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WAVELET ANALYSIS

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CHAPTER NO. 2

WAVELET ANALYSIS

Introduction:

As we have seen, Fourier transformation is a powerful tool in representing functions in time domain and in frequency domain. However the formula for transforming a signal from time domain to its frequency domain is quite inadequate, since to extract information about the signal f in a small neighborhood of some frequency value ω, full information about f in time domain must be acquired. In addition, a small change in signal would affect the entire frequency spectrum of the signal. Another drawback of Fourier transform is that the formula does not given any information about frequencies which involve with time. In many application, one only needs to have local information either in time or in frequency domain. In view of the above observations, Gabor in 1946, introduced a time localization method by using window function in the definition of transform. By shifting the window over the entire time domain localized information about the signal in frequency domain could be made available.

(2.1) Time Frequency Localization:

A function $f \in L^2(\mathbb{R})$ is used to represent an analog signal with finite energy and its Fourier transform is defined by

$$(\mathcal{F}f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt, \qquad \text{---(2.1.1)}$$

gives the spectral information of signal. This is similar to musical notation, which tells the player which notes (frequency information) to play at the given moment. Unfortunately, the formula (2.1.1) alone is not very useful for extracting information of the spectrum $\hat{f}$ from local observation of the signal f. Therefore, what is needed is good time window. Time localization can be achieved by windowing the signal f, so as to cut off only a well localized slice of f and then taking its Fourier transform.

$$(\mathcal{G}_b f)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)g(t - b)\ dt \qquad \text{---(2.1.2)}$$

This is a windowing Fourier transform which is standard technique for time-frequency localization. It is even more familiar to signal analysis in its discrete version. Many possible choices have been proposed for the window function g in the signal analysis, most of which have compact support and reasonable smoothness.

A very popular choice is a Gaussian function g_α. In all application g_α is supposed to be well concentrated in both time and frequency, if g_α and $\hat{g}_\alpha$ are both concentrated around zero (0), then $(\mathcal{G}_b f)(\omega)$ can be interpreted loosely as the "content" of f near time t and near frequency ω. The windowed Fourier transform thus provides a description of f in the time - frequency plane.

The "optimal window" for time localization is achieved by using any Gaussian function

$$g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{t^2}{4\alpha}} \qquad ---(2.1.3)$$

where $\alpha > 0$ is fixed, as a window function.

(2.2) Gabor Transform

Definition: For any fixed value $\alpha > 0$, the "Gabor transform" of an $f \in L^1(\mathbb{R})$ is defined by

$$(\mathcal{G}_b^\alpha f)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) g_\alpha(t - b)dt \qquad ---(2.2.1)$$

that is $(\mathcal{G}_b^\alpha f)(\omega)$ localizes the Fourier transform of f around $t = b$ Observe that in the integral

$$\int_{-\infty}^{\infty} g_\alpha(t - b)db$$

if we put $t - b = y \Rightarrow -db = dy$

Therefore,

$$\int_{-\infty}^{\infty} g_\alpha(t - b)db = \int_{-\infty}^{\infty} g_\alpha(y)dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{y^2}{4\alpha}} dy$$

$$\int_{-\infty}^{\infty} g_\alpha(t - b)db = \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4\alpha}} dy$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \sqrt{\frac{\pi}{(1/4\alpha)}}$$

$$\int_{-\infty}^{\infty} g_\alpha(t - b)db = 1 \qquad \text{---(2.2.2)}$$

so that

$$\int_{-\infty}^{\infty} (\mathcal{G}_b^\alpha f)(\omega)db = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) g_\alpha(t - b)\, dt\, db$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g_\alpha(t - b)\, db \right) e^{-i\omega t} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$$

$$= \hat{f}(\omega) \qquad \omega \in \mathbb{R}$$

That is, the set $\left\{ (\mathcal{G}_b^\alpha f) : b \in \mathbb{R} \right\}$ of Gabor transforms of f decomposes the Fourier transform $\hat{f}$ of f exactly, to gives its local spectral information

<u>Definition:</u> For a nontrivial function $w \in L^2(\mathbb{R})$, the center x^* and radius Δ_w are defined as

$$x^* = \frac{1}{\| w \|_2^2} \left\{ \int_{-\infty}^{\infty} t \,|\, w(t)|^2\, dt \right\} \qquad \text{---(2.2.3)}$$

$$\Delta_w = \frac{1}{\| w \|_2} \left\{ \int_{-\infty}^{\infty} (t - x^*)^2 |\, w(t)|^2\, dt \right\}^{1/2} \qquad \text{---(2.2.4)}$$

Geometrically the center of the window function, we mean the value in the time domain, around which g has maximum spectral energy. Width of the window function specifies that the energy contribution outside the interval $[\, t^* - \Delta_g \,,\, t^* + \Delta_g \,]$ will be always negligible.

<u>Theorem</u>(2.1): For any $\alpha > 0$

$$\Delta_{g_\alpha} = \sqrt{\alpha} \qquad \text{---(2.2.5)}$$

That is the width of the window function g_α is $2\sqrt{\alpha}$

PROOF: Recall that

$$\int_{-\infty}^{\infty} e^{-i\omega x} e^{-ax^2} dx = \sqrt{\pi/a} \cdot e^{-\frac{\omega^2}{4a}}$$

setting $\omega = 0$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a} \qquad \text{---(2.2.6)}$$

differentiate both side w.r.t. a

$$\int_{-\infty}^{\infty} x^2 \cdot e^{-ax^2} dx = \sqrt{\pi}/2 \; a^{-3/2} \qquad \text{---(2.2.7)}$$

