C h a p t e r - 4

MULTIRESOLUTION ANALYSIS

CH-4

MULTIRESOLUTION ANALYSIS

Introduction:

Wavelets have been found to be very useful in many scientific and engineering applications including computer graphics, scientific visualization, data compression and signal processing. Our objective here is to introduce wavelets via scaling function using the theory of multiresolution analysis. In this chapter, we try to describe some examples of scaling functions and their corresponding wavelets. The two-scale reconstruction and decomposition relations are described in order to gain quick working knowledge of wavelets.

4.1 Scaling Functions

Definition: A scaling function ϕ is essentially a function $\phi(x)$ which can be written as a linear combination of $\phi(2x - k)$ which are 1/2 scaled and k/2 translated version of $\phi(x)$. More precisely,

$$\phi(x) = \sum_{k \in \mathbb{Z}} P_k \, \phi(2x - k) \qquad \text{---(4.1.1)}$$

This is referred to as the two-scaled relation for the scaling function and sequence $\{p_k\}$ is called the two-scaled sequence of ϕ. We shall restrict our attention to those scaling functions for which only finitely many P_k's are nonzero in the above relationship. These scaling functions have compact support.

Suppose we define closed subspace V_o be the linear span of the integer translates of ϕ viz,

$$V_o = \text{clos}_{L^2}\langle \phi(\cdot - k) : k \in \mathbb{Z} \rangle \qquad ---(4.1.2)$$

and consider,

$$\phi_{j;k}(x) = \phi(2^j x - k) : j,k \in \mathbb{Z} \qquad ---(4.1.3)$$

which are the scaled and translated version of $\phi(x)$. Now we define

$$V_j = \text{clos}_{L^2}\langle \phi_{j,k} : k \in \mathbb{Z} \rangle j \in \mathbb{Z} \qquad ---(4.1.4)$$

Because of two-scale relation in (4.1.1), we have $V_o \subset V_1$. In fact two-scale relation generates a nested sequence of subspaces

$$\cdots \subset V_{-1} \subset V_o \subset V_1 \subset \cdots \qquad ---(4.1.5)$$

$\longleftarrow$ coarser finer $\longrightarrow$

Furthermore, we would like that every function on real line $\mathbb{R}$ should be representable in terms of $\phi_{j;k}$ for sufficiently large j or in other words,

$$\text{clos}_{L^2}\left(\bigcup_{j \in \mathbb{Z}} V_j \right) = L^2(\mathbb{R}) \qquad ---(4.1.6)$$

This property generates Multirsolution Analysis (MRA) defined below,

Definition: A function $\phi \in L^2(\mathbb{R})$ is said to generate a nested sequence of closed subspaces V_o that satisfy,

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$ (containment)
2. $\text{clos}_{L^2}\left(\bigcup_{j\in\mathbb{Z}} V_j \right) = L^2(\mathbb{R})$ (completeness)
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (unique)
4. $f(x) \in V_j \Longleftrightarrow f(2x) = V_{j+1};\ j \in \mathbb{Z}$ (scaling property)

It is in general not true that any function ϕ satisfying the two-scale relation (4.1.1) and property (4.1.6) generates MRA with all the above desired properties. That is why we restrict our attention to only those scaling functions which do generates MRA, that is, they do satisfy those properties above.

Definition: The family ϕ forms an orthonormal basis if

$$\langle \phi_{j;k} \cdot \phi_{l;m} \rangle = \delta_{j,l}\, \delta_{k,m} \quad \forall \quad j,k,l,m \in \mathbb{Z} \qquad \text{---(4.1.7)}$$

4.2 Wavelets

Given a nested sequences of subspaces V_j as in the containment property of MRA, there exists subspaces W_j , which are the orthogonal complements of V_j in V_{j+1} , that is,

$$V_{j+1} = V_j \oplus W_j \quad j \in \mathbb{Z} \qquad \text{---(4.2.1)}$$

and

$$W_j \perp W_i \quad \text{if } j \neq i \qquad \text{---(4.2.2)}$$

Since subspaces V_j are nested as (4.1.5) it follows that

$$V_i = V_j \oplus \sum_{k=0}^{i-j-1} W_{j+k} \quad \text{for } j < i \qquad \text{---(4.2.3)}$$

