CHAPTER 3

# GENERALISED BI-IDEALS IN NEAR-RINGS

# CHAPTER III

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## § 3.0 Introduction:

Throughout this chapter N denotes a right near-ring. The concept of bi-ideals in near-ring was introduced by T. Tamizh chelvam and N. Ganesan [3]. Generalization of bi-ideals in near-rings is done in this chapter, we name it generalised bi-ideal.

In § 3.1 we define a generalised bi-ideal in N and study some examples § 3.2 deals with the study of some properties of generalised bi-ideals. Mainly it is shown that in a zero-symmetric near-ring N, a semigroup G of N is a generalised bi-ideal if and only if $GNG \subseteq G$. . Also in case of generalised bi-ideals we prove:

Result 1 : Set of all generalised bi-ideals in N forms a Moore system in N.

Result 2 : Intersection of generalised bi-ideal G subnear-ring S of N is generalised bi-ideal of S.

In 3.2.6. We define G.B. simple near-ring and give the following result.

Result 3 : Let N be a near-ring with more than one element. Then the following conditions are equivalent :

(i) N is a near-field

(ii) N is G.B. simple, $N_d \neq \{0\}$ and for $0 \neq n \in N$ there exists an element $n' \in N$ such that $n'.n \neq 0$.

□

## § 3.1 Definition and examples :

In this section we define generalised bi-ideal and give some examples of generalised bi-ideal in a near-ring N.

**Definition 3.1.1 :**

Let <N,+, . > be a near-ring . A non-empty subset G of N is called generalised bi-ideal if it satisfies the following conditions.

(1) $a+b \in G$, $\forall\, a,b \in G$

(2) $GNG \cap (GN) * G \subseteq G$

□

Some examples of generalised bi-ideals of near-ring are given below.

**Example 3.1.2 : (Pilz, page – 408)**

Consider the near-ring N={0, a, b,c} with addition and multiplication defined by the Cayley tables.

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | b | 0 | b |
| b | 0 | 0 | 0 | 0 |
| c | 0 | b | 0 | b |

Let G={ 0,a} . Here G is generalised bi-ideal of N.

□

**Example 3.1.3 : (Clay, 2.2, 13)**

Consider the near-ring N={0,a,b,c} with addition and multiplication defined by the following tables.

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | 0 | 0 | 0 |
| c | 0 | a | b | c |

Let G = {0,b} . Here G is a generalised bi-ideal of N.

□

From the definition of bi-ideal of a near-ring (see 0.1.13) and the definition of generalised bi-ideal (by 3.1.1) it is clear that every bi-ideal in a near-ring is a generalised bi-ideal. But converse need not be true. This is established in the following example.

**Example 3.1.4 :**

Consider the set M of all 2x2 matrices over the set of all integers.M is a near-ring w.r.t. matrix addition and matrix multiplication . Let A={ $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ / a is positive integer }. It can be easily shown that A is a generalised bi-ideal. But $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin A$ . Hence A is not a bi-ideal.

□

By the definition of an ideal in a near-ring (see 0.1.10) it is clear that every ideal in a near-ring is a generalised bi-ideal. But converse need not be true. This is proved in the following example.

**Example 3.1.5 :**

In example 3.1.4 Consider A= { $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ /a is positive integer}. A is generalised bi-ideal of N. But as, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\notin$ A. Hence A is not an ideal of N.

□

We know that every quasi-ideal (see Def.0.1.14) in a near-ring is a bi-ideal (see Result 0.2.10) and every bi-ideal in a near-ring is a generalised bi-ideal. Hence every quasi-ideal in a near-ring is a generalised bi-ideal . But converse need not be true. This is proved in the following example.

**Example 3.1.6 : (Pilz, page – 408)**

Let N = { 0,a,b,c} be the near-ring defined by the Cayley tables.

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | b | 0 | b |
| b | 0 | 0 | 0 | 0 |
| c | 0 | b | 0 | b |

Let G={0,a} . G is a generalised bi-ideal of N.

