CHAPTER - III.

FARARO'S LAW OF ISOROTATION

"If space-time is considered from the point of view of its conformal structure only, points at infinity can be treated on the same basis as finite points"

- PENROSE (1964).

<u>Ferraro's Law of Isorotation</u> :

In the case of infinite electrical conductivity, Ferraro's law of isorotation states that, if the fluid is in steady azimuthal motion about an axis of symmetry, then the magnitude of the angular velocity is constant along magnetic field lines. This vital property of azimuthal motion proves to be of varied interest in the field of astrophysical discoveries.

It is proved by (Ciubotariu 1972) that

$$H_a \left[\pounds_{(u)} \omega^a \right] = 0, \qquad \ldots (3.1)$$

where, he has considered that the time-like flow is isometrio. This result is also valid for both finite and infinite electrical conductivity.

According to Yodzi's (1971)

$$\omega^2{}_{,a} H^a = \frac{1}{8} R_{abcd} U^a \omega^{bc} H^d. \qquad \ldots (3.2)$$

The aim here is to study these Ferraro's laws in context of F-magnetofluid following isometric motion.

We have Maxwell equation under the magnetohydrodynamics approximation of infinite electrical conductivity and constant magnetic permeability

$$\dot{H}^a + \theta H^a - H^b_{;b} U^a - H^b U^a_{;b} = 0. \qquad \ldots (3.3)$$

If we use isometry conditions in this equation, then we get

$$\dot{H}^a - H^b U_{a;b} = 0. \qquad \ldots (3.4)$$

We contract equation (3.4) by U_a which yields

$$\dot{H}^a U_a = 0. \qquad \ldots (3.5)$$

Also we have by (2.7, I)

$$H^a H_a = -\frac{1}{2} \dot{H}^2 = 0, \qquad \ldots (3.6)$$

Since we have $H^a H_a = - H^2$. $\qquad \ldots (3.7)$

We easily conclude from equations (3.5) and (3.6) that

$$\dot{H}^a = 0, \qquad \ldots (3.8)$$

because a vector cannot be normal to both a time-like vector and space-like vector at same time.

Therefore, equation (3.4) reduces to

$$H^b U_{a;b} = 0. \qquad \ldots (3.9)$$

We recall the expression for the covariant derivative of flow vector U_a as,

$$U_{a;b} = \sigma_{ab} + \frac{1}{3} \theta h_{ab} + \omega_{ab} + \dot{U}_a U_b \qquad (3.10)$$

The isometry conditions (5.8) with respect to U reduces this expression to

$$U_{a;b} = \omega_{ab} \qquad \ldots (3.11)$$

So that we arrive at the result

$$H^b \omega_{ab} = 0. \quad \text{(vide 3.9)} \qquad \ldots (3.12)$$

This shows that the magnetic field is normal to the plane of rotation.

Also by definition we write,

$$\omega_{ab} = \eta_{abcd}\, \omega^c U^d. \qquad \ldots (3.13)$$

Consequently equation (3.12) gives,

$$\eta_{abcd}\, H^b \omega^c U^d = 0. \qquad \ldots (3.14)$$

Here, η_{abcd} is the usual permutation symbol. We operate this with $\eta^{arst} U_t$ and use the expression

$$\eta_{abcd}\, \eta^{arst} = -\,3!\; \delta_b^{[r}\, \delta_c^{\ s}\, \delta_d^{\ t]}\,, \qquad \ldots (3.16)$$

in the resulting equation to get

$$\eta^{arst}\, \eta_{abcd}\, U_t H^b \omega^c U^d = 0. \qquad \ldots (3.17)$$

We already know that

$$H^a \omega^b = H^b \omega^a \quad . \qquad \ldots (3.18)$$

It follows immediately from this

$$H^a \omega^2 = (H^b \omega_b)\ \omega^a , \qquad \ldots (3.19)$$

and

$$H^b \omega_b = \pm H\omega , \qquad \ldots (3.20)$$

where,

$$\omega^2 = \omega_a \omega^a = \frac{1}{2} \omega_{ab}\, \omega^{ab} . \qquad \ldots (3.21)$$

