Chapter - 3

CARDINAL SPLINES

3.1) Cardinal Splines Spaces

3.2) B-Spline and their Properties

3.3) Two-scale Relation and Interpolatory Graphical Display Algorithm

CHAPTER NO. 3

CARDINAL SPLINE ANALYSIS

Introduction:

This chapter is devoted to the study of cardinal spline functions with emphasis on their basic properties. At the end of this chapter we develop "two-scale relation" for the cardinal splines of order m. Finally we develop an interpolatory graphical display algorithm.

(3.1) Cardinal Spline Spaces:

Notations:

Π_n: The collection of all algebraic polynomials of degree at most n

C^n: The collection of all functions f such that $f, f', f^{(2)}, f^{(3)}, \ldots, f^{(n)}$ are continuous everywhere with $C = C^o$ and C^{-1} is the space of piece wise continuous function.

Definition: For each positive integer m, the space S_m of cardinal splines of order m and with knot sequence $\mathbb{Z}$ is the collection of all functions $f \in C^{m-2}$ such that the restriction of f to any interval $[k, k+1)$, $k \in \mathbb{Z}$ are in Π_{m-1} that is,

$$f\Big|_{[k,k+1)} \in \Pi_{m-1}, \quad k \in \mathbb{Z}$$

S_1: The space of piece wise constant functions. The basis for S_1 can be $\{ N_1(x-k) : k \in \mathbb{Z} \}$ where N_1 is characteristics function of [0,1) defined by

$$N_1(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{---(3.1.1)}$$

To get basis for S_m: $m \ge 2$, we consider the space $S_{m;N}$ consisting of restriction of functions $f \in S_m$ to the interval $[-N,N]$, where N is any positive integer. That is $S_{m;N}$ is a subspace of functions $f \in S_m$ such that the restriction

$$f\Big|_{(-\infty,-N+1)} \quad \text{and} \quad f\Big|_{[N-1\ ,\ \infty)} \quad \text{of } f \text{ are in } \Pi_{m-1}$$

Setting $P_{m,j} = f\Big|_{[j,j+1)} \in \Pi_{m-1}$, $j = -N,\dots,N-1$ then

since $f \in C^{m-2}$ we have

$$\left(P^{(k)}_{m,j} - P^{(k)}_{m,j-1} \right)(j) = 0 \text{ for } k = 0,1,2,\dots,m-2 \ (m \ge 2)$$

The jumps C_j of f^{m-1} at the knot sequence $\mathbb{Z}$ are then given by

$$C_j = P^{(m-1)}_{m,j}(j + 0) - P^{(m-1)}_{m,j-1}(j - 0) \qquad \text{---(3.1.2)}$$

$$= \lim_{\varepsilon \to 0+} \left[f^{(m-1)}(j + \varepsilon) - f^{(m-1)}(j - \varepsilon) \right],$$

The adjacent polynomial pieces of f are related by

$$P_{m,j}(x) = P_{m,j-1}(x) + \frac{C_j}{(m-1)!}(x - j)^{m-1} \qquad \text{---(3.1.3)}$$

We introduce the new notation

$$x_+ = \max(x,0)$$

$$x^m_+ = (x_+)^m \qquad \text{for all } m \ge 1 \qquad \text{---(3.1.4)}$$

Therefore,

$$f(x) = f\Big|_{[-N,-N+1)}(x) + \sum_{j=-N+1}^{N-1} \frac{C_j}{(m-1)!}(x - j)_+^{m-1}$$

$$\text{for all } x \in [-N,N] \qquad ---(3.1.5)$$

This equation (3.1.5) is true for all $f \in S_{m;N}$ with constant C_j given by equation (3.1.2). Therefore, the collection

$$\{ 1,x,x^2,\ldots,x^{m-1},(x+N-1)_+^{m-1},\ldots,(x-N+1)_+^{m-1} \} \quad ---(3.1.6)$$

of (m + 2N - 1) functions is a basis of $S_{m,N}$. This collection consist of both monomials and truncated powers. We can replace monomials 1, x, x^2, ..., x^{m-1} by the truncated powers.

$$(x + N + m - 1)_+^{m-1}, \ \ldots, \ (x + N)_+^{m-1} \qquad ---(3.1.7)$$

Therefore, the following set of truncated powers, which are generated by using integer translates of a single function x_+^{m-1}, is also a basis of $S_{m;N}$

$$\{ (x - K)_+^{m-1} : k = -N - m + 1, \cdots , N - 1 \} \quad ---(3.1.8)$$

This basis is more powerful than (3.1.6) because,

i) Each function $(x - j)_+^{m-1}$ vanishes to the left of j

ii) All the basis in (3.1.8) are generated by a single function x_+^{m-1} which is independent of N

iii) Finally

$$S_m = \bigcup_{N=1}^{\infty} S_{m;N}$$

It follows from (iii) that the basis in (3.1.8) can be extended to a "basis" τ of the infinite dimensional space S_m

$$\tau = \left\{ (x - k)_+^{m-1} : k \in \mathbb{Z} \right\} \qquad ---(3.1.9)$$

Unfortunately, there is not a single function in τ that belongs to $L^2(\mathbb{R})$ as each $(x - k)_+^{m-1} \longrightarrow \infty$ as $x \longrightarrow \infty$

We therefore, have to create functions in $L^2(\mathbb{R})$ form those in τ_N, which can be done by controlling their growth. Since, in vector space, finite linear combination is the only operation, we use "differences" instead of derivatives in tamping polynomials growth.

Definition: Backward differences are defined recursively

$$(\Delta f)(x) = f(x) - f(x - 1)$$
$$(\Delta^n f)(x) = (\Delta^{n-1}(\Delta f))(x) \qquad ---(3.1.10)$$

where $f \in \Pi_{m-1}$

Clearly

$$\Delta^m f = 0 \qquad ---(3.1.11)$$

Definition: Let $M_1 = N_1$, where N_1 be characteristic function of $[0,1)$ defined as in (3.1.1) and for $m \geq 2$.