For a$\in$ G, a$\in$N, b=a.a $\in$ GN

For a$\in$G , a$\in$N, b=a.a $\in$ NG

and b=a.a =a.(0+a) = a.(0+a)-0

$= a.(0+a) - a.0 \in N^*G$ , For $a, 0 \in N$, $a \in G$

Hence $b \in GN \cap NG \cap N^*G$

But $b \notin G$.

Hence $GN \cap NG \cap N^*G \not\subset G$

Therefore , G is not a quasi-ideal in N.

□

**3.2 Properties of generalised bi-ideal :**

In this section we collect some properties of generalised bi-ideal of a near-ring N.

**Result 3.2.1** : The set of all generalised bi-ideal of a near-ring N form a Moore system on N.

**Proof** : By definition of generalised bi-ideal of N. N itself is a generalised bi-ideal . Let $\{G_i\}_{i \in I}$ be a set of generalised bi-ideals in N. Let $G = \cap_{i \in I} G_i$ .Obviously G is closed w.r.t addition . Since $G \subseteq G_i$ for every $i \in I$. Therefore $GNG \cap (GN)^*G \subseteq G_iNG_i \cap (G_iN)^*G_i \subseteq G_i$ for every $i \in I$. [Since each $G_i$ is a generalised bi-ideal ]

Therefore $GNG \cap (GN)^*G \subseteq \cap_{i \in I} G_i$

Therefore $GNG \cap (GN)^* G \subseteq G$

Hence G is a generalised bi-ideal of N. Therefore the set of all generalised bi-ideals of a near-ring N form a Moore system on N.

□

**Result 3.2.2** : If G is a generalised bi-ideal of a near-ring N. S is a sub near-ring of N. then $G \cap S$ is a generalised bi-ideal of S.

**Proof :** Since G is a generalised bi-ideal of N . Therefore G is closed w.r.t. addition and $GNG \cap (GN) * G \subseteq G$.

Let $C = G \cap S$.

Since S is a subnear-ring of N. Therefore <S,+> is a subgroup of <N,+>

Let $a, b \in C = G \cap S$

Therefore $a, b \in G$ and $a, b \in S$

Therefore $a+b \in G$ and $a+b \in S$

Therefore $a+b \in G \cap S$ $\forall$ a, $b \in G \cap S$

Hence $a+b \in C$, $\forall$ a, $b \in C$ ---------------- (1)

Now $CSC \cap (CS) * C = (G \cap S) S (G \cap S) \cap ((G \cap S)S) * (G \cap S)$

$\subseteq GSG \cap S \cap (GS) * G$ [Since $G \cap S \subseteq G$, $G \cap S \subseteq S$]

$\subseteq GSG \cap (GS) * G \cap S$

$\subseteq G \cap S = C$

Therefore $CSC \cap (CS) * C \subseteq C$ -------------- (2)

Hence form (1) and (2) . C is a generalised bi-ideal of S.

Therefore , $G \cap S$ is a generalised bi-ideal of S.

□

A necessary and sufficient condition for a semigroup G of N to be a generalised bi-ideal is given in the following result.

**Result 3.2.3 :** Let N be a zero-symmetric near-ring . A semigroup G of N is a generalised bi-ideal iff $GNG \subseteq G$.

**Proof :-** For a semigroup G of <N, +>

First suppose that $GNG \subseteq G$.

Since , $GNG \cap (GN)*G \subseteq GNG \subseteq G$

Hence , $GNG \cap (GN)*G \subseteq G$

Therefore G is a generalised bi-ideal of N .

Conversly suppose G is a generalised bi-ideal of N.

Therefore, $GNG \cap (GN)*G \subseteq G$.

Let $n.g \in NG$

Therefore $n.g = n.(0+g) +0 = n.(0+g) - n.0 \in N*G$

[Since N is a zero symmetric near-ring , therefore $n.0=0 \ \forall \ n \in N$]

Therefore $NG \subseteq N*G$.

We get $GNG = GNG \cap GNG \subseteq GNG \cap (GN)*G \subseteq G$ . Therefore $GNG \subseteq G$.