So that equations (3.20) and (3.19) give rise to

$$H^a \omega = H\, \omega^a . \qquad \ldots (3.22)$$

Hence we write,

$$\omega^a = (\omega/H)H^a. \qquad \ldots (3.23)$$

The divergence of ω^a is given by

$$\omega^a_{;a} = (\omega/H)_{;a}H^a + (\omega/H)\, H^a_{;a}. \qquad \ldots (3.24)$$

We know the divergence equation for vorticity as

$$\omega^a{}_{;a} = 2\, \omega_a\, \dot{U}^a \qquad \ldots (3.25)$$

But we have by (3.23),

$$\omega_a \dot{U}^a = (\omega/H)H_a\dot{U}^a = - (\omega/H)\dot{H}_a\, U^a = 0. \qquad \ldots (3.26)$$

Thus we conclude from these equations (3.25) and (3.26)

$$\omega^a{}_{;a} = 0. \qquad \ldots (3.27)$$

This implies that the vorticity vector is divergence free.

Along with isometry condition (II, 6.3) if we use equation (3.23) in equation (3.27), we get,

$$(\omega/H)_{;a}\, H^{a} = 0. \qquad \ldots (3.28)$$

Also in similar way we can prove that

$$(H/\omega)_{;a}\quad{}^{a} = 0. \qquad \ldots (3.29)$$

According to the definition of 'Lie derivative' we have,

$$\pounds(H)\,\omega^{a} = \omega^{a}{}_{;b}\, H^{b} - H^{a}{}_{;b}\, \omega^{b}. \qquad \ldots (3.30)$$

Finally when we use equations (3.23) and (3.28) in equation (3.30) we get,

$$\pounds(H)\,\omega^{a} = 0. \qquad \ldots (3.31)$$

Following the same procedure we can also prove that

$$\pounds(\omega)\, H^{a} = 0. \qquad \ldots (3.32)$$

On taking covariant derivative of equation (3.9) we obtain,

$$H^{b}_{;c}\, U_{a;b} + H^{b} U_{a;bc} = 0. \qquad \ldots (3.33)$$

Contracting it with ω^{ac} we get,

$$H^{b}{}_{;c}\, \omega^{ac}\, U_{a;b} + H^{b}\, \omega^{ac}\, U_{a;bc} = 0. \qquad \ldots (3.34)$$

The first term of equation (3.34) after substituting the value of $U_{a;b}$ from equation (3.10), gives

$$H^{b}_{;c}\ \omega^{ac}\ U_{a;b} = H^{b}_{;c}\ \omega^{ac}\ \omega_{ab}\ . \qquad \ldots (3.35)$$

We write from the expression (3.13)

$$\omega^{ac}\ \omega_{ab} = \eta_{abcd}\ \omega^{ac}\ \omega^{c}\ U^{d},$$

i.e., $$\omega^{ac}\ \omega_{ab} = (\ \omega^{c}\ \omega_{b} - \omega^{2}\ h^{c}_{b}\ ). \qquad \ldots (3.36)$$

From equations (3.35) and (3.36), we write

$$H^{b}_{;c}\ \omega^{ac}\ U_{a;b} = H^{b}_{;c}\ \omega^{c}\ \omega_{b} - H^{b}_{;b}\ \omega^{2}\ . \qquad \ldots (3.37)$$

If we use the isometry condition (II, 6.3) in this, we get,

$$H^{b}_{;c}\ \omega^{ac}\ U_{a;b} = H^{b}_{;c}\ \omega^{c}\ \omega_{b}. \qquad \ldots (3.38)$$

On substituting the value of H^{b} from equation (3.22) we write,

$$H^{b}_{;c}\ \omega^{ac}\ U_{a;b} = \frac{1}{2}\ \omega^{2}{}_{;c}\ H^{c}\ . \qquad \ldots (3.39)$$

We have the Ricci identities

$$U_{a;bc} - U_{a;cb} = R_{dabc}\ U^{d}\ , \qquad \ldots (3.40)$$

i.e., $U_{a;bc} = U_{a;cb} + R_{dabc}\, U^d$ . ... (3.41)

If we inner multiply this by $H^b \quad {}^{ac}$ we get,

$$H^b \ \omega^{ac}\, U_{a;bc} = H^b \ \omega^{ac} U_{a;cb} + R_{dabc}\, U^d H^b \ \omega^{ac} \ . \qquad \ldots(3.42)$$

By the use of expression (3.10) for $U_{a;c}$ here we have from

$$H^b \ \omega^{ac}\, U_{a;c} = H^b \ \omega^{ac} \ \omega_{ac} \ , \qquad \ldots (3.43)$$

As a result of application of isometry conditions.