Let

$$M_m(x) = \frac{1}{(m - 1)!} \Delta^m x_+^{m-1} \qquad ---(3.1.12)$$

Since

$$\Delta^2 x_+^1 = x_+^1 - 2(x-1)_+^1 + (x-2)_+^1$$

$$= \sum_{k=0}^{2} (-1)^k\ {}^2C_k\ (x-k)_+^1$$

$$\Delta^3 x_+^2 = x_+^2 - 3(x-1)_+^2 + 3(x-2)_+^2 - (x-3)_+^2$$

$$= \sum_{k=0}^{3} (-1)^k\ {}^3C_k\ (x-k)_+^2$$

In general

$$\Delta^m x_+^{m-1} = \sum_{k=0}^{m} (-1)^k\ {}^mC_k\ (x-k)_+^{m-1}$$

Therefore,

$$M_m(x) = \frac{1}{(m-1)!} \sum_{k=0}^{m} (-1)^k\ {}^mC_k\ (x-k)_+^{m-1} \qquad \text{---(3.1.13)}$$

$M_m(x) = 0$ for all $x \geq m$ and $M_m(x) = 0$ for all $x < 0$

Therefore, we have Supp $M_m \subseteq [0,m]$

Moreover, we can show that Supp $M_m = [0,m]$ ---(3.1.14)

Since M_m has compact support, $M_m(x) \in L^2(\mathbb{R})$. We now show that

$$B = \{ M_m(x-k) : k \in \mathbb{Z} \} \qquad \text{---(3.1.15)}$$

is a basis for S_m.

For instant, consider $S_{m;N}$,the dimension of $S_{m;N}$ is $(n + 2N - 1)$. Since Supp $M_m = [0,m]$, we see that each function in the collection

$$\{ M_m(x-k) : k = -N - m + 1, \ldots , N - 1 \} \qquad \text{---(3.1.16)}$$

is non-trivial on $[-N,N]$ and $M_m(x-k) = 0$ on $[-N,N]$ for

$k < (-N - m + 1)$ or $k > (N - 1)$.

Since the functions in the set (3.1.16) are linearly independent they form basis for $S_{m;N}$. Thus we have an another set of basis function for $S_{m;N}$. If we take union of these basis in (3.1.16) for N = 1,2,3,... we get B in (3.1.15) as a basis for S_m

Therefore,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k M_m(x - k) \qquad ---(3.1.17)$$

Since M_m has compact support, all except finite number of terms in (3.1.17) are zero and therefore RHS sum of (3.1.17) is convergent.

We are interested in only those cardinal splines that belong to $L^2(\mathbb{R})$, namely $S_m \cap L^2(\mathbb{R})$. Let V_0^m denote its closure in $L^2(\mathbb{R})$, that is, $\overline{S_m \cap L^2(\mathbb{R})} = V_0^m$. Observe that $B \subseteq V_0^m$. In fact B is a Riesz basis of V_0^m

The cardinal splines we have considered so far have the knot sequence $\mathbb{Z}$. If we consider the knot sequence $2^j\cdot\mathbb{Z}$, then the corresponding space of spline functions is denoted by S_m^j, since for $j_1 < j_2$ we have $2^{j_1}\cdot\mathbb{Z} \subset 2^{j_2}\cdot\mathbb{Z}$, we have $S_m^{j_1} \subset S_m^{j_2}$. Thus we have doubly infinite nested sequence.

$$\ldots \subset S_m^{-1} \subset S_m^{0} \subset S_m^{1} \ldots$$

of cardinal splines where $S_m^o = S_m$. Analogous to definition of

V_0^m, we let V_j^m denote the $L^2(\mathbb{R})$-closure of $S_m^j \cap L^2(\mathbb{R})$. Hence the nested sequence

$$\cdots \subset V_{-1}^m \subset V_0^m \subset V_1^m \subset \cdots \qquad ---(3.1.18)$$

of closed cardinal spline subspaces of $L^2(\mathbb{R})$. Then we have

$$\left.\begin{aligned} \left(\overline{\bigcup_{j\in\mathbb{Z}} V_j^m} \right) &= L^2(\mathbb{R}) \\ \bigcap_{j\in\mathbb{Z}} V_j^m &= \{0\} \end{aligned}\right\} \qquad ---(3.1.19)$$

Also, if B is a Riesz basis of V_0^m then for any $j \in \mathbb{Z}$ the collection

$$\left\{ 2^{j/2} M_m(2^j x - k) : k \in \mathbb{Z} \right\} \qquad ---(3.1.20)$$

is also Reisz basis of V_j^m with the same Riesz bounds.

(3.2) B-Splines and their properties:

Definition: The m^{th} order cardinal B-Spline $N_m(x)$ is defined by

$$N_m(x) = (N_{m-1} * N_1)(x) \quad \text{for all } m \geq 2$$

$$N_m(x) = \int_{-\infty}^{\infty} N_{m-1}(x - y)\, N_1(y)\, dy$$

$$N_m(x) = \int_0^1 N_{m-1}(x - y)\, dy \quad \text{for all } m \geq 2 \qquad ---(3.2.1)$$

In the definition of $M_m(x)$ we set $M_1 = N_1$, we can prove that $M_m(x) = N_m(x)$ for all x.