Now we calculate $\|g_\alpha\|_2$

$$\|g_\alpha\|_2 = \left\{ \int_{-\infty}^{\infty} g_\alpha^2(x) \; dx \right\}^{1/2}$$

$$\|g_\alpha\|_2 = \left\{ \int_{-\infty}^{\infty} \left[\frac{1}{2\sqrt{\pi\alpha}} \; e^{-\frac{x^2}{4\alpha}} \right]^2 dx \right\}^{1/2}$$

$$= \left\{ \int_{-\infty}^{\infty} \frac{1}{4\pi\alpha} \; e^{-\frac{x^2}{2\alpha}} \; dx \right\}^{1/2}$$

$$= \left\{ \frac{1}{4\pi\alpha} \sqrt{\frac{\pi}{1/2\alpha}} \right\}^{1/2}$$

$$\|g_\alpha\|_2^2 = (8\pi\alpha)^{-1/2} \qquad \text{---(2.2.8)}$$

Now by the definition (1.2.4)

$$\Delta_{g_\alpha} = \frac{1}{\|g_\alpha\|_2} \left\{ \int_{-\infty}^{\infty} (x - 0)^2 g_\alpha(x)^2 \, dx \right\}^{1/2}$$

$$= \frac{1}{(8\pi\alpha)^{-1/4}} \left\{ \int_{-\infty}^{\infty} x^2 \; g_\alpha(x)^2 \, dx \right\}^{1/2}$$

$$= \frac{1}{(8\pi\alpha)^{-1/4}} \left\{ \int_{-\infty}^{\infty} x^2 \left[\frac{1}{2\sqrt{\pi\alpha}} \; e^{-\frac{x^2}{4\alpha}} \right]^2 dx \right\}^{1/2}$$

$$= (8\pi\alpha)^{1/4} \cdot (4\pi\alpha)^{-1/2} \cdot \left\{ \int_{-\infty}^{\infty} x^2 \; e^{-\frac{x^2}{2\alpha}} \; dx \right\}^{1/2}$$

$$= (8\pi\alpha)^{1/4} \cdot (4\pi\alpha)^{-1/2} \left(\frac{\sqrt{\pi}}{2} (2\alpha)^{3/2} \right)^{1/2}$$

$$\Delta_{g_\alpha} = \sqrt{\alpha}$$

We may interpret the Gabor transform $(\mathcal{G}_b^\alpha f)$ in (2.2.1) different way by setting

$$G_{b,\omega}^\alpha(t) = e^{i\omega t} \; g_\alpha(t - b) \qquad \text{---(2.2.9)}$$

We have

$$(\mathcal{G}_b^\alpha f)(\omega) = \langle f, G_{b,\omega}^\alpha \rangle = \int_{-\infty}^{\infty} f(t) \; \overline{G_{b,\omega}^\alpha(t)} \; dt \qquad \text{---(2.2.10)}$$

One advantage of this formulation is that the Parseval Identity can be applied to relate the Gabor transform of f

with Gabor transform of $\hat{f}$ .We have

$$(G^{\alpha}_{b,\omega})\hat{}\,(\eta) = \int_{-\infty}^{\infty} e^{-i\eta t} G^{\alpha}_{b,\omega}(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-i\eta t}\; e^{i\omega t} g_{\alpha}(t - b)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-i(\eta - \omega)t}\; g_{\alpha}(t - b)\, dt$$

Put $t - b = x \Rightarrow dt = dx$

$$(G^{\alpha}_{b,\omega})\hat{}\,(\eta) = \int_{-\infty}^{\infty} e^{-i(\eta - \omega)(x + b)}\; g_{\alpha}(x)\, dx$$

$$= e^{-i(\eta - \omega)b} \int_{-\infty}^{\infty} e^{-i(\eta - \omega)x}\; g_{\alpha}(x)\, dx$$

$$= e^{-i(\eta - \omega)b} \int_{-\infty}^{\infty} e^{-i(\eta - \omega)x}\; \frac{1}{2\sqrt{\pi\alpha}}\; e^{-\frac{x^2}{4\alpha}}\, dx$$

$$= e^{-i(\eta - \omega)b}\; \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-i(\eta - \omega)x}\; e^{-\frac{x^2}{4\alpha}}\, dx$$

$$= \frac{e^{-i(\eta - \omega)b}}{2\sqrt{\pi\alpha}} \sqrt{\frac{\pi}{1/4\alpha}}\; e^{-\frac{(\eta - \omega)^2}{4(1/4\alpha)}}$$

$$(G^{\alpha}_{b,\omega})\hat{}\,(\eta) = e^{-i(\eta-\omega)b} \cdot e^{-\alpha(\eta-\omega)^2} \qquad \text{---(2.2.11)}$$

Now we know that

$$(\mathcal{G}^{\alpha}_{b}f)(\omega) = \langle f, G^{\alpha}_{b,\omega} \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{G}^{\alpha}_{b,\omega} \rangle$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\eta)\; \overline{\hat{G}^{\alpha}_{b,\omega}(\eta)}\, d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\eta) \cdot e^{\,i(\eta - \omega)b} \cdot e^{-\alpha(\eta - \omega)^2}\, d\eta$$

$$(\mathcal{G}_b^{\alpha}f)(\omega) = \frac{e^{-ib\omega}}{2\pi}\sqrt{\pi/\alpha}\int_{-\infty}^{\infty} e^{ib\eta}\,\hat{f}(\eta)\,\frac{1}{2\sqrt{\pi/4\alpha}}\,e^{-\frac{(\eta-\omega)^2}{4(1/4\alpha)}}\,d\eta$$

$$(\mathcal{G}_b^{\alpha}f)(\omega) = \left[\sqrt{\frac{\pi}{\alpha}}\,e^{-ib\omega}\right]\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ib\eta}\,\hat{f}(\eta)\,g_{1/4\alpha}(\eta-\omega)\,d\eta$$

---(2.2.12)

Let us interpret (2.2.12) from two different point of view. First, we consider

$$\int_{-\infty}^{\infty} e^{-i\omega t}\,f(t)\,g_{\alpha}(t-b)\,dt =$$

$$= \left[\sqrt{\frac{\pi}{\alpha}}\,e^{-ib\omega}\right]\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ib\eta}\,\hat{f}(\eta)\,g_{1/4\alpha}(\eta-\omega)\,d\eta$$

which says that, with the exception of the multiplication term $\sqrt{\pi/\alpha}\,e^{-ib\omega}$, the "window Fourier transform" of f with window function g_{α} at $t = b$ agrees with the " window Inverse Fourier transform " of $\hat{f}$ with window function $g_{1/4\alpha}$ at $\eta = \omega$. We know $\Delta_{g_{\alpha}}$ and $\Delta_{\hat{g}_{1/4\alpha}}$ are the radii for two windows. And by the Theorem(2.1) $\Delta_{g_{\alpha}} = \sqrt{\alpha}$, $\Delta_{\hat{g}_{1/4\alpha}} = \sqrt{1/4\alpha}$. Therefore, the product of the width of these two windows is

$$(2\,\Delta_{g_{\alpha}})\cdot(2\,\Delta_{\hat{g}_{1/4\alpha}}) = 2\sqrt{\alpha}\cdot\frac{1}{\sqrt{\alpha}} = 2 \qquad \text{---(2.2.13)}$$

