m\mu)_{;b} U^b + (\varrho + p)\ \theta + \mu (H^2 \theta + U_{a;b} H^a H^b) = 0. \ \ldots (1.4)$

By using two useful deductions of Maxwell equations (I, 6.6) and (I, 6.7) in the equation (1.4) we get,

$$\dot{\rho} + (\rho + p)\,\theta + \left(\frac{1}{2}\mu^2 H^2\right)^{\cdot} - \frac{\mu}{2}(H^2)^{\cdot} - H^2\dot{\mu} = 0,$$

i.e.,

$$\dot{\rho} + (\rho + p)\,\theta + \frac{\mu}{2}\dot{H}^2(\mu - 1) + \dot{\mu}H^2(\mu - 1) = 0,$$

i.e.,

$$\dot{\rho} + (\rho + p)\,\theta + (\mu-1)\left(\frac{\mu}{2}\dot{H}^2 + \dot{\mu}H^2\right) = 0. \qquad \ldots (1.5)$$

This is the equation of continuity for the F-magnetofluid space-time.

Remark : We observe from equation (1.5) that when the magnetic permeability is constant and equal to unity then the continuity equation does not preserve any term showing magnetic field effect in explicit form. The resulting equation of continuity is then of the form,

$$\dot{\rho} + (\rho + p)\,\theta = 0\,. \qquad \ldots (1.6)$$

This is the same equation as the equation of continuity for the relativistic perfect fluid.

2. Equations of stream lines :

We recall the equation (1.2)

$$(\rho + p + 2m)_{;b}\, U^a U^b + (\rho + p + 2m)\, U^a U^b_{\ ;b} +$$

$$+ (\rho + p + 2m)\,\dot{U}^a - (p + 2m - m\mu)_{;b}\, g^{ab} - \mu_{;b} H^a H^b -$$

$$- \mu H^a_{\ ;b} H^b - \mu H^a H^b_{\ ;b} = 0. \qquad \ldots (2.1)$$

By regrouping the terms in this equation and employing the continuity equation

$$(\rho + p + 2m)_{;b}\, U^b + (\rho + p + 2m) U^b_{;b} =$$

$$= (p + 2m - m\mu)_{;b}\, U^b + \mu H^a_{;b}\, U_a H^b , \qquad \ldots (2.2)$$

We obtain the result as given by,

$$(\rho + p + 2m)\, \dot{U}^a + (p + 2m - m\mu)_{;b}\, (U^a U^b - g^{ab}) -$$

$$- (\mu H^b)_{;b}\, H^a = 0. \qquad \ldots (2.3)$$

This with the use of equation (I, 2.3) provides

$$(\rho + p + 2m)\dot{U}^a + (p + 2m - m\mu)_{;b}(-h^{ab}) - (\mu H^b)_{;b} H^a = 0,$$

i.e., $(\rho + p + 2m)\dot{U}^a - (p + 2m - m\mu)_{;b} h^{ab} - (\mu H^b)_{;b} H^a = 0.$ ...(2.4)

These are the equations termed as the stream line equations, which exhibit the deviation of the fluid particle paths from the geodesic path.

By taking into account equations (1.2) and (I, 2.4), (I, 2.5) we deduce from $T^{ab}_{;b} H_a = 0$,

$$(\rho + p + 2m)\dot{H}^a U_a + (p - m\mu)_{;b} H^b - 2mH^b_{;b} + \frac{1}{2}\, \mu H^2_{;b} H^b = 0,$$

i.e. $(\varrho + p + 2m)\, \dot{H}^a U_a + (p - m\mu)_{;b} H^b - 2mH^b_{;b} + \frac{1}{2}\mu H^2_{;b} H^b = 0$ ... (2.5)

$$(\varrho + p)\dot{U}_a H^a + 2m\, \dot{U}_a H^a + 2mH^b_{;b} - (p - m\mu)_{;b} H^b -$$

$$- m_{;b} H^b + \frac{1}{2}\, \mu_{;b} H^2 H^b = 0. \qquad \text{... (2.6)}$$

This with the usage of Maxwell equation (I, 6.6) gives rise to,

$$(\varrho + p)\dot{U}^a H_a + \frac{1}{2}\, \mu_{;b} H^b H^2 - (p - m\mu + m)_{;b} H^b = 0. \qquad \text{... (2.7)}$$