The m^{th} order cardinal B-Spline N_m satisfies the properties:

Property (1): For any $f \in C$,

$$\int_{-\infty}^{\infty} f(x)\ N_m(x)\ dx =$$

$$= \int_0^1 \cdots \int_0^1 f(x_1+x_2+\ldots+x_m)dx_1 dx_2 \ldots dx_m \qquad ---(3.2.2)$$

Proof:We prove this result by method of mathematical induction

Step(1) We show that result is true for m = 1

$$\text{L.H.S} = \int_{-\infty}^{\infty} f(x)\ N_1(x)\ dx$$

$$= \int_0^1 f(x)\ dx$$

$$= \int_0^1 f(x_1)\ dx_1$$

$$= \text{R.H.S.}$$

Step(2) Let us assume that result is true for m-1 that is,

$$\int_{-\infty}^{\infty} f(x)\ N_{m-1}(x)\ dx = \int_0^1 \cdots \int_0^1 f(x_1+x_2+\ldots+x_{m-1})dx_1 dx_2 \ldots dx_{m-1}$$

Step(3) We prove that result is true for m

$$\int_{-\infty}^{\infty} f(x)\ N_m(x)\ dx = \int_{-\infty}^{\infty} f(x)\left[\int_0^1 N_{m-1}(x - t)\ dt\right] dx$$

$$= \int_0^1 \left[\int_{-\infty}^{\infty} f(x)\ N_{m-1}(x - t)\ dx\right] dt$$

Put $y = x - t \Rightarrow dy = dx$

$$\int_{-\infty}^{\infty} f(x)\ N_m(x)\ dx = \int_0^1 \left[\int_{-\infty}^{\infty} f(y + t)\ N_{m-1}(y)\ dy \right] dt$$

$$= \int_0^1 \left[\int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \ldots + x_{m-1} + t) dx_1 dx_2 \ldots dx_{m-1} \right] dt$$

$$\int_{-\infty}^{\infty} f(x)\ N_m(x)\ dx = \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \ldots + x_m) dx_1 dx_2 \ldots dx_m$$

Property (2): For any $g \in C^m$,

$$\int_{-\infty}^{\infty} g^{(m)}(x)\ N_m(x)\ dx = \sum_{k=0}^{m} (-1)^{m-k}\ {}^mC_k\ g(k) \qquad ---(3.2.3)$$

Proof: Since $g \in C^m \Rightarrow g^{(m)} \in C$ therefore, by applying above Property(1)

$$\int_{-\infty}^{\infty} g^{(m)}(x)\ N_m(x)\ dx = \int_0^1 \cdots \int_0^1 g^{(m)}(x_1 + x_2 + \ldots + x_m) dx_1 dx_2 \ldots dx_m$$

For $m = 1$,

$$\int_{-\infty}^{\infty} g'(x)\ N_m(x)\ dx = \int_0^1 g'(x_1) dx_1$$

$$= [g(x)]_0^1$$

$$= g(1) - g(0)$$

$$= \sum_{k=0}^{1} (-1)^{1-k}\ {}^1C_k\ g(k)$$

Thus by direct integration we get the required property

$$\int_{-\infty}^{\infty} g^{(m)}(x)\ N_m(x)\ dx = \sum_{k=0}^{m} (-1)^{m-k}\ {}^mC_k\ g(k)$$

Property (3): $N_m(x) = M_m(x)$ for all $x \in \mathbb{R}$

Proof: Fix $x \in \mathbb{R}$, By selecting

$$g(t) = \frac{(-1)^m}{(m-1)!}(x-t)_+^{m-1}$$

$$\therefore \quad g'(t) = \frac{(-1)^{m-1}}{(m-2)!}(x-t)_+^{m-2}$$

$$g''(t) = \frac{(-1)^{m-2}}{(m-3)!}(x-t)_+^{m-3}$$

and so on

$$g^{(m-1)}(t) = \frac{(-1)}{0!}(x-t)_+^{0}$$

$$g^{(m)}(t) = \delta(x-t) \qquad \text{---}(i)$$

where δ is delta distribution defined as

$\delta(x) = 0$ for all $x \neq 0$ and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Now by Property (2)

$$\int_{-\infty}^{\infty} g^{(m)}(x)\, N_m(x)\, dx = \sum_{k=0}^{m} (-1)^{m-k}\, {}^mC_k\, g(k)$$

$$= \sum_{k=0}^{m} (-1)^{m-k}\, {}^mC_k\, \frac{(-1)^m}{(m-1)!}(x-t)_+^{m-1}$$

$$= \frac{1}{(m-1)!}\sum_{k=0}^{m} (-1)^{k}\, {}^mC_k\, (x-t)_+^{m-1}$$

$$= M_m(x) \qquad \text{---}(ii)$$

And by (i)

$$\int_{-\infty}^{\infty} g^{(m)}(x)\, N_m(x)\, dx = \int_{-\infty}^{\infty} \delta(x)\, N_m(x)\, dx$$

$$\int_{-\infty}^{\infty} g^{(m)}(x)\, N_m(x)\, dx = N_m(x) \qquad ---(iii)$$

Finally, from (*ii*) and (*iii*) we have the required property

$$N_m(x) = M_m(x) \text{ for all } x \in \mathbb{R}$$

Property (4): Supp $N_m = [0\ ,\ m]$

Proof: We prove this result by method of mathematical induction

Step(1) We show that result is true for m = 1

The assertion is clearly true for m = 1 by the definition of N_1

Step(2) Let us assume that result is true for m-1 that is,

$$\text{Supp } N_{m-1} = [0\ ,\ m-1]$$

Step(3) We prove that result is true for m

Now we have

$$N_m(x) = \int_{-\infty}^{\infty} N_{m-1}(x - t)\, N_1(t)\, dt$$

$$= \int_0^1 N_{m-1}(x - t)\, dt$$

Since Supp $N_{m-1} = [0\ ,\ m-1]$

$$N_{m-1}(x - t) \neq 0 \text{ for all } 0 \le x - t \le m - 1$$

Let $x - t = y \Rightarrow -dt = dy$

$$N_m(x) = -\int_x^{x-1} N_{m-1}(y)\, dy$$

$$= \int_{x-1}^{x} N_{m-1}(y)\, dy$$

That is $x - 1 \leq y \leq m - 1 \leq x \leq m \Rightarrow N_{m-1}(y) \neq 0$

when $y = m - 1$, x will be m or $N_m(x) \neq 0$ for $0 \leq x \leq m$

Thus Supp $N_m = [0\ ,\ m]$

Hence by method of principle of mathematical induction result is true for all m

Property (5): $N_m(x) > 0$ for all $0 < x < m$

Proof: We prove this result by method of mathematical induction

Step(1) We show that result is true for m = 1

By the definition, $N_1(x) = 1 > 0 \quad 0 \leq x < 1$, therefore result holds for m = 1

Step(2) Let us assume that result is true for m-1, that is,

$$N_{m-1}(x) > 0$$

Step(3) We prove that result is true for m

$$N_m(x) = \int_0^1 N_{m-1}(x - t)\, dt$$

Therefore, by property (4) $N_m(x) > 0$ for all $0 < x < m$.