Where all these subspaces are orthogonal. From properties 2 and 3 of MRA, this gives rise to an orthogonal decomposition of $L^2(\mathbb{R})$,

$$L^2(\mathbb{R}) = \sum_{j \in \mathbb{Z}}^{\cdot} W_j = \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots \qquad \text{---(4.2.4)}$$

and by the property 4 of MRA

$$f(x) \in W_j \iff f(2x) \in W_{j+1} \qquad ; j \in \mathbb{Z} \qquad \text{---(4.2.5)}$$

Given a scaling function ϕ in V_o, the basis tenet of MRA is that there exists another function $\psi \in W_o$ called wavelet, such that $\{\psi_{j;k} : k \in \mathbb{Z}\}$ generates W_j where

$$\psi_{j;k}(x) = \psi(2^j x - k) \quad j,k \in \mathbb{Z} \qquad \text{---(4.2.6)}$$

Since $V_1 = V_o \oplus W_o$,

$\psi \in W_o$ can be written in terms of $\phi(2x - k)$, which forms basis of V_1. Therefore, analogous to the two-scale relation for scaling function (4.1.1), there exists the two-scale sequences $\{ q_k \}$ such that,

$$\psi(x) = \sum_{k \in \mathbb{Z}} q_k \, \phi(2x - k) \qquad \text{---(4.2.7)}$$

This relation (4.2.7) is called two-scale relation for wavelet.

4.3 Reconstruction and decomposition relations

Both of the two-scale relations (4.1.1) and (4.2.7) together are called the reconstruction relations. On the other hand since both $\phi(2x)$ and $\phi(2x - 1)$ are in V_1 and $V_1 = V_o \oplus W_o$, there are four sequences which are denoted by $\{a_{-2k}\}$, $\{b_{-2k}\}$, $\{a_{1-2k}\}$, $\{b_{1-2k}\}$, $k \in \mathbb{Z}$ such that

$$\phi(2x) = \sum_{k\in\mathbb{Z}} [\, a_{-2k}\, \phi(x - k) + b_{-2k}\, \psi(x - k) \,] \quad \text{---(4.3.1)}$$

$$\phi(2x - 1) = \sum_{k\in\mathbb{Z}} [\, a_{1-2k}\, \phi(x - k) + b_{1-2k}\, \psi(x - k) \,] \quad \text{---(4.3.2)}$$

Above two formulae (4.3.1) and (4.3.2) can be combined into a single formula, for $l \in \mathbb{Z}$

$$\phi(2x - l) = \sum_{k\in\mathbb{Z}} [\, a_{l-2k}\, \phi(x - k) + b_{l-2k}\, \psi(x - k) \,] \quad \text{---(4.3.3)}$$

which is called decomposition relation for ϕ and ψ. The two pairs of sequences $(\{p_k\}, \{q_k\})$ and $(\{a_k\}, \{b_k\})$ are used to formulate reconstruction and decomposition algorithms described below. $\{p_k\}$ and $\{q_k\}$ are called reconstruction sequences, while $\{a_k\}$ and $\{b_k\}$ are called decomposition sequences.

4.4 Reconstruction and decomposition algorithms

Let us consider the general structure of multiresolution analysis and wavelets as discussed in (4.2.3), where $\{ V_j \}$ is generated by translates of some scaling function $\phi_j \in L^2(\mathbb{R})$

and { W_j } is generated by translates of some wavelet $\psi_j \in L^2(\mathbb{R})$. In this case, by the property 2 of MRA, every function f in $L^2(\mathbb{R})$ can be approximated as closely as is desired by an $f_N \in V_N$ for some $N \in \mathbb{Z}$.

Since $V_j = V_{j-1} \oplus W_{j-1}$ for any $j \in \mathbb{Z}$, f_N has unique decomposition

$$f_N = f_{N-1} + g_{N-1} \qquad ---(4.4.1)$$

where, $f_{N-1} \in V_{N-1}$ and $g_{N-1} \in W_{N-1}$

By repeating this process, we have,

$$f_N = g_{N-1} + g_{N-2} + g_{N-3} + \cdots + g_{N-M} + f_{N-M} \qquad ---(4.4.2)$$

Where $f_j \in V_j$ and $g_j \in W_j$ for any $j \in \mathbb{Z}$ and M is so chosen that f_{N-M} is sufficiently "blurred" called "wavelet deco m position". In the following, we will discuss an algorithmic approach for expressing f_N as a direct sum of its components g_{N-1}, g_{N-2}, g_{N-3}, $\cdots$, g_{N-M} and f_{N-M}, and recovering f_N from these components.