3. <u>Equation of heat transfer</u> :

We start with the equation of continuity for F-magnetofluid in the form,

$$\dot{\varrho} + (\dot{\varrho} + p)\,\theta = (1 - \mu)\left(\frac{\mu \dot{H}^2}{2} + \dot{\mu} H^2\right). \qquad \text{... (3.1)}$$

We recall the interretations connecting the thermodynamical variables for polarised and magnetised fluid due to Maugin (1972) as

$$\varrho = \varrho_o (1 + \epsilon), \qquad \text{... (3.2)}$$

$$T_o dS = d\epsilon + pd(1/\varrho_o) - \frac{1}{2\varrho_o}\mu(1-\mu)\, dH^2, \qquad \text{... (3.3)}$$

$$\psi = 1 + \epsilon + p/\varrho_o\,. \qquad \text{... (3.4)}$$

where ψ is the generalised density. So we write from equation (3.2),

$$\dot{\rho} = \left[\rho_o (1 + \in)\right]_{;a} U^a, \qquad \ldots (3.5)$$

and also,

$$(\rho + p)\, \theta = \rho_o (1 + \in + p/\rho_o)\, U^a_{;a}. \qquad \ldots (3.6)$$

The addition of equations (8.5) and (3.6) yields

$$\dot{\rho} + (\rho + p)\theta = (\rho_o U^a)_{;a} \psi + \rho_o \in_{;a} U^a -$$

$$[illegible] - \rho_{o;a} (p / \rho_o)\, U^a. \qquad \ldots (3.7)$$

Further it follows from relation (3.3) that,

$$\rho_o T_o dS = \rho_o d\in - p/\rho_o \, d\rho_o - \tfrac{1}{2}\mu(1-\mu)\, dH^2. \qquad \ldots (3.8)$$

This further gives,

$$U^a \rho_o T_o \, dS = \rho_o U^a d\in - (p / \rho_o) U^a d\rho_o -$$

$$- \frac{1}{2} \mu(1 - \mu)\, U^a dH^2, \qquad \ldots (3.9)$$

i.e. $$\rho_o T_o S_{;a} U^a = \rho_o \in_{;a} U^a - (p/\rho_o)\, \rho_{o;a} U^a -$$

$$- \frac{1}{2} \mu(1-\mu) H^2_{;a} U^a. \qquad \ldots (3.10)$$

We combine this equation with equation (3.7) to produce,

$$\dot{\varrho} + (\varrho + p)\,\theta = \psi(\varrho_o U^a)_{;a} + \varrho_o T_o S_{;a} U^a + \frac{1}{2}\mu(1-\mu) H^2_{;a} U^a. \qquad \ldots (3.11)$$

We employ this equation (3.11) in continuity equation (3.1) to derive,

$$\psi(\varrho_o U^a)_{;a} + \varrho_o T_o S_{;a} U^a + \frac{1}{2}\mu(1-\mu) H^2_{;a} U^a = \left(\frac{\mu \dot{H}^2}{2} + \dot{\mu} H^2\right)(1-\mu). \qquad \ldots (3.12)$$

The law of conservation of baryon numbers which describes the equation of continuity for the rest mass (Misner and Sharp, 1964) is described by,

$$(\varrho_o U^a)_{;a} = 0. \qquad \ldots (3.13)$$

So that the equation (3.12) now can finally be put in the form,

$$\varrho_o T_o S_{;a} U^a + \frac{1}{2}\mu(1-\mu) H^2 = (1-\mu)\left(\frac{\mu \dot{H}^2}{2} + \dot{\mu} H^2\right),$$

i.e., $\varrho_o T_o S_{;a} U^a = (1-\mu)\,\dot{\mu} H^2. \qquad \ldots (3.14)$

Hence we conclude that in the space time of F-magnetofluid following the law of conservation of baryon numbers, the flow is adiabatic if and only if the magnetic permeability is constant.