Hence by method of principle of mathematical induction result is true for all $0 < x < m$.

Property (6): Partition of unity that is

$$\sum_{k=-\infty}^{\infty} N_m(x - k) = 1 \quad \text{for all } x$$

Proof:We prove this result by method of mathematical induction

Step(1) We show that result is true for m = 1

$$\sum_{k=-\infty}^{\infty} N_1(x - k) = 1 \quad \text{for all } x$$

as there is only one interval [k,k+1) such that $N_1(x - k) = 1$

Step(2) Let us assume that result is true for m-1 that is,

$$\sum_{k=-\infty}^{\infty} N_{m-1}(x - k) = 1 \quad \text{for all } x$$

Step(3) We prove that result is true for m

$$\sum_{k=-\infty}^{\infty} N_m(x - k) = \sum_{k=-\infty}^{\infty} \int_0^1 N_{m-1}(x - t - k)\, dt$$

$$= \int_0^1 \left[\sum_{k=-\infty}^{\infty} N_{m-1}(x - t - k) \right] dt$$

$$= \int_0^1 1\, dt \quad \text{for all } x \quad (\text{ by step(2) })$$

$$= 1 \quad \text{for all } x$$

Hence by method of principle of mathematical induction result

is true for all x

Property (7): $N'_m(x) = (\Delta N_{m-1})(x)$

Proof: We have by the definition

$$N'_m(x) = \int_0^1 N'_{m-1}(x - t)\, dt$$

$$N'_m(x) = -[N_{m-1}(x-t)]_0^1$$

$$= N_{m-1}(x) - N_{m-1}(x-1)$$

$$N'_m(x) = (\Delta N_{m-1})(x)$$

Property (8): The cardinal B-Spline N_m and N_{m-1} are related by the identity,

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1)$$

Proof: We have

$$M_m(x) = N_m(x) = \frac{1}{(m-1)!} \Delta^m x_+^{m-1}$$

and also Leibniz rule

$$(\Delta^m fg)(x) = \sum_{k=0}^{m} {}^mC_k \ (\Delta^k f)(x) \ (\Delta^{m-k} g)(x-k)$$

$$N_m(x) = \frac{1}{(m-1)!} \Delta^m x_+^{m-1}$$

$$= \frac{1}{(m-1)!} \Delta^m \{x(x_+^{m-2})\}$$

Applying Leibniz rule,

$$N_m(x) = \frac{1}{(m-1)!} \left[x \cdot \Delta^m x_+^{m-2} + m \cdot (1) \cdot \Delta^{m-1} (x-1)_+^{m-2} \right]$$

$$= \frac{1}{(m-1)!} \left[x \, [\Delta^{m-1} (\Delta \, x_+^{m-2})] + m \cdot \Delta^{m-1} (x-1)_+^{m-2} \right]$$

$$= \frac{1}{(m-1)!} \left[x \, [\Delta^{m-1} (x_+^{m-2} - (x-1)_+^{m-2})] + m \cdot \Delta^{m-1} (x-1)_+^{m-2} \right]$$

$$= \frac{1}{(m-1)!} \left[x \, \Delta^{m-1} x_+^{m-2} - x \, \Delta_+^{m-1} (x-1)_+^{m-2} + m \cdot \Delta^{m-1} (x-1)_+^{m-2} \right]$$

$$= \frac{1}{(m-1)!}\left[x\,\Delta^{m-1} x_+^{m-2} + (m-x)\cdot\Delta^{m-1}(x-1)_+^{m-2} \right]$$

$$= \frac{1}{(m-1)}\left[x\,\frac{\Delta^{m-1}}{(m-2)!}\,x_+^{m-2} + (m-x)\,\frac{\Delta^{m-1}}{(m-2)!}\,(x-1)_+^{m-2} \right]$$

$$N_m(x) = \frac{x}{(m-1)}\,N_{m-1}(x) + \frac{(m-x)}{(m-1)}\,N_{m-1}(x-1)$$

Hence the property.

Property(9): N_m is symmetric with respect to the center of its support,

$$N_m\left(\frac{m}{2} + x\right) = N_m\left(\frac{m}{2} - x\right)$$

Proof:We prove this result by method of mathematical induction

Step(1) We show that result is true for m = 1

$$N_m\left(\frac{1}{2} + x\right) = \begin{cases} 1 & \text{for } 0 \le \frac{1}{2} + x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_m\left(\frac{1}{2} + x\right) = \begin{cases} 1 & \text{for } -\frac{1}{2} \le x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$N_m\left(\frac{1}{2} - x\right) = \begin{cases} 1 & \text{for } 0 \le \frac{1}{2} - x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_m\left(\frac{1}{2} - x\right) = \begin{cases} 1 & \text{for } -\frac{1}{2} \le x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Hence result is true for m = 1.