To describe decomposition and reconstruction algorithms, let us first note that both $f_j \in V_j$ and $g_j \in W_j$ have unique series representation.

$$\left. \begin{aligned} f_j(x) &= \sum_{k\in\mathbb{Z}} c_k^j \phi(2^j x - k) \\ &\text{with} \quad c^j = \{ c_k^j \} \in l^2(\mathbb{Z}) \end{aligned} \right\} \qquad ---(4.4.3)$$

$$g_j(x) = \sum_{k\in\mathbb{Z}} d_k^j \, \psi(2^j x - k) \quad \text{with} \quad d^j = \{ d_k^j \} \in l^2(\mathbb{Z}) \qquad \text{---(4.4.4)}$$

In the following decomposition and reconstruction algorithms, the function f_j and g_j are represented by the sequences c^j and d^j as defined in (4.4.3) and (4.4.4)

Decomposition Algorithm:

By using (4.3.3), (4.4.3) and (4.4.4) we have

$$f_j(x) = \sum_{k\in\mathbb{Z}} c_k^j \, \phi(2^j x - k)$$

$$= \sum_{k\in\mathbb{Z}} c_k^j \sum_{l\in\mathbb{Z}} [\, a_{k-2l} \, \phi(2^{j-1}x - l) + b_{k-2l} \, \psi(2^{j-1}x - l) \,]$$

$$= \sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{k-2l} \, c_k^j \right) \phi(2^{j-1}x - l) + \sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} b_{k-2l} \, c_k^j \right) \psi(2^{j-1}x - l) \qquad \text{---(4.4.5)}$$

Since $f_j(x) = f_{j-1}(x) + g_{j-1}(x)$

Therefore, using the equations (4.4.5), (4.4.3) and (4.4.4) we have,

$$\sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{k-2l} \, c_k^j \right) \phi(2^{j-1}x - l) + \sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} b_{k-2l} \, c_k^j \right) \psi(2^{j-1}x - l) =$$

$$= \sum_{l\in\mathbb{Z}} c_l^{j-1} \, \phi(2^{j-1}x - l) + \sum_{l\in\mathbb{Z}} d_l^{j-1} \, \psi(2^{j-1}x - l)$$

$$\sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{k-2l} \, c_k^j - c_l^{j-1} \right) \phi(2^{j-1}x - l) +$$

$$+ \left(\sum_{k\in\mathbb{Z}} b_{k-2l} \, c_k^j - d_l^{j-1} \right) \psi(2^{j-1}x - l) = 0$$

From l^2-linear independence of $\{ \phi_{j-1;k} : k \in \mathbb{Z} \}$ and $\{ \psi_{j-1;k} : k \in \mathbb{Z} \}$ and the fact that $V_{j-1} \cap W_{j-1} = \{0\}$

$$\sum_{k\in\mathbb{Z}} a_{k-2l}\, c_k^j - c_l^{j-1} = 0$$

$$\Rightarrow \quad c_l^{j-1} = \sum_{k\in\mathbb{Z}} a_{k-2l}\, c_k^j \qquad \text{---(4.4.6)}$$

and

$$\sum_{k\in\mathbb{Z}} b_{k-2l}\, c_k^j - d_l^{j-1} = 0$$

$$\Rightarrow \quad d_l^{j-1} = \sum_{k\in\mathbb{Z}} b_{k-2l}\, c_k^j \qquad \text{---(4.4.7)}$$

$$\begin{array}{ccccccccccc} & \nearrow & d^{N-1} & \nearrow & d^{N-2} & \nearrow & d^{N-3} & \nearrow & & \nearrow & d^{N-M} \\ c^N & \longrightarrow & c^{N-1} & \longrightarrow & c^{N-2} & \longrightarrow & c^{N-3} & \longrightarrow & \cdots & \longrightarrow & c^{N-M} \end{array}$$

Here both c^{j-1} and d^{j-1} are obtained from c^j by moving average schemes using the decomposition sequences as "weights" with the exception that those moving averages are sampled only at the even integers. This is called down sampling. Therefore, each of the arrows in above figure indicates a moving averages followed by down sampling at the even indices.