This we rewrite, by taking covariant derivative of both sides with respect to b, and making use of equation (3.21) as in the form

$$H^b\, \omega^{ac}\, U_{a;cb} = \omega^2{}_{;b}\, H^b. \qquad \ldots (3.44)$$

Consequently we write equation (3.42) as,

$$H^b \ \omega^{ac}\, U_{a;bc} = H^b \ \omega^{ac}\, U^d\, R_{dabc} + \omega^2{}_{;b}\, H^b,$$

i.e.,

$$H^b \ \omega^{ac}\, U_{a;bc} = \omega^2{}_{;b}\, H^b - R_{abcd}\, U^a \ \omega^{bc}\, H^d. \qquad \ldots(3.45)$$

By making use of equations (3.45) and (3.39) in equation (3.34) we deduce,

$$\frac{1}{2}\ \omega^2{}_{;c} H^c + \omega^2{}_{;b}\, H^b - R_{abcd}\, U^a \ \omega^{bc}\, H^d = 0, \qquad \ldots(3.46)$$

i.e., $\frac{3}{2}\,\omega^2{}_{;b}\,H^b = R_{abcd}\,U^a\,\omega^{bc}\,H^d\,,$ ... (3.47)

i.e., $\omega^2{}_{;a}\,H^a = \frac{2}{3}\,R_{abcd}\,U^a\,\omega^{bc}\,H^d\,.$ ... (3.48)

This clearly exhibits the effect of gravitational field of F-magnetofluid on the magnitude of angular velocity.

We have the expression for the conformal curvature tensor in terms of Riemannian curvature tensor, Ricci tensor and curvature scalar given by

$$C_{abcd} = R_{abcd} - \Big[\frac{1}{2}\,(g_{ac}\,R_{bd} + g_{bd}\,R_{ac} - g_{bc}\,R_{ad} - g_{ad}\,R_{bc}) + \frac{1}{6}\,(g_{ac}\,g_{bd} - g_{ad}g_{bc})\,R\Big]\,. \quad \ldots (3.49)$$

We write this as,

$$R_{abcd} = C_{abcd} + \frac{1}{2}\,(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}) + \frac{1}{6}\,(g_{ac}g_{bd} - g_{ad}g_{bc})\,R \quad \ldots (3.50)$$

This, when contracted with U^a gives

$$R_{abcd}\,U^a = C_{abcd}\,U^a + \frac{1}{2}\,(U_cR_{bd} + g_{bd}R_{ac}\,U^a - g_{bc}R_{ad}U^a - U_dR_{bc}) + \frac{R}{6}(U_cg_{bd}-U_dg_{bc}) \quad \ldots (3.51)$$

Again innermultiplying both sides by $^{bc}\,H^{d}$ we obtain,

$$R_{abcd}U^{a}\ \omega^{bc}\ H^{d} = C_{abcd}\ U^{a}\ \omega^{bc}H^{d} + \frac{1}{2}\ (U_{c}R_{bd}\ \omega^{bc}\ H^{d} + $$

$$+ g_{bd}R_{ac}U^{a}\ \omega^{bc}\ H^{d} - g_{bc}R_{ad}U^{a}\ \omega^{bc}H^{d}) +$$

$$+ \frac{R}{6}\ (U_{c}\ \omega^{bc}\ H_{b}\ ). \qquad \ldots (3.52)$$

As ω_{ab} is antisymmetric and R_{ab} is symmetric, we conclude from above equation,

$$R_{abcd}\ U^{a}\ \omega^{bc}\ H^{d} = C_{abcd}\ U^{a}\ \omega^{bc}\ H^{d} +$$

$$+ \frac{1}{2}\ (H_{b}R_{ac}\ \omega^{bc}\ U^{a}\ ). \qquad \ldots (3.53)$$

For F-magnetofluid we have the value of R_{ac} from equation (I, 6.4), substituting it in equation (3.53), we write equation (3.53) as

