CHAPTER 1

DEFINITIONS AND STATEMENTS OF KNOWN RESULTS

ABSTRACT

In this first chapter, we list a few definitions related to q-valent functions and several explanatory statements which we are going to use during our research work. The references used, are given at the end of the chapter.

DEFINITIONS AND STATEMENTS

STATEMENT - We consider throughout the discussion the domain E, as the unit disc , defined by

$$E = \{ Z : |z| < 1 , z \text{ is any complex number} \}.$$

DEFINITION - A complex valued function f(z) is said to be holomorphic in a domain D in the complex plane $\mathbb{C}$ if it has a uniquely determined derivative at each point of D.

DEFINITION - A single valued function f is said to be univalent (or Schlicht) in a domain $D \subset \mathbb{C}$ if it never takes the same value twice, that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_1 \neq z_2$.

STATEMENT - Let S denote the class of all functions f(z) holomorphic and univalent in E and normalised by the conditions $f(0) = 0$ and $f'(0) = 1$.

DEFINITION - The radius of univalence or the region of univalence is defined to be the largest value of r such that f(z) is holomorphic and univalent for $|z| < r$.

Accordingly the same can be generalised to region of q-valence.

We note that in particular region of starlikeness and convexity can be established.

DEFINITION - The domain which contains the origin is said to be starlike with respect to the origin if it is intersected by any straight line through the origin in a linear segment.

Starlike with respect to origin is referred to as simply starlike.

<u>Remark</u> - We also note that a starlike function is a conformal mapping of the unit disc onto a domain starlike with respect to the origin.

STATEMENT - Let K be the subclass of S whose members transform every disc $|z| \leq \rho$, $\rho \in (0,1)$ onto a convex domain. We note that K is the family of convex functions in S.

DEFINITION - A function $f(z) \in S$ is said to be close-to-convex with respect to the convex function $e^{i\alpha}g(z)$, where $g(z)$ is convex, $\alpha \in [o, 2\pi)$ if

$$\operatorname{Re} \left\{ \frac{f'(z)}{e^{i\alpha} g'(z)} \right\} > 0 \text{, for } z \in E.$$

This class is denoted by 'C'.

Let $f(z)$ be holomorphic at $z = 0$ and $f(0) = 0$, $f'(0) \neq 0$. We define the region of close-to-convexity as follows -

DEFINITION - The radius of close-to-convexity is defined to be the largest value of r such that $f(z)$ is holomorphic and close-to-convex for $|z| < r$.

STATEMENT - Let $p(z) = a \prod_{k=1}^{n} (z - z_k)$ be a polynomial of degree n, where n is a positive integer, all of whose zeros lie outside or on the unit circle.

The set of all such polynomials, we denote by $Q(z)$.

$\wp(n,1)$ denotes a polynomial of degree n and all the n zeros lie outside or on the circle with centre at origin and unit radius.

DEFINITION – The holomorphic function $f(z)$ in E is said to be subordinate to $g(z)$ if $g(z)$ is univalent in E, $f(0) = g(0)$ and $f(E) \leq g(E)$. We denote this relation by $f(z) \ll g(z)$.

DEFINITION – Suppose that we are given a function $f(z)$ holomorphic in the unit disc E and that the equation $f(z) = w$ has there never more than q-solutions, as w moves over the open plane, then $f(z)$ is said to be q-valent in $|z| < 1$.

STATEMENT – Let S_q (q , a fixed integer greater than zero) denote the class of functions $f(z) = z^q + \sum_{k=q+1}^{\infty} a_k z^k$ which are holomorphic in $E = \{z : |z| < 1\}$.

DEFINITION – A function $f(z) \in S_q$ is called q-valent starlike function of order α if it satisfies

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \, , \quad 0 \leq \alpha < q$$

We denote this type of function by $s_q^*(\alpha)$.

Remark – For $\alpha = 0$, we get

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0 .$$

We call these functions ,as simply q-valent starlike functions , denoted by S_q^*.

DEFINITION –A Function $f(z) \in S_q$ is said to be q-valent starlike function of order α and type β if the following condition is satisfied,

$$\left| \frac{\left(\frac{z f'(z)}{f(z)} - q \right)}{2\beta \left[\frac{zf'(z)}{f(z)} - \alpha \right] - \left[\frac{zf'(z)}{f(z)} - q \right]} \right| < 1$$

for $0 \leq \alpha < q, \quad 0 < \beta \leq 1$, $z \in E$,

We denote this class by $S_q^*(\alpha,\beta)$. This can be found in Aouf [1].

DEFINITION -Extreme point - The point 'a' in a convex set A is called an extreme point of A if and only if 'a' cannot be expressed as convex combination of any other two distinct points of A.i.e. $tx_1 + (1-t)\,x_2$ is the convex combination of any two distinct points x_1, x_2 of A then $a \neq tx_1 + (1-t)\,x_2$ for $0 < t < 1$.

REFERENCES

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