On the other hand, by considering

$$H_{b,\omega}^{\alpha}(\eta) = \frac{1}{2\pi}(G_{b,\omega}^{\alpha})\hat{}\,(\eta) = \frac{e^{ib\omega}}{2\sqrt{\pi\alpha}}\,e^{-ib\eta}\,g_{1/4\alpha}(\eta-\omega) \qquad \text{---(2.2.14)}$$

We have

$$\langle f, G^{\alpha}_{b,\omega} \rangle = \langle \hat{f}, H^{\alpha}_{b,\omega} \rangle \qquad \text{---(2.2.15)}$$

This identity says that the information obtained by investigating an analog signal $f(t)$ at $t = b$ by using the window function $G^{\alpha}_{b,\omega}$ can be obtained by observing the spectrum $\hat{f}(\eta)$ of the signal in a neighborhood of the frequency $\eta = \omega$ by using the window function $H^{\alpha}_{b,\omega}$. Again the product of the width of the time-window $G^{\alpha}_{b,\omega}$ and that of frequency window $H^{\alpha}_{b,\omega}$ is

$$(2\Delta_{G^{\alpha}_{b,\omega}})\cdot(2\,\Delta_{H^{\alpha}_{b,\omega}}) = (2\,\Delta_{g_{\alpha}})\cdot(2\,\Delta_{\hat{g}_{1/4\alpha}}) = 2 \qquad \text{---(2.2.16)}$$

The cartesian product

$$[b - \sqrt{\alpha}\,, b + \sqrt{\alpha}\,]\ X\ [\omega - \frac{1}{\sqrt{4\alpha}}\,, \omega + \frac{1}{\sqrt{4\alpha}}\,]$$

of these two windows is called a rectangular time-frequency window. The width $2\sqrt{\alpha}$ of the time window is called the "width of the time-frequency window" and the width $1/\sqrt{\alpha}$ of the frequency window is called "Height of the time-frequency window". Observe that the width of the time and frequency window is unchanged for observing the spectrum at all frequencies. This restricts the application of the Gabor transform to study signal with high and low frequencies.

(2.3) SHORT TIME FOURIER TRANSFORM (STFT):

The Gabor transform is a window Fourier transform with any Gaussian function g_α as the window function. For various reasons such as computational efficiency or convenience in implementation other functions may also be used as a window function instead of g_α.

<u>Definition</u>: A nontrivial function $g \in L^2(\mathbb{R})$ is called window function if

$$x\, g(x) \in L^2(\mathbb{R}) \qquad \text{---(2.3.1)}$$

We called this condition (2.3.1) as window condition.

Since $t \cdot g(t) \in L^2(\mathbb{R}) \Rightarrow |t|^{1/2} g(t) \in L^2(\mathbb{R})$

$$\Rightarrow (1 + |t|)^{-1} \in L^2(\mathbb{R}) \text{ and } (1 + |t|) \in L^2(\mathbb{R}).$$

By Property of Fourier transform $\hat{g}$ is continuous and by the parseval identity $\hat{g}(\omega) \in L^2(\mathbb{R})$ but does not necessarily satisfy window condition and hence may not be a (frequency) window function. Since Fourier transform of Gaussian function is itself a Gaussian function, so that g_α and $\hat{g}_\alpha$ can be used for time-frequency localization.

Ex. Both the first order B-Spline

$$N_1(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and Haar function

$$\psi_1(t) = \begin{cases} 1 & \text{for } 0 \le t < 1/2 \\ -1 & \text{for } 1/2 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

are window functions but their Fourier transforms $\hat{N}_1$ and $\hat{\psi}_1$ do not satisfies window condition. Hence N_1 and ψ_1 can not be used for time - frequency localization.

The Gabor transform defined in (2.2.1) can be generalized to any "window Fourier transform" of an $f \in L^2(\mathbb{R})$, by using function g that satisfies window condition.

$$(\tilde{\mathcal{G}}_b f)(\omega) = \int_{-\infty}^{\infty} (e^{-i\omega t} f(t)) \overline{g(t-b)}\, dt \qquad \text{---(2.3.2)}$$

Hence by setting

$$W_{b,\omega}(t) = e^{i\omega t} g(t-b) \qquad \text{---(2.3.3)}$$

We have

$$(\tilde{\mathcal{G}}_b f)(\omega) = \langle f , W_{b,\omega} \rangle = \int_{-\infty}^{\infty} f(t)\, \overline{W_{b,\omega}(t)}\, dt$$

The time - frequency window can be obtained as

$$[x^* + b - \Delta_g , x^* + b + \Delta_g] \times [\omega^* + \omega - \Delta_{\hat{g}} , \omega^* + \omega - \Delta_{\hat{g}}]$$

where x^* and ω^* are centers of g and $\hat{g}$ respectively. Such a transform in which both g and $\hat{g}$ are window functions is called short time Fourier transform (STFT). For an arbitrary window function, the window area is given by $4\,\Delta_g \cdot \Delta_{\hat{g}}$.

It is indeed remarkable to note that the window area for

Gabor transform turns out to be minimal.

Theorem(2.2): (Uncertainty Principle)

Let g and $\hat{g}$ be both window functions, then

$$\Delta_g \cdot \Delta_{\hat{g}} \geq 1/2,$$

with equality if and only if

$$g(t) = c \cdot e^{iat} g_{\alpha}(t - b)$$

where c ($\neq 0$) : constant and $a, b \in \mathbb{R}$, $\alpha > 0$

Short time Fourier transform (STFT) gives a way to localized time - frequency analysis. However, one finds that there is still a scope for improvement in this time - frequency analysis. Since in Gabor transform or Short time Fourier Transform the width of the window remains unchanged. One needs to have a window function such that the spectrum of the signal can be analyzed locally more efficiently. By this we mean that the width of the window should be relatively small for high frequency levels and be wide enough to cover low levels. This motivates us to define Integral Wavelet Transform (IWT).