<u>Remark</u> : The flow is adiabatic also even if μ is unity.

4. <u>Raychaudhari's equation</u> :

We write by definition (I, 5.1) of expansion parameter, the result

$$\frac{d\theta}{d\tau} = U^b (U^a{}_{;a})_{;b} = U^b U^a{}_{;ab} \; . \qquad \ldots (4.1)$$

The famous Ricci identities are

$$U^a_{;ba} - U^a_{;ab} = R^a{}_{bcd} \; U^d. \qquad \ldots (4.2)$$

So that we write from equation (4.1),

$$U^b (U^a_{;ab}) = \frac{d\theta}{d\tau} = U^b U^a{}_{;ba} - R^a{}_{bcd} \; U^d. \qquad \ldots (4.3)$$

Further we can write,

$$U^a_{;ba} \; U^b = (U^a_{;b} U^b)_{;a} - U^a{}_{;b} U^b{}_{;a} \; . \qquad \ldots (4.4)$$

This can be rewritten in the form,

$$U^a_{;ba}\, U^b = (U^a_{\ ;b} U^b)_{;a} - (\ \omega^a_b + \sigma^{\ a}_b + \frac{1}{3}\,\theta\, h^a_b -$$

$$- \dot{U}^a U_b)\ \times\ (\ \omega^{\ b}_a + \sigma^{\ b}_a + \frac{1}{3}\,\theta\, h^b_a - \dot{U}^b U_a\ ).\ \quad \ldots(4.5)$$

By using the symmetry properties of ω_{ab}, σ_{ab} and h_{ab} and also keeping in mind that they are orthogonal to U , we obtain from the equations (4.4) and (4.5),

$$\frac{d\theta}{d\tau} = \dot{\theta} = R_{ab}\, U^a U^b - \frac{1}{3}\,\theta^2 + 2(\ \sigma^{\ 2} - \ \omega^2) - \dot{U}^a_{;a}\ .\ \quad \ldots(4.6)$$

On substituting the value of the Ricci tensor R_{ab} from equation (I,6.4), we finally write the equation (4.6) as,

$$\dot{\theta} = -\frac{K}{2}\left[\rho + 3p + 4m - 2m\mu\right] - \frac{1}{3}\,\theta^2 + 2(\ \sigma^{\ 2} - \omega^2) -$$

$$- \dot{U}^a_{;a}\ . \qquad \ldots\ (4.7)$$

This is known as the Raychaudhari's equation in the space-time of F-magnetofluid.

5. Isometry with respect to the flow vector U :

The unitary 4-velocity is said to define an isometric motion along U (Trautman, 1964), when

$$\mathcal{L}(U)\ g_{ab} = 0. \qquad \ldots\ (5.1)$$

Equivalently we write

$$U_{a;b} + U_{b;a} = 0. \qquad \dots (5.2)$$

These are known as killing equations in literature. On contracting equation (5.2) by g^{ab} we get,

$$U^{a}_{;a} = 0,$$

$$\text{i.e.,}\ \theta = 0. \qquad \dots (5.3)$$

<u>This shows that isometric flow lines are expansion free.</u>

Further when, we contract equation (5.2) by U^{a}, then we get,

$$U_{a;b}\, U^{b} = 0, \qquad \dots (5.\ )$$

$$\text{i.e.,}\quad \dot{U}_{a} = 0. \qquad \dots (5.4)$$

<u>This states that the flow is geodesic</u>.

