Chapter - 1

FOURIER ANALYSIS

1.1) Fourier Transform

1.2) Fourier Series

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FOURIER ANALYSIS

Introduction:

In process of communication, one often needs to represent and analyze an electrical signal or an image. A musical note, an electrical signal, satellite images are some of the examples of signals. When one think about Fourier analysis, one usually refers to Fourier transform and Fourier series. A Fourier transform is the Fourier integral of some function f defined on the real line $\mathbb{R}$. When f is thought of as an analog signal, then its domain of definition $\mathbb{R}$ is called continuous time domain. In this case the Fourier transform $\hat{f}$ of f describe the spectral behavior of the signal f. Since the spectral information is given in terms of frequency, the domain of definition of the Fourier transform $\hat{f}$, which is again $\mathbb{R}$, is called the frequency domain. On the other hand a Fourier series is a transformation of bi-infinite sequences to periodic functions. Hence, when a bi-infinite sequence is thought of as a digital signal, then its domain of definition, which is the set $\mathbb{Z}$ of integers, is called the discrete time-domain. In this case, its Fourier series again describes the spectral behavior of the digital signal and the domain of definition of a Fourier series is again the real line $\mathbb{R}$ which is the frequency domain.

The importance of both the Fourier transform and Fourier series stems not only from the significance of their physical interpretation, but also from the fact that Fourier analytic techniques are extremely powerful. For instant, in the study of wavelet analysis the Poisson summation formula, Parseval identities for both Fourier transform and Fourier series, Fourier transform of Gaussian function and the delta distribution etc. are often encountered.

1.1) Fourier Transform

Definition: The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) \; dt \qquad \text{---(1.1.1)}$$

The Fourier transform is a mathematical procedure by which a function is split into its different frequencies, like a prism breaking lights into its component colours. But Fourier transform goes further and tells both how much of each frequency the function contain (the amplitude of frequency) and the phase of the signal at each frequency.

The Fourier transform also describes the result of that operation. The Fourier transform of particular function (that varies with time) is a new function (that varies with frequency).

Properties: If $f \in L^1(\mathbb{R})$ Then its Fourier transform $\hat{f}$ satisfies

1) $\hat{f} \in L^\infty(\mathbb{R})$ with $\| \hat{f} \|_\infty \leq \| f \|_1$

2) $\hat{f}$ is uniformly continuous on $\mathbb{R}$

3) If the derivative f' of f also exists and is in $L^1(\mathbb{R})$ then

$$\hat{f'}(\omega) = i\omega \hat{f}(\omega) \text{ and}$$

4) $\hat{f}(\omega) \longrightarrow 0$ as $\omega \longrightarrow \pm\infty$

Although, $\hat{f}(\omega) \longrightarrow 0$ as $\omega \longrightarrow \pm\infty$, for every $f \in L^1(\mathbb{R})$, it does not mean $\hat{f}$ is necessarily in $L^1(\mathbb{R})$

<u>A counter Example</u>:

Consider Heaviside unit step function defined by

$$u_a(x) = \begin{cases} 1 & \text{for } x \geq a; \\ 0 & \text{for } x < a. \end{cases}$$

where $a \in \mathbb{R}$. Then the function $f(x) = e^{-x} u_0(x)$ belongs to $L^1(\mathbb{R})$, but its Fourier transform

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx$$

$$= \int_0^{\infty} e^{-x} e^{-i\omega x}\, dx$$

$$= \int_0^{\infty} e^{-(1 + i\omega)x}\, dx$$

$$= \left[\frac{e^{-(1 + i\omega)x}}{-(1 + i\omega)} \right]_0^{\infty}$$

$$= \frac{1}{(1 + i\omega)} \left[-e^{-(1 + i\omega)x} \right]_0^{\infty}$$

$$= \frac{1}{(1 + i\omega)} (1 - 0)$$

$$\mathcal{F} \{f(x)\} = \frac{1}{(1 + i\omega)}$$

which behaves likes $O(|\omega|^{-1})$ at ∞ and hence, does not belongs to $L^1(\mathbb{R})$

Definition: Let $\hat{f} \in L^1(\mathbb{R})$ be the Fourier transform of some function $f \in L^1(\mathbb{R})$ then the inverse Fourier transform of $\hat{f}$ is defined by,

$$(\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega)\, d\omega \qquad ---(1.1.2)$$

Now important question is that, can f be recovered from $\hat{f}$ by using the operator $\mathcal{F}^{-1}$ or under what condition $(\mathcal{F}^{-1}\hat{f})(x) = f(x)$?

This is possible only because of the following theorem.

Theorem(1.1): If $\hat{f} \in L^1(\mathbb{R})$ be the Fourier transform of some function $f \in L^1(\mathbb{R})$ then $f(x) = (\mathcal{F}^{-1}\hat{f}))(x)$ at every point x where f is continuous.