Reconstruction Algorithm:

Using two-scale relation (4.1.1) and (4.2.7) we have

$$f_{j-1}(x) + g_{j-1}(x) = \sum_{l\in\mathbb{Z}} [\, c_l^{j-1} \phi(2^{j-1}x - l) + d_l^{j-1}\, \psi(2^{j-1}x - l)]$$

$$= \sum_{l \in \mathbb{Z}} c_l^{j-1} \sum_{k \in \mathbb{Z}} p_k \phi(2^j x - 2l - k) + \sum_{l \in \mathbb{Z}} d_l^{j-1} \sum_{k \in \mathbb{Z}} q_k \phi(2^j x - 2l - k)$$

$$= \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} [\, c_l^{j-1} \, p_{k-2l} + d_l^{j-1} \, q_{k-2l} \,] \, \phi(2^j x - k)$$

$$= \sum_{k \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} p_{k-2l} \, c_l^{j-1} + q_{k-2l} \, d_l^{j-1} \right) \phi(2^j x - k)$$

Since

$$f_{j-1}(x) + g_{j-1}(x) = f_j(x)$$

$$\sum_{k \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} p_{k-2l} \, c_l^{j-1} + q_{k-2l} \, d_l^{j-1} \right) \phi(2^j x - k) = \sum_{k \in \mathbb{Z}} c_k^j \, \phi(2^j x - k)$$

and because of l^2-linear independence of $\{ \phi_{j;k} : k \in \mathbb{Z} \}$

$$\Rightarrow \quad c_k^j = \left(\sum_{l \in \mathbb{Z}} p_{k-2l} \, c_l^{j-1} + q_{k-2l} \, d_l^{j-1} \right) \text{---(4.4.8)}$$

$$\begin{array}{ccccccccc}
d^{N-M} & \searrow & d^{N-M+1} & \searrow & d^{N-M+2} & \searrow & d^{N-1} & \searrow & \\
c^{N-M} & \longrightarrow & c^{N-M+1} & \longrightarrow & c^{N-M+2} & \longrightarrow \cdots & c^{N-1} & \longrightarrow & c^N
\end{array}$$

Here c^j is obtained from c^{j-1} and d^{j-1} by two moving averages, using the reconstruction sequences as "weights" with the exception that an upsampling is required before the moving averages are performed. More precisely, the samples c_l^{j-1} and d_l^{j-1} are used at the even indices $m = 2l$ and zeros are used at the odd indices $m = 2l + 1$, when the (discrete) convolutions are taken with respect to $\{p_n\}$ and $\{q_n\}$.

4.5 Examples

1) Haar Wavelets

1.1) Scaling function

We discuss first example as Haar function and Haar wavelet which are very simple but useful to illustrate many nice properties of scaling functions and for practical use. Haar scaling function is defined by

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{---(4.5.1)}$$

The subspace V_o is spanned by scaling functions $\phi(x - k)$ which are integer translates of piece-wise constant functions on unit interval. The subspace V_1 is spanned by $\phi(2x - k)$ which are k/2 translates of piece-wise constant on 1/2 interval. In general, V_j is spanned by $k/2^j$ translates of piece-wise constant functions on $1/2^j$ interval. The two-scale relation (4.1.1) for Haar scaling function is,

$$\phi(x) = \sum_{k \in \mathbb{Z}} P_k \, \phi(2x - k)$$

$$\phi(x) = \phi(2x) + \phi(2x - 1) \qquad \text{---(4.5.2)}$$

Therefore, the two-scale sequences $\{ p_k \}$ for Haar scaling function have non-zero values $p_o = p_1 = 1$ and 0's for other p_j's.

1.2) Wavelets

The Haar wavelet $\psi(x)$ corresponding to the Haar scaling function $\phi(x)$ is given by

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \le x < 1/2 \\ -1 & \text{for } 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

We can easily construct the two-scale sequences $\{ q_k \}$ in (4.2.7) are 0's except for $q_o = 1$ and $q_1 = -1$

1.3) Decomposition Relations:

The two-scale relations (4.5.2) for Haar scaling functions express $\phi(x)$ in terms of $\phi(2x)$ and $\phi(2x - 1)$, while the two-scale relation (4.5.4) for Haar wavelets express $\psi(x)$ also in terms of $\phi(2x)$ and $\phi(2x - 1)$. Both of the two-scale relations together are called the reconstruction relations.

$$\begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \phi(2x) \\ \phi(2x-1) \end{bmatrix}$$

Where as, the decomposition relation for Haar wavelets are the expression of $\phi(x)$ and $\psi(x)$ which are the inverse of the reconstruction relations (4.5.5).