$$R_{abcd}\ U^{a}\ \omega^{bc}\ H^{d} = C_{abcd}\ U^{a}\ \omega^{bc}\ H^{d} -$$

$$- \frac{k}{2}\ H_{b}U_{c}\ \omega^{bc}\ \left[(\varrho+p+2m) - \frac{1}{2}(\ \varrho-p+2m\mu)\right].$$

$$\ldots (3.54)$$

This leads to

$$R_{abcd}\ U^{a}\ \omega^{bc}\ H^{d} = C_{abcd}\ U^{a}\ \omega^{bc}\ H^{d}. \qquad \ldots (3.55)$$

If we combine this with equation (3.48) we get,

$$\omega^2{}_{;a}\, H^a = \frac{2}{3}\, C_{abcd}\, U^a\, \omega^{bc}\, H^d \,. \qquad \ldots (3.56)$$

This exhibits the effect of free gravitational field on the divergence of the magnitude of angular velocity.

<u>Remark</u> : We infer from equation (3.56) that for the F-magnetofluid admitting isometric motion, the magnitude of the angular velocity ω is invariant along the magnetic lines provided the space-time is conformally flat.

We know that isometry with respect to U is described through,

$$\mathfrak{L}(U)\; g_{ab} = 0. \qquad \ldots (3.57)$$

Aslso we have the defining expression for the conformal motion with respect to U as,

$$\mathfrak{L}(U)\; g_{ab} = \lambda\, g_{ab} \,. \qquad \ldots (3.58)$$

We have from equation (3.58)

$$U_{a;b} + U_{b;a} = \lambda\, g_{ab} \qquad \ldots (3.59)$$

This implies,

$$\lambda = 0, \quad \text{since } U^a_{;b}\, U_a = 0. \qquad \ldots (3.60)$$

so that equation (3.58) reduces to the form of equation (3.57).

Hence we conclude that the conformal motion with respect to U directs isometry with respect to U. That means conformal motion satisfies all the conditions (II, 5.8) of isometry with respect to U.

From equation $T^{ab}_{;b} H_a = 0$, we get

$$\frac{1}{2} \mu_{;b} H^b H^2 - (p + m - m\mu)_{;b} H^b = 0. \qquad \ldots (3.61)$$

We write the Raychaudhari's equation under the condition of conformal motion with respect to U as,

$$\omega^2{}_{;b} H^b = - \frac{k}{4} (\varrho + 3p + 4m - 2m\mu)_{;b} H^b. \qquad \ldots (3.62)$$

Further from equation (3.61), we write,

$$p_{;b} H^b = (m\mu - m)_{;b} H^b + \frac{1}{2} \mu_{;b} H^b H^2. \qquad \ldots (3.63)$$

Also we put equation (3.62) as,

$$p_{;b} H^b = \frac{1}{3} \quad \frac{4}{k} \omega^2{}_{;b} H^b + (\varrho + 4m - 2m\mu)_{;b} H^b . \qquad \ldots (3.64)$$

Consequently equations (3.63) and (3.64) imply,

$$\frac{1}{2} \mu_{;b} H^b H^2 + (m\mu - m)_{;b} H^b = \frac{4}{3k} \omega^2{}_{;b} H^b + \frac{1}{3} (\varrho + 4m - 2m\mu)_{;b} H^b$$

i.e., $\omega^2_{;b}H^b = -\frac{3k}{8}\mu_{;b}H^bH^2 + \frac{k}{4}(5m\mu - 7m - \rho)_{;b}H^b$,

i.e., $\omega^2_{;b}H^b = -\frac{k}{4}\rho_{;b}H^b + \frac{k}{8}(10\,m\mu - 14m)_{;b}H^b +$

$$+ 3\mu_{;b}H^bH^2 \quad . \qquad \ldots (3.65)$$

This is the dynamical form of Ferraro's law of isorotation, in the space-time of F-magnetofluid under the conditions of conformal motion with respect to flow vector U.

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