Also by contracting equation (5.2) with $H^{a}H^{b}$ gives rise to,

$$U_{a;b}H^{a}H^{b} = 0. \qquad \dots (5.5)$$

By recalling the gradient expression

$$U_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\,\theta h_{ab} + \dot{U}_{a}U_{b}\ , \qquad \dots (5.6)$$

and taking into account that the shear tensor is symmetric and rotation tensor is anti-symmetric we derive from (5.2)

$$\sigma_{ab} = 0. \qquad \dots (5.7)$$

<u>This shows that the flow is shearfree.</u>

Finally we conclude that the isometric flow generates the severior restrictions on it as described via kinematical parameters,

$$\theta = 0, \quad \dot{U}_b = 0, \quad U_{a;b}H^aH^b = 0, \quad \sigma_{ab} = 0. \qquad \ldots (5.8)$$

The consequences of Maxwell equations (I, 6.6) and (I,6.7) under the conditions of isometric flow (5.8) yields the result,

$$(\mu\, H^a)_{;a} = 0, \qquad \ldots (5.9)$$

i.e., $B^a_{\ ;a} = 0$ and ... (5.9a)

$$\frac{\mu}{2}\,(H^2)^{\cdot} + H^2\,\dot{\mu} = 0. \qquad \ldots (5.9b)$$

Hence we infer that,

(i) The magnetic induction is divergence free (vide 5.9)

(ii) The sum of the magnitude of magnetic field and the magnetic permeability is invariant along the isometric flow (vide 5.9b).

Further the equations (5.9) and (5.9a) demand that the magnetic permeability solely depends on the magnitude of magnetic field.

We find from the equation

$$(T^{ab}\, U_a)_{;b} = 0, \qquad \ldots (5.10)$$

$$\left[(\,\rho + p + 2m - p - 2m + m\mu)\; U^b\,\right]_{;b} = 0,$$

i.e., $$(\,\rho + m\mu)_{;b}\, U^b + (\,\rho + m\mu)\; U^b_{;b} = 0. \qquad \ldots (5.11)$$

The equation (5.3) reduces equation (5.11) in the form,

$$(\,\rho + m\mu)_{;b}\, U^b = 0. \qquad \ldots (5.12)$$

We have the continuity equation as,

$$\dot{\rho} + (\,\rho + p)\;\theta = (\,1 - \mu)\;\left(\frac{\mu \dot{H}^2}{2} + \dot{\mu} H^2\right). \qquad \ldots (5.13)$$

This under isometry conditions (5.8) and equation (5.9b) implies,

$$\dot{\rho} = 0. \qquad \ldots (5.14)$$

By the use of equation (5.13) in (5.12) and equation (3.10b), we conclude

$$\dot{\mu} = 0 = \dot{H}^2\,. \qquad \ldots (5.15)$$

We infer from this that the matter density ρ and the magnetic permeability μ are constant along the isometric flow.

We recall the result $T^{ab}_{;b}\, H_a = 0$ given by (2.7) in the form,

$$(\varrho + p)\dot{U}^a H_a = (p + m - m\mu)_{;b} H^b - \frac{1}{2}\,\mu_{;b} H^b H^2. \qquad \dots (5.16)$$

This under the isometry conditions (5.8) reduces to,

$$(p + m - m\mu)_{;b}\, H^b - \frac{1}{2}\,\mu_{;b}\, H^b H^2 = 0. \qquad \dots (5.17)$$

Also we employ the isometry conditions (5.8) and (5.9) in stream line equations (2.4) and write,

$$p_{;b}\, h^{ab} = (m\mu - 2m)_{;b}\, h^{ab} - \mu_{;b}\, H^a H^b\ . \qquad \dots (5.18)$$

This reveals that the magnetic field effects on the spatial gradient of the isotropic pressure. From the equation (5.17), we conclude that the space-time of the relativistic F-magnetofluid following isometry condition demands that the magnetic permeability is constant along the magnetic lines if and only if,

$$p_{;b}\, H^b = \left[\, m(\mu - 1)\,\right]_{;b}\, H^b. \qquad \dots (5.19)$$

6. <u>Isometry with respect to the magnetic field vector</u> :

The magnetic field vector H^a is said to obey isometric motion if and only if

$$\mathcal{L}(H) g_{ab} = 0. \qquad \dots (6.1)$$

This provides

$$H_{a;b} + H_{b;a} = 0. \qquad \ldots (6.2)$$

The contraction of (6.2) with g^{ab}, U^aU^b, H^aH^b, U^aH^b respectively produce successively the results

$$H^a_{;a} = 0, \qquad \ldots (6.3)$$

$$\dot{U}^a H_a = 0, \qquad \ldots (6.4)$$

$$H^2_{;a} H^a = 0, \qquad \ldots (6.5)$$

$$\theta = 0. \qquad \ldots (6.6)$$

<u>We conclude from these results that the isometry with respect to the magnetic field implies,</u>

(i) The magnetic lines are divergence free.