Let us begin with a window function $\psi \in L^2(\mathbb{R})$ such that both ψ and its Fourier transform $\hat{\psi}$ are window function. Let the center and width of ψ and $\hat{\psi}$ be given by t^*, $2\Delta_\psi$ and ω^*, $2\Delta_{\hat{\psi}}$ respectively. We define for $a, b \in \mathbb{R}$ and $a \neq 0$

$$\psi_{b;a}(t) = |\alpha|^{-1/2} \psi\left(\frac{t - b}{a}\right) \qquad \text{---(2.3.4)}$$

The integral wavelet transform of a signal $f \in L^2(\mathbb{R})$ may be defined as

$$(W_\psi f)(b,a) = \langle f, \psi_{b,a} \rangle$$

$$= \int_{-\infty}^{\infty} f(t)\, \overline{\psi_{b,a}(t)}\, dt \qquad \text{---(2.3.5)}$$

Notice that the introduction of parameters 'a' allows $\psi_{b,a}$ to compress or expand depending on the choice of 'a'. The normalizing factor $|a|^{-1/2}$ automatically take care of the relative frequency band. For example if $a > 1$, $\psi_{b,a}$ is stretched by a factor 'a' in the horizontal direction, where as if $0 < a < 1$ then it is compressed in the same horizontal direction, when $\psi_{b,a}$ is dilated in horizontal direction, the factor $|a|^{-1/2}$ automatically reduces its size in the vertical direction and for small 'a' with the compression of $\psi_{b,a}$ in the horizontal direction, ψ is enlarged in the vertical direction. In fact the total energy of $\psi_{b,a}$ remains independent of 'b' and 'a', that is $\| \psi_{b,a} \|^2 = \| \psi \|^2$

Now let us proceed to the effect of parameter 'a' in the time-frequency window of wavelet transform. Since the center and width of ψ are given by t^* and $2\,\Delta_\psi$ respectively. these quantities for $\psi_{b,a}$ can be seen to be $\left(b + a\ t^*\right)$ and $\left(2a\ \Delta_\psi\right)$ respectively.

Therefore, the time window is given by ,

$$[\ b + at^{*} - a\,\Delta_{\psi}\ ,\ b + at^{*} + a\,\Delta_{\psi}\ ].$$

We now proceed to determine the frequency window. We have

$$(W_{\psi}f)(b,a) = \langle f, \psi_{b,a}\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{\psi}_{b,a}\rangle \qquad \text{---(2.3.6)}$$

Also we see that

$$\frac{1}{2\pi}\hat{\psi}_{b,a}(\omega) = \frac{|\ a\ |^{-1/2}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,\psi\left(\frac{t-b}{a}\right)dt$$

$$= \frac{a|\ a\ |^{-1/2}}{2\pi}\ e^{-ib\omega}\,\hat{\psi}(a\omega) \qquad \text{---(2.3.7)}$$

Setting

$$\eta(\omega) = \hat{\psi}(\omega + \omega^{*}) \qquad \text{---(2.3.8)}$$

We can shift the center of the window function to the origin for convenience. Using (2.3.6) we get

$$(W_{\psi}f)(b,a) = \frac{a|\ a\ |^{-1/2}}{2\pi}\int_{-\infty}^{\infty} e^{ib\omega}\,\hat{f}(\omega)\ \eta(a\omega - \omega^{*})\,d\omega$$

width of $\eta(a\omega - \omega^{*})$ will be given by $\frac{1}{a}\ \Delta_{\hat{\psi}}$. Therefore, the frequency window in this case is given by

$$[\ \frac{\omega^{*}}{a} - \frac{1}{a}\,\Delta_{\hat{\psi}}\ ,\ \frac{\omega^{*}}{a} + \frac{1}{a}\,\Delta_{\hat{\psi}}\ ].$$

Hence the rectangular time - frequency window is given by

$$[b+at^{*} - a\Delta_{\psi}\ ,\ b+at^{*} + a\Delta_{\psi}\ ]\ \mathrm{X}\ [\frac{\omega^{*}}{a} - \frac{1}{a}\,\Delta_{\hat{\psi}}\ ,\ \frac{\omega^{*}}{a}+\frac{1}{a}\,\Delta_{\hat{\psi}}\ ].$$

--(2.3.9)

(2.4) Reconstruction

Theorem(2.3): Let $w \in L^2(\mathbb{R})$ be so chosen that $\| w \|_2 = 1$ and both w and $\hat{w}$ satisfies the window condition. Also let $W_{b,\omega}(t) = e^{i\omega t} w(t - b)$. Then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} < f , W_{b,\omega} > < \overline{g , W_{b,\omega}} > db\, d\omega = 2\pi < f \quad g >$$

Proof: For any $f \in L^2(\mathbb{R})$, let $\check{f}$ denote the inverse Fourier transform of f; that is, $\check{f}(x) = \frac{1}{2\pi} \hat{f}(-x)$. Then by the Parseval Identity,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} <f,W_{b,\omega}> < \overline{f , W_{b,\omega}} > db\, d\omega =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tilde{\mathcal{G}}_b f)(\omega) \, \overline{(\tilde{\mathcal{G}}_b f)(\omega)} \, db\, d\omega$$

$$= 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tilde{\mathcal{G}}_b f)^{\vee}(x) \, \overline{(\tilde{\mathcal{G}}_b f)^{\vee}(x)} \, dx\, db$$

$$= 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \, \overline{w(t - b)} \, \overline{g(t) \, \overline{w(t - b)}} \, dt\, db$$

$$= 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \, \overline{g(t)} \, | w(t - b) |^2 \, dt\, db$$

$$= 2\pi \int_{-\infty}^{\infty} f(t) \, \overline{g(t)} \left[\int_{-\infty}^{\infty} |w(t - b)|^2 \, db \right] dt$$

$$= 2\pi < f , g > \quad (\text{ Since } \| w \|_2 = 1)$$

Hence,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} < f , W_{b,\omega} > < \overline{f , W_{b,\omega}} > db\, d\omega = 2\pi < f \quad g >$$

Theorem(2.4): Let ψ be a basic wavelet which defines an IWT W_{ψ}. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\ (W_{\psi} f)(b,a)\ \overline{(W_{\psi} g)(b,a)}\ ]\ \frac{da}{a^2}\ db = C_{\psi} \langle f\ ,\ g\rangle$$

for all $f\ ,\ g \in L^2(\mathbb{R})$. Furthermore, for any $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ at which f is continuous,

$$f(x) = \frac{1}{C_{\psi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_{\psi} f)(b,a)\ \psi_{b;a}(x)\ \frac{da}{a^2}\ db$$

where $\psi_{b;a}(x)$ given by the equation (2.3.4).