□

In the following result we give a property of a generalised bi-ideal in a zero-symmetric near-ring.

**Result 3.2.4** : Let N be a zero -symmetric near-ring . If G is a generalised bi-ideal of N. then Gn and n'G are generalised bi-ideals of N where $n, n' \in N$ and n' is distributive element in N.

**Proof** : Let $x \in G$, $y \in G$. Therefore $x+y \in G$.

Therefore $x.n \in Gn$ and $y.n \in Gn$.

Therefore $x.n + y.n = (x+y).n \in Gn$.

Therefore Gn is closed w.r.t . addition. And $GnNGn \subseteq GNGn \subseteq Gn$ [Since G is a generalised bi-ideal , therefore from Result 3.2.3 $GNG \subseteq G$ and $nN \subseteq N$]

Therefore from result 3.2.3 Gn is a generalised sed bi-ideal of N.

Since n′ is a distributive element in N.

Therefore n′(a+b) =n′.a+n′.b, ∀ a, b ∈ N.

Let x , y ∈ G. Therefore x+y∈G .Hence n′.x+n′.y= n′ (x+y) ∈ n′G [Since n′ is a distributive element in N.]

Thus n′G is closed w.r.t. addition and n′GNn′G ⊆ n′GNG ⊆ n′G [Since N is zero -symmetric near-ring and G is a generalised bi-ideal of N, therefore from Result 3.2.3 GNG ⊆ G and Nn′ ⊆ N.]

Therefore from result 3.2.3 n′G is a generalised bi-ideal of N.

□

As a corollary of Result 3.2.4 we get.

**Corollary 3.2.5** : Let G be a generalised bi-ideal of a zero-symmetric near-ring N. b is a distributive element in N then bGc is a generalised bi-ideal of N, where c∈N.

**Proof :** Since G is a generalised bi-ideal of a zero symmetric near-ring N. Therefore from result 3.2.4 Gc is a generalised bi-ideal where c∈N Again bGc is a generalised bi-ideal of N, where b is a distributive element in N.

□

Now we define G.B. simple near-ring and give necessary and sufficient condition for a near-ring to be a near-field.

**Definition 3.2.6 :**

A near-ring N is said to be G.B- simple if it has no proper generalised bi-ideals

**Resultl 3.2.7** : Let N be a near-ring with more than one element . Then the following conditions are equivalent.

(I) N is a near-field.

(2) N is G.B. simple , $N_d \neq \{0\}$ and for $0 \neq n \in N$ there exists an element $n' \in N$ such that $n'n \neq 0$.

**Proof :**

(I) $\Rightarrow$ (ii)

Let N be a near-field . To pove that N is G. B. simple . i.e To prove that $\{0\}$ and N are the only generalised bi-ideals of N.

If $\{0\} \neq B$ is a generalised bi-ideals of N, then for $0 \neq b \in B$.Now prove that N = Nb and N=bN.

Let $n \in N$. Therefore $n=n.1 = n.(b^{-1}.b) = (n.b^{-1})\, b \in Nb$. Hence $N \subseteq Nb$. But $Nb \subseteq N$. Therefore N=Nb, Similary N = bN. Now $N=N^2 = (N)(N) = (bN)(Nb) = bN^2b \subseteq bNb \subseteq B$; Since B is a generalised bi-ideal of N. Therefore N=B . Hence N is G.B-simple and for $0 \neq n \in N$, there exist an element $1 \in N$ such that $1.n = n \neq 0$

(ii) $\Rightarrow$ (i)

Since $N_d \neq \{0\}$ we get N is not constant . We know that $N_o$ is a generalised bi-ideal of N and since N is G.B simple we get $N = N_o$ . Let

$0 \neq n \in N$ by Result 3.2.4 Nn is a generalised bi-ideal of N and $0 \neq n'.n \in Nn$ for some $n' \in N$ Hence Nn=N.

Therefore N is a near-field (From Result 0.2.8)

□

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