Since the support of the scaling function and the wavelets are within the same interval. the decomposition relations are easily derived by just inverting the reconstruction relation as follows;

$$\begin{bmatrix} \phi(2x) \\ \phi(2x-1) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}$$

Hence the non-zero elements of decomposition sequences { a_k } and { b_k } are $a_o = 1/2$, $b_o = 1/2$, $a_1 = 1/2$, $b_1 = -1/2$.

Now we see a PASCAL program for Haar wavelet.

```
program Haar;

uses crt;

var

   i,j,l,k,n : integer;

   x         : real;

   a,b,p,q   : array [-200..200] of real;

   c,d       : array [0..50,0..50] of real;

   f1        : text;

function power(base,index : integer): integer;

var

   temp,ii : integer;

begin

     temp := 1;

     for ii := 1 to index do

        temp := temp * base;

     power := temp;

end;
```

```
Begin
    Assign(f1,'\mphil\haar.op');
    rewrite(f1);
    clrscr;
    x := -2; n := 4;
    writeln(f1,'-----------------------------------------');
    writeln(f1,'Given input data,that is,the sequence {c_k}');
    writeln(f1,' X ':6,' F(x)':13);
    writeln(f1,'-----------------------------------------');
    j := power(2,n)-1;
    for i := -j to j do
    begin
        a[i] := 0;b[i] := 0;
        p[i] := 0;q[i] := 0;
    end;
    a[0] := 0.5; a[1] := 0.5;b[0] := 0.5; b[1] := -0.5;
    p[0] := 1; p[1] := 1;q[0] := 1; q[1] := -1;
    for i := 0 to j do
    begin
        c[n,i] := exp(-x);
        writeln(f1,x:6:2,c[n,i]:15:7);
        x := x + 0.25;
```

```
end;
clrscr;
writeln(f1,'----------------------------------------');
writeln(f1,'Decomposition of given data into sequences
{c_k} and {d_k} using the formulae (4.4.6) and (4.4.7)');
writeln(f1,' C(i,k)':11,'D(i,k)':13);
writeln(f1,'----------------------------------------');
for j := n-1 downto 0 do
begin
     i := power(2,j) - 1;
     for k := 0 to i do
     begin
          c[j,k] := 0;d[j,k] := 0;
          for l := 0 to power(2,j+1)-1 do
          begin
               c[j,k] := c[j,k] + a[l-2*k]*c[j+1,l];
               d[j,k] := d[j,k] + b[l-2*k]*c[j+1,l];
          end;
          writeln(f1,c[j,k]:11:8,d[j,k]:15:7);
     end;
end;
clrscr;
```

```
    writeln(f1,'----------------------------------------');
    writeln(f1,'Reconstruction by using the formula(4.4.8)');
    writeln(f1,' C(i,k)':11);
    writeln(f1,'----------------------------------------');
    writeln(f1,c[0,0]:0:8,'  ');
    for j := 1 to n do
    begin
        i := power(2,j)-1;
        for k := 0 to i do
        begin
            c[j,k] := 0;
            for l := 0 to power(2,j - 1) - 1 do
              c[j,k] := c[j,k] + p[k-2*l]*c[j-1,l] +
                        + q[k-2*l]*d[j-1,l];
              writeln(f1,c[j,k]:0:8,'      ');
        end;
    end;
    close(f1);
end.
```

Output of the above programme:

Given input data, that is, the sequence $\{c_k\}$

X	F(x)
-2.00	7.3890561
-1.75	5.7546027
-1.50	4.4816891
-1.25	3.4903430
-1.00	2.7182818
-0.75	2.1170000
-0.50	1.6487213
-0.25	1.2840254
0.00	1.0000000
0.25	0.7788008
0.50	0.6065307
0.75	0.4723666
1.00	0.3678794
1.25	0.2865048
1.50	0.2231302
1.75	0.1737739

Decomposition of given data into sequences $\{c_k\}$ and $\{d_k\}$

C(i,k)   D(i,k) using the formulae (4.4.6) and (4.4.7)

C(i,k)	D(i,k)
6.57182939	0.8172267
3.98601601	0.4956731
2.41764092	0.3006409
1.46637334	0.1823479
0.88940039	0.1105996
0.53944861	0.0670821
0.32719212	0.0406873
0.19845205	0.0246781
5.27892270	1.2929067
1.94200713	0.4756338
0.71442450	0.1749759
0.26282209	0.0643700
3.61046492	1.6684578
0.48862329	0.2258012
2.04954410	1.5609208