<u>Proof</u>: In order to prove the theorem we use the following notation

$$F(x) = \hat{f}(x)\cdot\ \overline{\hat{\psi}(ax)}$$

$$G(x) = \hat{g}(x)\cdot\ \overline{\hat{\psi}(ax)}$$

We have,

$$\int_{-\infty}^{\infty} [\ (W_{\psi} f)(b,a)\ \overline{(W_{\psi} g)(b,a)}\ ]\ db =$$

$$=\int_{-\infty}^{\infty}\left[\ \int_{-\infty}^{\infty} |a|^{-1/2} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}dt\ \int_{-\infty}^{\infty} |a|^{-1/2}\overline{g(s)}\ \psi\left(\frac{s-b}{a}\right)ds\right]\ db$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty}\left[\ \overline{\int_{-\infty}^{\infty} \overline{f(t)}\ \psi\left(\frac{t-b}{a}\right)}\ dt\ \int_{-\infty}^{\infty} \overline{g(s)}\ \psi\left(\frac{s-b}{a}\right)\ ds\right]\ db$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty}\left[\ \frac{a}{2\pi}\ \overline{\int_{-\infty}^{\infty} \bar{\hat{f}}(x)\ e^{-ibx}\ \hat{\psi}(ax)\ dx}\ \right]\cdot$$

$$\cdot\left[\ \frac{a}{2\pi}\int_{-\infty}^{\infty} \bar{\hat{g}}(y)\ e^{-iby}\ \hat{\psi}(ay)\ dy\ \right]\ db$$

$$= \frac{a^2}{|a|}\int_{-\infty}^{\infty}\left[\ \frac{1}{2\pi}\ \overline{\int_{-\infty}^{\infty} e^{-ibx}\ \overline{F(x)}\ dx}\ \right]\cdot\left[\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iby}\ \overline{G(y)}\ dy\right]\ db$$

$$= \frac{a^2}{|a|} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \overline{\hat{\overline{F}}(b)} \right) \cdot \left(\frac{1}{2\pi} \hat{\overline{G}}(b) \right) db$$

$$= \frac{a^2}{2\pi |a|} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\overline{G}}(b)\ \overline{\hat{\overline{F}}(b)}\ db \right]$$

$$= \frac{a^2}{2\pi |a|} \int_{-\infty}^{\infty} \overline{G}(x)\ F(x)\ dx$$

Thus,

$$\int_{-\infty}^{\infty} [(W_\psi\ f)(b,a)\ \overline{(W_\psi\ g)(b,a)}\ ]db = \frac{a^2}{2\pi |a|} \int_{-\infty}^{\infty} \overline{G}(x)\ F(x)\ dx$$

Now integrate above equation from $-\infty$ to ∞ w. r. t. a

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\ (W_\psi\ f)(b,a)\ \overline{(W_\psi\ g)(b,a)}\ ]\ db\ \frac{da}{a^2} =$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi |a|} \int_{-\infty}^{\infty} \overline{G}(x)\ F(x)\ dx \right] da$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi |a|} \int_{-\infty}^{\infty} \overline{\hat{g}(x)\ \hat{\psi}(ax)}\ \ \hat{f}(x)\ \hat{\psi}(ax)\ dx \right] da$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\ \ \overline{\hat{g}(x)}\ \hat{f}(x) \left[\int_{-\infty}^{\infty} \frac{|\hat{\psi}(ax)|^2}{|a|}\ da \right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\ \ \overline{\hat{g}(x)}\ \hat{f}(x) \left[\int_{-\infty}^{\infty} \frac{|\hat{\psi}(y)|^2}{|y|}\ dy \right] dx$$

$$= C_\psi \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{g}(x)}\ \hat{f}(x)\ dx$$

$$= C_\psi < f , g >$$

(2.5) FRAMES AND FRAME BOUNDS:

<u>Definition</u>: A family of functions $\{ \phi_j \}_{j\in\mathbb{Z}}$ in a Hilbert space $\mathbb{H}$, is called a frame if there exists A and B ($0 < A < B < \infty$) so that, for all $f \in \mathbb{H}$

$$A \| f \|^2 \le \sum_{j\in\mathbb{Z}} |\langle f , \phi_j \rangle|^2 \le B \| f \|^2 \qquad \text{---(2.5.1)}$$

Here A and B are called frame bounds. If A = B then frame is called tight frame. In this case,

$$\sum_{j\in\mathbb{Z}} |\langle f , \phi_j \rangle|^2 = B \| f \|^2$$

<u>Definition</u>: If $\{ \phi_j \}_{j\in\mathbb{Z}}$ is a frame in $\mathbb{H}$, then the frame operator T is the linear operator from $\mathbb{H}$ to $\ell^2(\mathbb{Z})$ ($T : \mathbb{H} \longrightarrow \ell^2(\mathbb{Z})$), defined by $(Tf)_j = \langle f , \phi_j \rangle$

Since

$$\| Tf \| \le \sqrt{B}\ \| f \| \qquad \text{---(2.5.2)}$$

the operator T is clearly bounded and is called as "frame operator" associated with the frame $\{ \phi_j \}_{j\in\mathbb{Z}}$. The adjoint T^* of T is also an frame operator which maps from $\ell^2(\mathbb{Z})$ to $\mathbb{H}$ ($T^* : \ell^2(\mathbb{Z}) \longrightarrow \mathbb{H}$) and can be easily calculated

$$\langle T^{*}c\ ,\ f\rangle = \langle c\ ,Tf\rangle \qquad \text{where } c = \{\ c_j\ \}_{j\in\mathbb{Z}}$$

$$= \sum_{j\in\mathbb{Z}} c_j\ \overline{(Tf)}_j$$

$$= \sum_{j\in\mathbb{Z}} c_j\ \overline{\langle f\ ,\ \phi_j\rangle}$$

$$= \sum_{j\in\mathbb{Z}} c_j\ \langle \phi_j\ ,\ f\rangle$$

$$\langle T^{*}c\ ,\ f\rangle = \sum_{j\in\mathbb{Z}} \langle c_j\ \phi_j\ ,\ f\rangle$$

$$\langle T^{*}c\ ,\ f\rangle - \sum_{j\in\mathbb{Z}} \langle c_j\ \phi_j\ ,\ f\rangle = 0$$

$$\Rightarrow \quad \langle T^{*}c\ ,\ f\rangle - \langle \sum_{j\in\mathbb{Z}} c_j\ \phi_j\ ,\ f\rangle = 0$$

$$\Rightarrow \quad \left\langle \left[\ T^{*}c - \sum_{j\in\mathbb{Z}} c_j\ \phi_j\ \right]\ ,\ f\right\rangle = 0$$

$$\Rightarrow \quad T^{*}c - \sum_{j\in\mathbb{Z}} c_j\ \phi_j = 0$$

Therefore,

$$T^{*}c = \sum_{j\in\mathbb{Z}} c_j\ \phi_j \qquad \text{---(2.5.3)}$$

Thus T^{*} is an adjoint operator defined by $T^{*}c = \sum_{j\in\mathbb{Z}} c_j\ \phi_j$

where $c = \{\ c_j\ \}_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $\{\ \phi_j\ \}_{j\in\mathbb{Z}} \in \mathbb{H}$