(ii) The 4-acceleration is orthogonal to magnetic field vector.

(iii) The magnitude of the magnetic field vector is preserved along itself.

(iv) The stream lines are expansion free.

We recall back the consequences of Maxwell equations

$$\mu(\dot{U}_a H^a + H^a_{;a}) = -\mu_{;a} H^a \text{ and} \qquad \ldots (6.7)$$

$$\mu \left\{H^2 \theta + \frac{1}{2} \dot{H}^2 + U_{a;b} H^a H^b\right\} + H^2 \dot{\mu} = 0. \qquad \ldots (6.8)$$

If we utilise the isometry conditions with respect to H (6.3) and (6.4) in equation (6.7) then we obtain

$$\mu_{;a} H^a = 0. \qquad \ldots (6.9)$$

From this we conclude that magnetic permeability is invariant along the magnetic lines which follow isometry.

Furthermore from equation (6.8) we have

$$(\mu H^2)^{\cdot} + 2\mu \, \sigma_{ab} H^a H^b = 0. \qquad \ldots (6.10)$$

By employing isometry conditions in the continuity equation (1.3) for F-magnetofluid we find

$$(\rho + m\mu)_{;b} U^b = 0. \qquad \ldots (6.11)$$

Hence we comment that for the F-magnetofluid satisfying isometry conditions with respect to H, the total internal energy density ($\rho + m\mu$) is invariant along the flow.
We also derive from $T^{ab}_{;b} H_a = 0$, with the help of isometry condition (6.4) the result,

$$P_{;b} H^b = 0. \qquad \ldots (6.12)$$

This establishes that the isotropic pressure of F-magnetofluid following the isometry conditions with respect to H is left invariant along the magnetic field vector if and only if the magnetic permeability is maintained constant along the magnetic lines.

Remark : The pressure is constant along the magnetic lines.

The isometry condition with respect to H reduces stream line equation (2.4) in the form

$$(\rho + p + 2m)\ \dot{U}^a - (p + 2m - m\mu)_{;b}\ h^{ab} - \mu_{;b} H^a H^b = 0. \qquad \ldots (6.13)$$

The famous Raychaudhari's equation (4.7) under the isometry condition (6.6) becomes

$$\dot{U}^a_{;a} = 2(\sigma^2 - \omega^2)\ \frac{k}{2}\ (\rho + 3p + 4m - 2m\mu). \qquad \ldots (6.14)$$

The above equation (6.14) can be written as

$$2\ \omega^2 = 2\ \sigma^2 - \frac{k}{2}\ (\rho + 3p + 4m - 2m\mu) - \dot{U}^a_{;a}\ ,$$

$$\text{i.e., } \omega^2 = \sigma^2 - \frac{k}{4}\ (\rho + 3p + 4m - 2m\mu) - \frac{1}{2}\ \dot{U}^a_{;a}\ ,$$

$$\text{i.e., } \omega^2_{;b}\ H^b = \sigma^2_{;b} H^b - \frac{k}{4}\ (\rho + 3p + 4m - 2m\mu)_{;b}\ H^b - \frac{1}{2}\ \dot{U}^a_{;ab}\ H^b\ . \qquad \ldots (6.15)$$

Equation (6.15) under the effect of equations (6.5) and (6.9) becomes

$$\omega^2_{;b} H^b = \sigma^2_{;b}\ H^b - \frac{k}{4}\ (\rho + 3p)_{;b} H^b - \frac{1}{2}\ \dot{U}^a_{;ab} H^b\ . \qquad \ldots (6.16)$$

If the flow is geodesic and shear free then this gives

$$\omega^2_{;b}\ H^b = - \frac{k}{4}\ (\rho + 3p)_{;b}\ H^b. \qquad \ldots (6.17)$$

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