Reconstruction by using the formula (4.4.8)

C(i,k)

2.04954410

3.61046492
0.48862329

5.27892270
1.94200713
0.71442450
0.26282209

6.57182939
3.98601601
2.41764092
1.46637334
0.88940039
0.53944861
0.32719212
0.19845205

7.38905610
5.75460268
4.48168907
3.49034296
2.71828183
2.11700002
1.64872127
1.28402542
1.00000000
0.77880078
0.60653066
0.47236655
0.36787944
0.28650480
0.22313016
0.17377394

2) General Order B-Spline wavelet

2.1) Scaling function

We have already defined m^{th} order B-Spline N_m recursively by convolution

$$N_m(x) = \int_{-\infty}^{\infty} N_{m-1}\; N_1(t)\; dt$$

where

$$N_1(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_m(x) = \int_{0}^{1} N_{m-1}(x - t)\; dt \qquad \text{---(4.5.3)}$$

The two-scale relation (4.1.1) for B-Spline scaling functions of general order m is written as

$$N_m(x) = \sum_{k=\mathbb{Z}}^{m} P_k\; N_m(2x - k) \qquad \text{---(4.5.4)}$$

Where the two-scale sequence { p_k } for B-Spline scaling function are given by

$$p_k = 2^{-m+1}\; {}^{m}C_k \qquad \text{for } 0 \leq k < m \qquad \text{---(4.5.5)}$$

2.2) Wavelets

Since B-Spline N_m are scaling functions, it follows the general theory that the B-Spline subspaces V_j form a nested sequence. Moreover, the complimentary subspaces W_j are mutually orthogonal and any function in $L^2(\mathbb{R})$ can be represented as a linear sum of function in W_j.

In general, $\psi_m \in W_j$ are not unique. For the moment we focus on the unique wavelets $\psi_m \in W_o$ that has the smallest support. These will be referred to as B-Spline wavelets for general order m and is given by

$$\psi_m(x) = \sum_{k=0}^{3m-2} q_k\ N_m(2x - k) \qquad ---(4.5.6)$$

where

$$q_k = (-1)^k\ 2^{1-m} \sum_{l=0}^{m} {}^mC_l\ N_{2m}(k + l - 1) \qquad ---(4.5.7)$$

2.3) Decomposition Relations:

The decomposition relation for m^{th} order B-Spline is

$$\phi(2x - l) = \sum_{k\in\mathbb{Z}} [a_{l-2k}\phi(x-k) + b_{l-2k}\psi(x-k)]\quad l \in \mathbb{Z} \qquad ---(4.5.8)$$

Where, the decomposition sequences $\{ a_k \}$ & $\{ b_k \}$ are given in following forms

$$a_k = \frac{1}{2}\ (-1)^{k+1} \sum_{l\in\mathbb{Z}} q_{-k+2m-1-2l}\ C_{l,2m} \qquad ---(4.5.9)$$

$$b_k = -\frac{1}{2}\ (-1)^{k+1} \sum_{l\in\mathbb{Z}} p_{-k+2m-1-2l}\ C_{l,2m} \qquad ---(4.5.10)$$

and $\{ p_k \}$ and $\{ q_k \}$ are two-scale sequences given by (4.5.5) and (4.5.7).

3) Linear B-Spline Wavelets

3.1) Scaling function:

Linear B-Spline $N_2(x)$ is derived from the recurrence (4.5.3) and (4.5.4) as the case of m = 2 for general B-Splines as follows.

$$N_2(x) = \phi(x) = \begin{cases} x & \text{for } 0 \leq x < 1 \\ 2 - x & \text{for } 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{---(4.5.11)}$$

Then the functions $\phi(2x - k)$ in V_1 subspace are expressed explicitly as

$$\phi(2x - k) = \begin{cases} 2x-k & \text{for } k/2 \leq x < (k+1)/2 \\ 2+k-2x & \text{for } (k+1)/2 \leq x < (k+2)/2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{---(4.5.12)}$$