Since T^{*} is adjoint operator $\|\ T^{*}\ \| = \|\ T\ \|$

Therefore, $\quad \|\ T^{*}c\ \| \leq \sqrt{B}\ \|\ c\ \|$ (by (2.5.2))

And hence T^{*} is also a bounded operator.

We have

$$\sum_{j\in\mathbb{Z}} |\langle f , \phi_j\rangle|^2 = \| Tf \|^2$$

$$= \langle Tf , Tf\rangle$$

$$= \langle T^* Tf , f\rangle$$

$$\sum_{j\in\mathbb{Z}} |\langle f , \phi_j\rangle|^2 = \langle T^* Tf , f\rangle \qquad \text{---}(2.5.4)$$

Thus by (2.5.1) and (2.5.4)

$$A \| f \|^2 \leq \langle T^* Tf , f \rangle \leq B \| f \|^2 \qquad \text{---}(2.5.4a)$$

Now we recall an important result

Result(1): If G is Hermitian operator on $\mathbb{H}$ such that $\langle G^* Gf , f \rangle \geq 0$ for all $f \in \mathbb{H}$, then all the eigenvalues of G are necessarily nonnegative. We then say that the operator G itself is nonnegative and write this as an operator inequality $G \geq 0$.

Now since inequality (2.5.4a) can be written as,

$$A \langle f , f \rangle \leq \langle T^* Tf , f \rangle \leq B \langle f , f \rangle$$

that is

$$\langle T^* Tf , f \rangle - A \langle f , f \rangle \geq 0$$

$$\Rightarrow \quad \langle [T^* Tf - Af] , f \rangle \geq 0$$

$$\Rightarrow \quad \langle [T^* T - A\, Id] \cdot f , f \rangle \geq 0$$

Using above result(1), we have

$$T^* T - A\, Id \geq 0 \Rightarrow T^* T \geq A\, Id \qquad \text{---}(a)$$

Similarly,

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$$B \langle f , f \rangle - \langle T^* T f , f \rangle \geq 0$$

$$\langle [B\, Id - T^* T] \cdot f , f \rangle \geq 0$$

Using above result(1), we have

$$B\, Id - T^* T \geq 0 \Rightarrow B\, Id \geq T^* T \qquad \text{---(b)}$$

Combining (a) and (b) we have

$$A\, Id \leq T^* T \leq B\, Id \qquad \text{---(2.5.5)}$$

Inequality (2.5.5) shows that ($T^* T$) is inversible, by result (2)

$$B^{-1}\, Id \leq (T^* T)^{-1} \leq A^{-1}\, Id \qquad \text{---(2.5.6)}$$

Result(2): If a positive bounded linear operator T on $\mathbb{H}$ is bounded below by a strictly positive constant α, then T is inversible and its inverse T^{-1} is bounded by α^{-1}

Applying the operator $(T^* T)^{-1}$ to the vector $\{\phi_j\}_{j \in \mathbb{Z}}$ gives a new family of vectors, which is denoted by $\tilde{\phi}_j$

$$\tilde{\phi}_j = (T^* T)^{-1} \phi_j \qquad \text{---(2.5.7)}$$

Then the family $\{\tilde{\phi}_j\}_{j \in \mathbb{Z}}$ constitute another frame, More precisely,

I) The family $\{\tilde{\phi}_j\}_{j \in \mathbb{Z}}$ with $\tilde{\phi}_j = (T^* T)^{-1} \phi_j$ constitute frame with bounds B^{-1} and A^{-1}

Proof: For any $f \in \mathbb{H}$ we have,

$$\langle f , \tilde{\phi}_j \rangle = \langle f , (T^* T)^{-1} \phi_j \rangle$$

$$= \langle [(T^* T)^{-1}]^* f , \phi_j \rangle$$

$$\langle f , \tilde{\phi}_j \rangle = \langle [(T^* T)^*]^{-1} f , \phi_j \rangle$$

$$\langle f , \tilde{\phi}_j \rangle = \langle (T^* T)^{-1} f , \phi_j \rangle$$

Hence

$$\sum_{j \in \mathbb{Z}} | \langle f , \tilde{\phi}_j \rangle |^2 = \sum_{j \in \mathbb{Z}} | \langle (T^* T)^{-1} f , \phi_j \rangle |^2$$

$$= \sum_{j \in \mathbb{Z}} | (T(T^* T)^{-1} f)_j |^2 \quad (\text{by } (Tf)_j = \langle f , \phi_j \rangle)$$

$$= \| T(T^* T)^{-1} f \|^2$$

$$= \langle T(T^* T)^{-1} f , T(T^* T)^{-1} f \rangle$$

$$= \langle (T^* T)^{-1} f , (T^* T)(T^* T)^{-1} f \rangle$$

$$\sum_{j \in \mathbb{Z}} | \langle f , \tilde{\phi}_j \rangle |^2 = \langle (T^* T)^{-1} f , f \rangle \qquad \text{---(2.5.8)}$$

From (2.5.4a), (2.5.6) and (2.5.8)

$$B^{-1} \| f \|^2 \le \sum_{j \in \mathbb{Z}} |\langle f , \tilde{\phi}_j \rangle|^2 \le A^{-1} \| f \|^2 \qquad \text{---(2.5.9)}$$

which shows that the family $\{\tilde{\phi}_j\}_{j \in \mathbb{Z}}$ constitute a frame, with frame bounds B^{-1} and A^{-1}.