Step(2) Let us assume that result is true for m-1 that is,

$$N_{m-1}\left(\frac{m-1}{2} + x \right) = N_{m-1}\left(\frac{m-1}{2} - x \right)$$

Step(3) We prove that result is true for m

Let us start with

$$N_{m-1}\left(\frac{m}{2} + x \right) = N_{m-1}\left(\frac{m-1}{2} + \frac{1}{2} + x \right)$$

$$= N_{m-1}\left(\frac{m-1}{2} - \frac{1}{2} - x \right)$$

$$= N_{m-1}\left(\frac{m}{2} - 1 - x \right) \qquad \text{---}(i)$$

$$N_{m-1}\left(\frac{m}{2} - x \right) = N_{m-1}\left(\frac{m-1}{2} + \frac{1}{2} - x \right)$$

$$= N_{m-1}\left(\frac{m-1}{2} - \frac{1}{2} + x \right)$$

$$= N_{m-1}\left(\frac{m}{2} - 1 + x \right) \qquad \text{---}(ii)$$

By Property (8)

$$N_m\left(\frac{m}{2} + x\right) = \frac{\frac{m}{2} + x}{(m - 1)} N_{m-1}\left(\frac{m}{2} + x\right) + \frac{m - \left(\frac{m}{2} + x\right)}{(m - 1)} N_{m-1}\left(\frac{m}{2} + x - 1\right)$$

By equation (i) and (ii), we have

$$N_m\left(\frac{m}{2} + x\right) = \frac{\frac{m}{2} + x}{(m - 1)} N_{m-1}\left(\frac{m}{2} -1- x\right) + \frac{m - \left(\frac{m}{2} + x\right)}{(m - 1)} N_{m-1}\left(\frac{m}{2} - x\right)$$

$$= \frac{\frac{m}{2} - x}{(m - 1)} N_{m-1}\left(\frac{m}{2} - x\right) + \frac{m - \left(\frac{m}{2} - x\right)}{(m - 1)} N_{m-1}\left(\frac{m}{2} - x - 1 \right)$$

$$= N_m\left(\frac{m}{2} - x\right) \text{ for all } x$$

Thus

$$N_m\left(\frac{m}{2} + x\right) = N_m\left(\frac{m}{2} - x\right) \text{ for all } x$$

Hence by method of principle of mathematical induction result is true for all x.

Now we proceed to show that the cardinal B-Splines basis,

$$B = \{ N_m(x - k) : k \in \mathbb{Z} \} \qquad \text{---(3.2.4)}$$

is a Riesz basis for V_0^m in the sense that there exists constant A and B with $0 < A \leq B < \infty$ such that for any sequence $\{ C_k \} \in \ell^2(\mathbb{Z})$, we have

$$A \, \| \{ C_k \} \|^2_{\ell^2(\mathbb{Z})} \leq \Big\| \sum_{k=-\infty}^{\infty} C_k N_m(x - k) \Big\|^2_{L^2(\mathbb{R})} \leq B \, \| \{ C_k \} \|^2_{\ell^2(\mathbb{Z})}$$

---(3.2.5)

Condition (3.2.5) is equivalent to frequency domain condition

$$A \leq \sum_{k=-\infty}^{\infty} \left| \hat{N}_m(\omega - 2\pi k) \right|^2 \leq B \quad \text{a.e.} \qquad \text{---(3.2.6)}$$

We will work with this frequency domain condition to obtain A and B. Replacing ω by $2x$ in above equation (3.2.6) we get,

$$A \leq \sum_{k=-\infty}^{\infty} \left| \hat{N}_m(2x - 2\pi k) \right|^2 \leq B \quad \text{a.e.}$$

Since

$$\begin{aligned} N_m(x) &= (N_{m-1} * N_1)(x) \\ &= ((N_{m-2} * N_1) * N_1)(x) \quad \text{and so on} \\ &= (N_1 * N_1 * \cdots * N_1)(x) \end{aligned}$$

Thus N_m is an m-fold convolution of N_1 and

$$\begin{aligned} \hat{N}_1(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} N_1(t)\, dt \\ &= \int_0^1 e^{-i\omega t}\, dt \qquad (\text{ By the definition of } N_1(t) \text{ }) \end{aligned}$$

$$= \left[\frac{e^{-i\omega t}}{i\omega} \right]_0^1$$

$$\hat{N}_1(\omega) = \left[\frac{1 - e^{-i\omega}}{i\omega} \right]$$

We have,

$$\left| \hat{N}_m(\omega) \right|^2 = \left| \frac{1 - e^{-i\omega}}{i\omega} \right|^{2m}$$

We have,

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right|^2 = \left(\frac{1 - e^{-i\omega}}{i\omega} \right) \cdot \overline{\left(\frac{1 - e^{-i\omega}}{i\omega} \right)}$$

$$= \left(\frac{1 - e^{-i\omega}}{i\omega} \right) \cdot \left(\frac{1 - e^{i\omega}}{-i\omega} \right)$$

$$= \frac{1 - e^{i\omega} - e^{-i\omega} + 1}{\omega^2}$$

$$= \frac{2 - 2 \left[\frac{e^{i\omega} + e^{-i\omega}}{2} \right]}{\omega^2}$$

$$= \frac{2 - 2 \cdot \cos(\omega)}{\omega^2}$$

$$= \frac{2[1 - \cos(\omega)]}{\omega^2}$$

$$= \frac{2 \cdot 2 \cdot \sin^2(\omega/2)}{\omega^2}$$

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right|^2 = \frac{\sin^2(\omega/2)}{(\omega/2)^2}$$

Thus,

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right|^2 = \frac{\sin^2(\omega/2)}{(\omega/2)^2}$$

Using this relation, we get

$$\left| \hat{N}_m(\omega) \right|^2 = \left| \frac{1 - e^{-i\omega}}{i\omega} \right|^{2m} = \frac{\sin^{2m}(\omega/2)}{(\omega/2)^{2m}}$$

Therefore,

$$\sum_{k=-\infty}^{\infty} \left| \hat{N}_m(2x - 2\pi k) \right|^2 = \sum_{k=-\infty}^{\infty} \frac{\sin^{2m}(x + \pi k)}{(x + \pi k)^{2m}}$$

$$\sum_{k=-\infty}^{\infty} \left| \hat{N}_m(2x - 2\pi k) \right|^2 = \sin^{2m}(x) \sum_{k=-\infty}^{\infty} \frac{1}{(x + \pi k)^{2m}} \qquad \text{---(3.2.7)}$$

Now we have,

$$\cot(x) = \lim_{n \to \infty} \sum_{k=-n}^{n} \frac{1}{(x + \pi k)} \qquad \text{---(3.2.8)}$$