The linear B-Spline $N_2(2x - k)$ that is $\phi(2x - k)$ in V_1 subspace for $k \in \mathbb{Z}$. Since the support of $\phi(x)$ is [0 , 2], its two-scale relation is in the form

$$\phi(x) = \sum_{k=0}^{2} p_k \, \phi(2x - k) \qquad \text{---(4.5.13)}$$

By substituting the expressions (4.5.11) and (4.5.12) for each 1/2 interval between [0 , 2] into (4.5.13) the coefficients $\{ p_k \}$ are obtained and the two-scale relation for Linear B-Spline is given by

$$\phi(x) = \frac{1}{2}\,\phi(2x) + \phi(2x - 1) + \frac{1}{2}\,\phi(2x - 2) \qquad \text{---(4.5.14)}$$

Here $\{ p_o, p_1, p_2 \} = \{ 1/2 , 1 , 1/2 \}$

3.2) Wavelets:

Now, let us study Linear B-Spline wavelets denoted by $\psi_2(x)$.

$$\psi_2(x) = \sum_{k=0}^{4} q_k \, N_2(2x - k) \qquad \text{---(4.5.15)}$$

where

$$q_k = (-1)^k \, 2^{-1} \sum_{l=0}^{2} {}^2C_l \; N_4(k + 1 - l)$$

$$q_k = \frac{(-1)^k}{2} \{ N_4(k+1) + 2\, N_4(k) + N_4(k - 1) \} \quad ---(4.5.16)$$

The term $N_4(k)$ in above equation (4.5.16) can be calculated by using relation

$$N_m(\frac{m}{2} + x) = \frac{\frac{m}{2} + x}{(m - 1)} N_{m-1}(\frac{m}{2} + x) + \frac{m - (\frac{m}{2} + x)}{(m - 1)} N_{m-1}(\frac{m}{2} + x - 1)$$

We have non-zero $N_m(k)$ values for m = 2 , 3 , 4 are

$N_m(k)$		k					
		0	1	2	3	4	5
m	2	0	1/2	0	...		
	3	0	1/2	1/2	0	...	
	4	0	1/6	2/3	1/6	0	...

Non-zero values of $N_m(k)$; $k \in \mathbb{Z}$ for some small m are summarized in above table. Then the two-scale sequence { q_k } for $\psi_2(x)$ is computed as follows:

q_0 = (1/2) {$N_4(1)$ + 2 $N_4(0)$ + $N_4(-1)$ } = (1/2) · (1/6) = 1/12

q_1 = (-1/2) { $N_4(2)$ + 2 $N_4(1)$ + $N_4(0)$ } = (-1/2) · (1) = -1/2

q_2 = (1/2) { $N_4(3)$ + 2 $N_4(2)$ + $N_4(1)$ } = (1/2) · (5/3) = 5/6

q_3 = (-1/2) { $N_4(4)$ + 2 $N_4(3)$ + $N_4(2)$ } = (-1/2) · (1) = -1/2

q_4 = (1/2) { $N_4(5)$ + 2 $N_4(4)$ + $N_4(3)$ } = (1/2) · (1/6) = 1/12

Thus the two-scale relation for linear B-Spline wavelet is

$$\psi_2(x) = \frac{1}{12} N_2(2x) - \frac{1}{2} N_2(2x - 1) + \frac{5}{6} N_2(2x - 2) - \frac{1}{2} N_2(2x - 3) + \frac{1}{12} N_2(2x - 4) \quad \text{---(4.5.17)}$$

3.3) Decomposition relation:

The decomposition sequences $\{ a_k \}$ and $\{ b_k \}$ given in (4.5.9) and (4.5.10) are written for m = 2 as,

$$a_k = \frac{1}{2} (-1)^{k+1} \sum_{l \in \mathbb{Z}} q_{-k+3-2l} \, C_{1,4} \quad \text{---(4.5.18)}$$

$$b_k = -\frac{1}{2} (-1)^{k+1} \sum_{l \in \mathbb{Z}} p_{-k+3-2l} \, C_{1,4} \quad \text{---(4.5.19)}$$

We already know the reconstruction sequences $\{ p_k \}$ and $\{ q_k \}$ for linear B-Spline (m = 2) and above $\{ a_k \}$ and $\{ b_k \}$ are decomposition sequences for linear B-Spline.

4) Daubechies Wavelets:

4.1) Scaling functions

Another example of compactly supported wavelets defined on real line is Daubechies wavelets. Daubechies scaling function ϕ_3^D is defined by the following two-scale relation;

$$\phi_3^D(x) = \sum_{k=0}^{3} p_k \, \phi(2x - k) = \frac{1 + \sqrt{3}}{4} \phi_3^D(2x) + \frac{3 + \sqrt{3}}{4} \phi_3^D(2x - 1) + \frac{3 - \sqrt{3}}{4} \phi_3^D(2x - 2) + \frac{1 - \sqrt{3}}{4} \phi_3^D(2x - 3) \quad \text{---(4.5.20)}$$

that is, non-zero values of the two-scale sequence $\{ p_k \}$ are $\{ p_o, p_1, p_2, p_3 \} = \{ \frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4} \}$.