II) Associated operator $\tilde{T}$ to the frame $\{\tilde{\phi}_j\}_{j \in \mathbb{Z}}$ is given by

i) $\tilde{T} = T(T^* T)^{-1}$ and it satisfies

ii) $\tilde{T}^* \tilde{T} = (T^* T)^{-1}$

iii) $\tilde{T}^* T = Id = (T^* \tilde{T})$ ---(2.5.10)

<u>Proof</u>: We have the definition of $\tilde{\phi}_j = (T^* T)^{-1} \phi_j$

Also, We have definition of $(Tf)_j = \langle f , \phi_j \rangle$

By applying these definition,

$$(\tilde{T}f)_j = \langle f , \tilde{\phi}_j \rangle$$

$$= \langle f , (T^*T)^{-1}\phi_j \rangle$$

$$= \langle (T^*T)^{-1} f , \phi_j \rangle \quad (\text{Since } (Tf)_j = \langle f , \phi_j \rangle)$$

$$= (T(T^*T)^{-1}f)_j$$

$$\tilde{T} = T(T^*T)^{-1} \qquad \text{---(i)}$$

Now

$$\tilde{T}^* \tilde{T} = \left[T(T^*T)^{-1} \right]^* \left[T(T^*T)^{-1} \right]$$

$$= \left[(T^*T)^{-1}T^* \right] \left[T(T^*T)^{-1} \right]$$

$$= \left[(T^*T)^{-1}(T^*T) \right] \left[(T^*T)^{-1} \right]$$

$$= \left[Id \right] \left[(T^*T)^{-1} \right]$$

$$\tilde{T}^* \tilde{T} = (T^*T)^{-1} \qquad \text{---(ii)}$$

Finally,

$$\tilde{T}^* T = \left[T(T^*T)^{-1} \right]^* T$$

$$= \left[(T^*T)^{-1} T^* \right] T$$

$$= \left[(T^*T)^{-1}(T^*T) \right]$$

$$\tilde{T}^* T = Id$$

$$T^* \tilde{T} = T^* \left[T(T^*T)^{-1} \right]$$

$$= (T^* T)(T^*T)^{-1}$$

$$T^* \tilde{T} = Id$$

Therefore,

$$\tilde{T}^{*} T = T^{*} \tilde{T} \qquad \text{---(iii)}$$

III) $\tilde{T}T^{*} = T\tilde{T}^{*}$ is the orthogonal projection operator in $\ell^2(\mathbb{Z})$ on the range of T

Proof: $\tilde{T}T^{*} = T(T^{*}T)^{-1} T^{*} \qquad \text{---(iv)}$

$$T\tilde{T}^{*} = T \left[T \, (T^{*}T)^{-1} \right]^{*}$$

$$T\tilde{T}^{*} = T(T^{*}T)^{-1} T^{*} \qquad \text{---(v)}$$

From equation (iv) and (v), we have,

$$\tilde{T}T^{*} = T\tilde{T}^{*}$$

Now we have to only prove that,

i) $(\tilde{T} \, T^{*})(c) = c$ where c is in Ran(T) and

ii) $(\tilde{T} \, T^{*})(c) = 0$ for all c orthogonal to Ran(T)

Proof:

i) $(\tilde{T} \, T^{*})(c) = (\tilde{T} \, T^{*})(Tf)$ where $c \in \mathrm{Ran}(T)$

$$= \left[T(T^{*}T)^{-1}T^{*} \right](Tf)$$

$$= \left[T(T^{*}T)^{-1}(T^{*}T) \right] f$$

$$= Tf$$

$$(\tilde{T} \, T^{*})(c) = c$$

ii) $c \perp \mathrm{Ran}(T) \Rightarrow \langle c \, , Tf \rangle = 0$ for all $f \in \mathbb{H}$

$\Rightarrow \langle T^{*}c \, , f \rangle = 0$ for all $f \in \mathbb{H}$

$\Rightarrow T^{*}c = 0$

$\Rightarrow (\tilde{T} \, T^{*})(c) = 0$

The operation $\{ \phi_j : j \in \mathbb{Z} \} \longrightarrow \{\tilde{\phi}_j : j \in \mathbb{Z} \}$ defines, in a sense, a duality operation. The same procedure applied to the frame $\{\tilde{\phi}_j : j \in \mathbb{Z} \}$ gives the original frame $\{\phi_j : j \in \mathbb{Z} \}$ back again. We shall therefore, call $\{\tilde{\phi}_j : j \in \mathbb{Z} \}$ the dual frame of $\{\phi_j : j \in \mathbb{Z} \}$. The duality

$\phi_j \longleftrightarrow \tilde{\phi}_j$ is also expressed by

$$\tilde{T}^* T = Id = T^* \tilde{T}$$

that is $(\tilde{T}^* T)f = f$

$$\Rightarrow \tilde{T}^* (Tf) = f$$

$$\Rightarrow \tilde{T}^* \Big(\sum_{j\in\mathbb{Z}} \langle f , \phi_j\rangle\Big) = f$$

$$\Rightarrow \sum_{j\in\mathbb{Z}} \langle f , \phi_j\rangle \tilde{\phi}_j = f$$

OR

$$(T^* \tilde{T})f = f$$

$$\Rightarrow T^*(\tilde{T} f) = f$$

$$\Rightarrow T^* \Big(\sum_{j\in\mathbb{Z}} \langle f , \tilde{\phi}_j\rangle \Big) = f$$

$$\Rightarrow \sum_{j\in\mathbb{Z}} \langle f , \tilde{\phi}_j\rangle \phi_j = f$$

That is

$$\sum_{j\in\mathbb{Z}} \langle f , \phi_j\rangle \tilde{\phi}_j = f = \sum_{j\in\mathbb{Z}} \langle f , \tilde{\phi}_j\rangle \phi_j \qquad \text{---(2.5.11)}$$

OR

$$\langle f , g\rangle = \sum_{j\in\mathbb{Z}} \langle f , \phi_j\rangle \langle\tilde{\phi}_j , g\rangle \qquad \text{---(2.5.12)}$$

This mean that we have a reconstruction formula for f from the $\langle f , \phi_j \rangle$. At the same time we have also obtained a recipe for writting f as a superposition of ϕ_j

We introduce new notation for T^*T operator

$$\mathbb{T} = T^*T \qquad \text{and}$$

$$\begin{aligned}
\tilde{\mathbb{T}} &= \tilde{T}^* \tilde{T} \\
&= \tilde{T}^* [\, T(T^*T)^{-1}] \\
&= [\, T(T^*T)^{-1}]^* [\, T(T^*T)^{-1}] \\
&= [\, (T^*T)^{-1} T^*] [\, T(T^*T)^{-1}] \\
&= (T^*T)^{-1} (T^*T) (T^*T)^{-1} \\
&= (T^*T)^{-1} \\
&= \mathbb{T}^{-1}
\end{aligned}$$