Differentiating equation (3.2.8) w.r.t. x, (2m -1) times we get,

$$\sum_{k=-\infty}^{\infty} \frac{1}{(x + \pi k)^{2m}} = - \frac{1}{(2m-1)!} \cdot \frac{d^{2m-1}}{dx^{2m-1}} \{ \cot(x) \} \qquad \text{---(3.2.9)}$$

Using equations (3.2.8) and (3.2.9) in equation (3.2.7) we get,

$$\sum_{k=-\infty}^{\infty} \left| \hat{N}_m(2x - 2\pi k) \right|^2 = \frac{-\sin^{2m}(x)}{(2m-1)!} \frac{d^{2m-1}}{dx^{2m-1}} (\cot(x)) \qquad \text{---(3.2.10)}$$

We evaluate R.H.S. of equation (3.2.10) for $\hat{N}_1$ and $\hat{N}_2$

I] For m = 1 in equation (3.2.10)

$$\sum_{k=-\infty}^{\infty} \left| \hat{N}_1(2x - 2\pi k) \right|^2 = \frac{-\sin^2(x)}{(2-1)!} \frac{d}{dx} \{ \cot(x) \}$$

$$= -\sin^2(x) \, [-\operatorname{cosec}^2(x)]$$

$$= 1$$

Thus in this case when m = 1, A = B = 1 and the B-Spline basis are orthonormal basis of $L^2(\mathbb{R})$

II] For m = 2 in equation (3.2.10)

$$\sum_{k=-\infty}^{\infty} \left| \hat{N}_2(2x - 2\pi k) \right|^2 = \frac{-\sin^4(x)}{3!} \frac{d^3}{dx^3} \{ \cot(x) \}$$

But the value of

$$\frac{d^3}{dx^3} \{ \cot(x) \} = \frac{-2}{\sin^4(x)} \left(1 + 2\cos^2 x \right)$$

$$\sum_{k=-\infty}^{\infty} \left| \hat{N}_2(2x - 2\pi k) \right|^2 = \frac{-\sin^4(x)}{3!} \left[\frac{-2}{\sin^4(x)} (1 + 2\cos^2 x) \right]$$

$$= \frac{1}{3} + \frac{2}{3}\cos^2 x$$

Since $0 \le \cos^2(x) \le 1$, we have,

$$\frac{1}{3} \le \sum_{k=-\infty}^{\infty} \left| \hat{N}_2(2x - 2\pi k) \right|^2 \le 1$$

Although the formula (3.2.10) is explicit and provide a formula for optimal Riesz bounds. In general it is quite cumbersome to calculate, we therefore, use following theorem.

Theorem(3.1) Let $f \in L^2(\mathbb{R})$ satisfies any one of the following three conditions:

(i) $f(x) = O(|x|^{-\beta})$, $\beta > 1$; and $\hat{f}(x) = O(|x|^{-\alpha})$, $\alpha > \frac{1}{2}$, as $|x| \longrightarrow \infty$

(ii) $\hat{f}$ is of compact support, and belongs to class *Lip*(γ) for some $\gamma > 0$, meaning:

$$\sup_{x} \sup_{0<t\leq h} \left| \hat{f}(x+t) - \hat{f}(x) \right| = O(h^{\gamma}), \text{ as } h \longrightarrow 0^{+}$$

(iii) $\hat{f}$ is a continuous function of compact support, and is of bounded variation in its support.

Then if follows that

$$\sum_{k=-\infty}^{\infty} \left| \hat{f}(x + 2\pi k)\right|^2 = \sum_{k=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(y + k)\, \overline{f(y)}\, dy \right\} e^{-ikx}$$

for all $x \in \mathbb{R}$.

Now using the relation

$$\int_{-\infty}^{\infty} f(x)\, \overline{N_m(x)}\, dx = \int_0^1 \cdots \int_0^1 f(x_1+x_2+\ldots+x_m)dx_1dx_2\ldots dx_m$$

We take $f(x) = N_m(x + k)$

$$\int_{-\infty}^{\infty} N_m(x + k)\, \overline{N_m(x)}\, dx = \int_0^1 \cdots \int_0^1 N_m(x_1+x_2+\ldots+x_m+k)dx_1dx_2\ldots dx_m$$

Since,

$$N_m(x) = \int_0^1 N_m(x - t)dt$$

$$= \int_{x-1}^{x} N_{m-1}(y)\, dy$$

Set x = 1, we have

$$N_m(1) = \int_0^1 N_{m-1}(x)dx$$

Thus we have,

$$\int_{-\infty}^{\infty} N_m(x + k)\ N_m(x)\ dx =$$

$$= \int_0^1 \ldots \int_0^1 N_{m+1}(x_1 + \ldots + x_{m-1} + k + 1)dx_1 dx_2 \ldots dx_{m-1}$$

$$= \int_0^1 \ldots \int_0^1 N_{m+2}(x_1 + \ldots + x_{m-2} + k + 2)dx_1 dx_2 \ldots dx_{m-2}$$

$$= N_{m+m}(m + k)$$

$$= N_{2m}(m + k)$$

To evaluate $N_{2m}(x + k)$ at $k \in \mathbb{Z}$ we use $N_2(k) = \delta_{k1} \quad k \in \mathbb{Z}$

$$N_{n+1}(k) = \frac{k}{n} N_n(k) + \frac{n-k+1}{n} N_n(k - 1) \text{ for } k = 1, 2, \ldots$$

$N_{n+1}(k) = 0$ for $k \leq 0$ and $k \geq n + 1$

Therefore,

$$\sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2\pi k)|^2 = \sum_{k=-m+1}^{m-1} N_{2m}(m + k)\ e^{-ik\omega} \leq 1$$

(By Property (5) and (6))