Here $p_o + p_2 = 1$ and $p_1 + p_3 = 1$. In general, two-scale sequence $\{ p_k \}$ for any scaling functions has the property

$$\sum_k p_{2k} = \sum_k p_{2k+1} = 1 \qquad \text{---(4.5.21)}$$

4.2) Wavelets:

The two-scale relation for the Daubechies wavelets is in the form

$$\psi_3^D(x) = \sum_k q_k \, \phi_3^D(2x - k) \qquad \text{---(4.5.22)}$$

where

$$q_k = (-1)^k \, \bar{p}_{-k+1} \qquad \text{---(4.5.23)}$$

and $\bar{p}_k$ is the complex conjugate of p_k. Since the two-scale coefficients $\{ p_k \}$ ar all real for $\phi_3^D(x)$, we simply have $\bar{p}_{-k+1} = p_{-k+1}$. Therefore the non-zero values of the two-scale sequence $\{ q_k \}$ are

$\{ q_{-2}, q_{-1}, q_0, q_1 \} = \{ p_3, -p_2, p_1, -p_0 \} =$

$= \{ \frac{1-\sqrt{3}}{4}, -\frac{3-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, -\frac{1+\sqrt{3}}{4} \}$, and the explicit formula for (4.5.22) is

$$\psi_3^D(x) = \sum_{k=-2}^{1} q_k \, \phi_3^D(2x - k)$$

$$\psi_3^D(x) = \frac{1 - \sqrt{3}}{4}\,\phi_3^D(2x + 2) - \frac{3 - \sqrt{3}}{4}\,\phi_3^D(2x + 1) + + \frac{3 + \sqrt{3}}{4}\,\phi_3^D(2x) - \frac{1 + \sqrt{3}}{4}\,\phi_3^D(2x - 1)$$

---(4.5.24)

4.3) Decomposition relations:

The decomposition relations for Daubechies Wavelets are rather simple and given by;

$$\phi_3^D(2x - l) = \sum_k [\, a_{l-2k}\,\phi_3^D(x - k) + b_{l-2k}\,\psi_3^D(x - k)\,] \qquad l \in \mathbb{Z}$$

---(4.5.25)

where

$$a_{l-2k} = \frac{1}{2}\,\bar{p}_{l-2k}$$ ---(4.5.26)

$$b_{l-2k} = \frac{1}{2}\,\bar{q}_{l-2k}$$ ---(4.5.27)

Since only four { p_k } and { q_k } are non-zero, the decomposition relation (4.5.25) is written in the following more explicit form;

$$\phi_3^D(2x) = \frac{3 - \sqrt{3}}{8}\,\phi_3^D(x + 1) + \frac{1 + \sqrt{3}}{8}\,\phi_3^D(x) + + \frac{3 + \sqrt{3}}{8}\,\psi_3^D(x) + \frac{1 - \sqrt{3}}{8}\,\psi_3^D(x - 1)$$ ---(4.5.28)

$$\phi_3^D(2x - 1) = \frac{1 - \sqrt{3}}{8}\,\phi_3^D(x + 1) + \frac{3 + \sqrt{3}}{8}\,\phi_3^D(x) + - \frac{1 + \sqrt{3}}{8}\,\psi_3^D(x) - \frac{3 - \sqrt{3}}{8}\,\psi_3^D(x - 1)$$ ---(4.5.29)

Hence the non-zero elements of decomposition sequences $\{ a_k \}$ and $\{ b_k \}$ are

$$a_o = \frac{1 + \sqrt{3}}{8}, \quad a_1 = \frac{3 + \sqrt{3}}{8}, \quad a_2 = \frac{3 - \sqrt{3}}{8}, \quad a_3 = \frac{1 - \sqrt{3}}{8}$$

$$b_{-2} = \frac{1 - \sqrt{3}}{8}, \quad b_{-1} = - \frac{3 - \sqrt{3}}{8}, \quad b_o = \frac{3 + \sqrt{3}}{8},$$

$$b_1 = - \frac{1 + \sqrt{3}}{8}$$