Thus

$$\tilde{\mathbb{T}} = \mathbb{T}^{-1}$$

In particular,

$$\mathbb{T} = T^*T = \sum_{j \in \mathbb{Z}} \langle \cdot , \phi_j \rangle \phi_j$$

$$\mathbb{T}(f) = \sum_{j \in \mathbb{Z}} \langle f , \phi_j \rangle \phi_j$$

By inequality (2.5.5)

$$A\, Id \le \mathbb{T} \le B\, Id \qquad \text{---(2.5.13)}$$

If the elements $f \in \mathbb{H}$ are characterized by mean of the inner product $\left\{ \langle f , \phi_j \rangle : j \in \mathbb{Z} \right\}$ then f can be reconstructed from

$$f = \sum_{j\in\mathbb{Z}} \langle \phi_j , f \rangle \tilde{\phi}_j$$

The vectors $\tilde{\phi}_j$ are defined by $\tilde{\phi}_j = (T^* T)^{-1} \phi_j = \mathbb{T}^{-1} \phi_j$.

If the frame contains "more" vectors then a basis would, there exists other vectors in $\mathbb{H}$ that could equally well play the role of the $\tilde{\phi}_j$ and lead to a reconstruction formula. This is due to the fact that the ϕ_j are not linearly independent in the general case. This phenomenon can be illustrated with the following example with $\mathbb{H} = \mathbb{R}^2$.

Define

$$\phi_1 = e_1, \quad \phi_2 = -\frac{1}{2} e_1 + \frac{\sqrt{3}}{2} e_2 \quad \phi_3 = -\frac{1}{2} e_1 - \frac{\sqrt{3}}{2} e_2$$

where $e_1 = (1,0)$, $e_2 = (0,1)$ constitute the standard orthonormal basis in $\mathbb{R}^2$ for $u \in \mathbb{H}$

$$\sum_{j=1}^{3} |\langle u , \phi_j \rangle|^2 = |\langle u , \phi_1 \rangle|^2 + |\langle u , \phi_2 \rangle|^2 + |\langle u , \phi_3 \rangle|^2 \qquad \text{---(2.5.14)}$$

Now

$$|\langle u , \phi_1 \rangle|^2 = \langle u , \phi_1 \rangle \cdot \langle u , \phi_1 \rangle$$

$$= u_1 \cdot u_1$$

$$|\langle u , \phi_1 \rangle|^2 = | u_1 |^2 \qquad \text{---(2.5.15)}$$

$$|\langle u , \phi_2 \rangle|^2 = \langle u , \phi_2 \rangle \cdot \langle u , \phi_2 \rangle$$

$$= \left(-\frac{1}{2} e_1 + \frac{\sqrt{3}}{2} e_2 \right) \cdot \left(-\frac{1}{2} e_1 + \frac{\sqrt{3}}{2} e_2 \right)$$

$$|\langle u , \phi_2 \rangle|^2 = \frac{1}{4} |u_1|^2 + \frac{3}{4} |u_2|^2 + \frac{\sqrt{3}}{4} u_1 \cdot u_2 - \frac{\sqrt{3}}{4} u_2 \cdot u_1$$

$$|\langle u , \phi_2 \rangle|^2 = \frac{1}{4} |u_1|^2 + \frac{3}{4} |u_2|^2 \qquad ---(2.5.16)$$

$$|\langle u , \phi_3 \rangle|^2 = \langle u , \phi_3 \rangle \cdot \langle u , \phi_3 \rangle$$

$$|\langle u , \phi_3 \rangle|^2 = \frac{1}{4} |u_1|^2 + \frac{3}{4} |u_2|^2 \qquad ---(2.5.17)$$

From equations (2.5.14),(2.5.15),(2.5.16) and (2.5.17)

$$\begin{aligned}\sum_{j=1}^{3} |\langle u , \phi_j \rangle|^2 &= | u_1 |^2 + \frac{1}{4} |u_1|^2 + \frac{3}{4} |u_2|^2 + \\ &\quad + \frac{1}{4} |u_1|^2 + \frac{3}{4} |u_2|^2 \\ &= \frac{3}{2} \left[|u_1|^2 + |u_2|^2 \right]\end{aligned}$$

$$\sum_{j=1}^{3} |\langle u , \phi_j \rangle|^2 = \frac{3}{2} \| u \|^2$$

This implies that $\{ e_1, e_2, e_3 \}$ is a tight frame, but definitely not an ortonormal basis; the three vectors e_1, e_2, e_3 are clearly not linearly independent.

So that $\mathbb{T} = \frac{3}{2} \mathrm{Id}$

hence

$$\tilde{\phi}_j = \mathbb{T}^{-1} \phi_j = \frac{3}{2} \phi_j \quad \text{and}$$

$$u = \frac{2}{3} \sum_{j=1}^{3} \langle u , \phi_j \rangle \phi_j$$

Since

$$\sum_{j=1}^{3} \phi_j = \phi_1 + \phi_2 + \phi_3$$

$$\sum_{j=1}^{3} \phi_j = (1,0) + (-\frac{1}{2}, \frac{\sqrt{3}}{2}) + (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

$$= (0, 0)$$

$$\sum_{j=1}^{3} \phi_j = \overline{0}$$

It is clear that for any choice of $a \in \mathbb{H}$ an equality valid reconstruction formula is given by

$$u = \frac{2}{3} \sum_{j=1}^{3} \langle u, \phi_j \rangle (\phi_j + a)$$

For a = 0, corresponding to

$$u = \frac{2}{3} \sum_{j=1}^{3} \langle u, \phi_j \rangle \phi_j$$

is the "minimal solution" in the sense that the image $\mathbb{R}^3$ to $\mathbb{R}^2$ under the frame operator T is the two dimensional subspace with equation $x_1 + x_2 + x_3 = 0$. We denote this subspace by Ran(T) vectors in $\mathbb{R}^3$ orthogonal to Ran(T) are all of the type $c = \lambda(1,1,1)$ when the components of such a vector are substituted for the $\langle u, \phi_j \rangle$ in (2.5.11) then the reconstruction leads to zero. Since

$$\sum_{j=1}^{3} \tilde{\phi}_j c_j = \frac{2}{3} \lambda \sum_{j=1}^{3} \phi_j = 0$$