This gives the smallest upper bound B = 1

To determine lower bound A, consider the Euler-Frobenious polynomials

$$E_{2m-1}(z) = (2m - 1)!\ z^{2m-1} \sum_{k=-m+1}^{m-1} N_{2m}(m + k)\ z^k$$

This is a polynomial of order (2m - 1) and has 2m - 2 roots

$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{2m-2}$. These are simple, real and negative roots, that is ,

$$\lambda_1 \cdot \lambda_{2m-2} = \lambda_2 \cdot \lambda_{2m-3} = \cdots = \lambda_{m-1} \cdot \lambda_m = 1$$

Hence we have,

$$A_m = \frac{1}{(2m-1)!} \prod_{k=1}^{m-1} \frac{(1+\lambda_k)^2}{\lambda_k} > 0$$

Now, we can write

$$\sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2\pi k)|^2 = \frac{1}{(2m-1)!} \prod_{k=1}^{2m-2} | e^{i\omega} - \lambda_k |$$

Since λ_i are simple roots in reciprocal pairs

$$\lambda_1 \cdot \lambda_{2m-2} = 1 \Rightarrow \lambda_{2m-2} = \frac{1}{\lambda_1}$$

$$\lambda_2 \cdot \lambda_{2m-3} = 1 \Rightarrow \lambda_{2m-3} = \frac{1}{\lambda_2}$$

$$\lambda_m \cdot \lambda_{m-1} = 1 \Rightarrow \lambda_{m-1} = \frac{1}{\lambda_m}$$

$$\sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2\pi k)|^2 =$$

$$= \frac{1}{(2m-1)!} \prod_{k=1}^{m-1} | e^{i\omega} - \lambda_k | \, | e^{i\omega} - \frac{1}{\lambda_k} |$$

$$= \frac{1}{(2m-1)!} \prod_{k=1}^{m-1} \frac{|e^{i\omega}(1 - \lambda_k e^{-i\omega})| \, | \lambda_k e^{i\omega} - 1|}{|\lambda_k|}$$

$$= \frac{1}{(2m-1)!} \prod_{k=1}^{m-1} \frac{| 1 - \lambda_k e^{-i\omega} | \, | 1 - \lambda_k e^{i\omega} |}{|\lambda_k|}$$

$$= \frac{1}{(2m-1)!} \prod_{k=1}^{m-1} \frac{|(1-\lambda_k e^{-i\omega})(1-\lambda_k e^{i\omega})|}{|\lambda_k|}$$

$$= \frac{1}{(2m-1)!} \prod_{k=1}^{m-1} \frac{|1-\lambda_k e^{-i\omega} - \lambda_k e^{i\omega} + \lambda_k^2|}{|\lambda_k|}$$

$$= \frac{1}{(2m-1)!} \prod_{k=1}^{m-1} \frac{|1-2\cdot\lambda_k \cos(\omega) + \lambda_k^2|}{|\lambda_k|}$$

Since $|\cos(\omega)| \le 1$

$$\sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega + 2\pi k)|^2 \ge \sum_{k=-\infty}^{\infty} |\hat{N}_m(\pi + 2\pi k)|^2 = A_m$$

Thus cardinal B-Spline basis B is a Riesz basis of V_0^m with Riesz bounds $A = A_m$ and $B = 1$.

(3.3) THE TWO SCALE RELATION & INTERPOLATORY GRAPHICAL DISPLAY ALGORITHM:

Since $B = \{ M_m(x - k) : k \in \mathbb{Z} \}$ is a Riesz basis of V_o^m, then for any $j \in \mathbb{Z}$, the collection

$$B_j = \{ 2^{j/2} M_m(2^j x - k) : k \in \mathbb{Z} \} \qquad \text{---(3.3.1)}$$

is also a Riesz basis of V_j^m with the same Riesz bounds as those of B. Obviously, for $j = 0$, B_j reduces to B and in the construction of computational algorithms, it is more convenient to drop the normalization constant $2^{j/2}$ in B_j. This only changes the Riesz bounds by a factor of 2^{-j}. Hence for each j, since $N_m(2^j x) \in V_j^m$ and $V_j^m \subset V_{j-1}^m$,

We have from (3.3.1),

$$N_m(2^j x) = \sum_{k\in\mathbb{Z}} P_{m,k}\, N_m(2^{j+1}x - k) \qquad \text{---(3.3.2)}$$

where

$$\{ P_{m,k} : k \in \mathbb{Z} \} \text{ is some sequence in } \ell^2(\mathbb{Z}).$$

Now, replacing $2^j x$ by y, we have,

$$N_m(y) = \sum_{k\in\mathbb{Z}} P_{m,k}\, N_m(2y - k)$$

Taking Fourier transform of both the side,

$$\hat{N}_m(\omega) = \frac{1}{2}\left(\sum_{k\in\mathbb{Z}} P_{m,k}\, e^{-ik\omega/2}\, \hat{N}_m\left(\frac{\omega}{2}\right) \right) \qquad \text{---(3.3.3)}$$

Since,

$$\hat{N}_m(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^m$$

Then by the equation (3.3.3)

$$\left(\frac{1 - e^{-i\omega}}{i\omega}\right)^m = \frac{1}{2}\left(\sum_{k\in\mathbb{Z}} P_{m,k}\, e^{-ik\omega/2} \left[\frac{1 - e^{-i\omega/2}}{i\omega/2} \right]^m \right)$$

Therefore,

$$\frac{1}{2}\left(\sum_{k\in\mathbb{Z}} P_{m,k}\, e^{-ik\omega/2} \right) = \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^m \left(\frac{1 - e^{-i\omega/2}}{i\omega/2} \right)^{-m}$$

$$= \left(\frac{(1-e^{-i\omega/2})\,(1+e^{-i\omega/2})}{i\omega} \right)^m \left(\frac{i\omega/2}{(1-e^{-i\omega/2})} \right)^m$$

$$= 2^{-m}\left(1 + e^{-i\omega/2} \right)^m$$

$$= 2^{-m} \sum_{k=0}^{m} {}^mC_k\, e^{-ik\omega/2}$$

Thus

$$\sum_{k\in\mathbb{Z}} P_{m,k}\ e^{-ik\omega/2} = 2^{-m+1} \sum_{k=0}^{m} {}^{m}C_k\ e^{-ik\omega/2}$$

Therefore,

$$P_{m,k} = \begin{cases} 2^{-m+1}\ {}^{m}C_k & \text{for } 0 \le k \le m \\ 0 & \text{otherwise} \end{cases} \qquad \text{---(3.3.4)}$$

Consequently,

$$\hat{N}_m(x) = \sum_{k\in\mathbb{Z}} 2^{-m+1}\ {}^{m}C_k\ e^{-ik\omega/2}\ \hat{N}_m(2x - k) \qquad \text{---(3.3.5)}$$

which is called "two-scale relation" for the cardinal splines of order m.

Consider a cardinal spline function

$$f_{j_o}(x) = \sum_i a_i^{j_o}\ N_m(2^{j_o}x - i) \qquad \text{---(3.3.6)}$$

of order m with knot sequence $2^{-j_o}\mathbb{Z}$ where j_o is any (fixed) integer. Suppose that $\{ a_i^{j_o} \}$ is a " Causal " sequence of known real numbers, where causality means that $a_i^{j_o} = 0$ for all $i < i_o$(some constant).

The object is to compute all the values of the sequence

$$f_{j_o}\left(\frac{k}{2^{j_1}}\right) \quad k \in \mathbb{Z} \qquad \text{for all } j_1 \ge j_o$$

To display the graph of $f(x)$, it is adequate to display the sequence $f_{j_o}\left(\frac{k}{2^{j_1}}\right)$ $k \in \mathbb{Z}$, provided that the real (fixed) integer j_1 is sufficiently larger. For each $j \geq j_o$ let us the notation

$$\left. \begin{aligned} f_j(x) &= \sum_i a_i^j \, N_m(2^j x - i) \\ a^j &= \{ a_i^{(j)} \} \; i \in \mathbb{Z} \end{aligned} \right\} \quad \text{---(3.3.7)}$$

By applying the two-scale relation is the identity

$$f_{j+1}(x) = f_j(x)$$

$$\Rightarrow \quad \sum_i a_i^{j+1} \, N_m(2^{j+1}x - i) = \sum_i a_i^j \, N_m(2^j x - i)$$

Using (3.3.2)

$$\sum_i a_i^{j+1} \, N_m(2^{j+1}x - i) = \sum_i a_i^j \quad \sum_k P_{m,k} \, N_m(2^{j+1}x - 2i - k)$$

$$\sum_i a_i^{j+1} \, N_m(2^{j+1}x - i) = \sum_i \left[\sum_k P_{m,i} - 2k \, a_k^j \right] N_m(2^{j+1}x - i)$$

Hence, since the collection $N_m(2^{j+1}x - i) : i \in \mathbb{Z}$ is a Riesz basis of V_{j+1}^m the identity $f_{j+1}(x) = f_j(x)$ is precisely described by the formula

$$a_i^{j+1} = \sum_k P_{m,i} - 2k \, a_k^{(j)} \quad \text{---(3.3.8)}$$

where

$a^j = \{ a_k^{(j)} \}$ and $a^{j+1} = \{ a_k^{(j+1)} \}$ are the coefficient of sequences of $f_j(x)$ and $f_{j+1}(x)$ respectively.

Finally, from the sequence $a^{j_1} = \{ a_k^{j_1} \}$ we still have to compute the values of $f_{j_o}(\frac{k}{2^{j_1}})$ $k \in \mathbb{Z}$,

Since

$$N_2(k) = \delta_{k,1} \quad k \in \mathbb{Z}$$

$$N_m(k) = \frac{k}{m-1} N_{m-1}(k) + \frac{m-k}{m-1} N_{m-1}(k-1) \quad k = 1,2,3,\dots,m-1$$

For $k \in \mathbb{Z}$ we have,

$$f_{j_o}(\frac{k}{2^{j_1}}) = \sum_i a_i^{(j_1)} N_m(2^{j_1} \frac{k}{2^{j_1}} - i) \qquad \text{---(3.3.9)}$$

$$= \sum_i a_i^{(j_1)} N_m(k - i)$$

$$= \sum_i W_{m,k-1} a_i^{(j_1)}$$

where $W_{m,k} = N_m(k)$, $k \in \mathbb{Z}$ ---(3.3.10)

Note that both (3.3.8) and (3.3.9) are only "moving average"(M.A.) formulae except that the sequence a^j in (3.3.8) needs "upsampling". This means that a zero term must be inserted in between any two consecutive terms of the sequence a^j. To be precise, let us set

$$\begin{cases} \tilde{a}^j = \{\tilde{a}_i^j\} \quad i \in \mathbb{Z}; \text{ with} \\ \tilde{a}_{2k}^j = a_k^j \quad \text{and} \\ \tilde{a}_{2k-1}^j = 0, \; k \in \mathbb{Z} \end{cases} \qquad \text{---(3.3.11)}$$

Then the formula (3.3.8) becomes

$$a_i^{j+1} = \sum_k P_{m,i-k} - 2k \, \tilde{a}_k^{(j)} \quad i \in \mathbb{Z} \qquad \text{---(3.3.12)}$$

Algorithm:(Interpolatory graphical display algorithm)

Let f_{j_o} be a cardinal spline function with causal coefficient sequence

$$a^{j_o} = \{ a_i^{j_o} : i = i_o, i_o + 1, \ldots \}$$

as in (3.3.6). Select any $j_1 > j_o$. Then for $j = j_o, \ldots, j_1 - 1$, compute

S_1: $\tilde{a}^j$ using (3.3.11)

S_2: a^{j+1} using S_1 and (3.3.12)

Finally,

S_3: $\{ f_{j_o}(\frac{k}{2^{j_o}}) : k \in \mathbb{Z} \}$. using (3.3.9) and

S_2 for $j = j_1 - 1$

(Skip S_1 and S_2 if $j_1 = j_o$).