Now we prove an important result, Fourier transform of Gaussian function is again a Gaussian function. This is one of the function whose Fourier transform is of the same type. The Gabor transform is a window Fourier transform with any Gaussian function g_α as the window function. For various reasons other functions may be used as window function. But they must have to satisfy window function condition $tw(t) \in L^2(\mathbb{R})$ for $w \in L^2(\mathbb{R})$, then we can use w as time window function. Now if $w \in L^2(\mathbb{R})$ satisfies the window condition then its Fourier transform $\hat{w} \in L^2(\mathbb{R})$ need not necessarily satisfies window function condition and hence it may not a frequency window function. Since Fourier transform of Gaussian function is again a Gaussian function, we can use g_α and $\hat{g}_\alpha$ for time - frequency localization.

Result:

$$\int_{-\infty}^{\infty} e^{-i\omega x}\, e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}\; e^{-(\omega^2/4a)} \qquad \text{---(1.1.3)}$$

Proof: Consider the function

$$f(y) = \int_{-\infty}^{\infty} e^{-ax^2 + xy}\, dx \qquad y \in \mathbb{R}$$

Now

$$\begin{aligned} -ax^2 + xy &= -a\left(x^2 - \frac{xy}{a} \right) \\ &= -a\left(x^2 - \frac{xy}{a} + \frac{y^2}{4a^2} \right) + \frac{y^2}{4a} \\ &= -a\left(x - \frac{y}{2a} \right)^2 + \frac{y^2}{4a} \end{aligned}$$

Therefore,

$$f(y) = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{y}{2a}\right)^2 + \frac{y^2}{4a}}\, dx \qquad y \in \mathbb{R}$$

$$= e^{(y^2/4a)} \int_{-\infty}^{\infty} e^{-a\left(x - \frac{y}{2a}\right)^2}\, dx$$

Put $t^2 = a\left(x - \frac{y}{2a}\right)^2 \Rightarrow dt = \sqrt{a}\, dx$

$$f(y) = e^{(y^2/4a)} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{a}}$$

$$= e^{(y^2/4a)} \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= e^{(y^2/4a)} \frac{1}{\sqrt{a}} \sqrt{\pi}$$

$$= \sqrt{\frac{\pi}{a}}\; e^{(y^2/4a)}$$

This can be extended to be entire (analytic) function and since they agree on $\mathbb{R}$, they must agree in the complex plane $\mathbb{C}$.

By setting $y = -i\omega$,

$$\int_{-\infty}^{\infty} e^{-i\omega x} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}\; e^{-(\omega^2/4a)}$$

Now we switch towards the important concept in Fourier analysis what is called as CONVOLUTION.

Continuous - time convolution:

Definition: Let f and g be function in $L^1(\mathbb{R})$. Then the (continuous time) convolution of f and g is also an $L^1(\mathbb{R})$ function h defined by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x - y)\, g(y)\, dy \qquad \text{---}(1.1.4)$$

If $h(x) = (f * g)(x)$ then $h \in L^1(\mathbb{R})$. Moreover,

$$\| h \|_1 \leq \| f \|_1 \| g \|_1$$

Proof:

$$\| h \|_1 = \left\{ \int_{-\infty}^{\infty} |h(x)|\, dx \right\}$$

$$= \left\{ \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x - y)\, g(y)\, dy \right| dx \right\}$$

$$\leq \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | f(x - y)\, g(y) |\, dy\, dx \right\}$$

$$= \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | f(x - y) | \cdot | g(y) |\, dy\, dx \right\}$$

$$= \left\{ \int_{-\infty}^{\infty} | g(y) | \left[\int_{-\infty}^{\infty} | f(x - y) |\, dx \right] dy \right\}$$

$$= \left\{ \int_{-\infty}^{\infty} | g(y) | \left[\int_{-\infty}^{\infty} | f(t) |\, dt \right] dy \right\}$$

$$= \left\{ \int_{-\infty}^{\infty} | \ g(y) \ | \left[\| f \|_1 \right] dy \right\}$$

$$= \| f \|_1 \ \| g \|_1$$

$$\| h \|_1 \leq \| f \|_1 \ \| g \|_1$$

We can prove

$$f * g = g * f \qquad \text{for all} \quad f, g \in L^1(\mathbb{R}) \qquad \text{---(1.1.5)}$$

$$(f * g) * u = f * (g * u)$$

Thus convolution operator is commutative and associative. Now important question is that, does there exists some function $d \in L^1(\mathbb{R})$ such that

$$f * d = d * f = f \quad \text{for all} \quad f \in L^1(\mathbb{R}) \ ? \qquad \text{---(1.1.6)}$$

The answer is NO, there does not exists such a function $d \in L^1(\mathbb{R})$ that satisfies (1.1.6). Since approximation of the convolution identity is very useful technique in Fourier analysis. We also wish to approximate d in (1.1.6).

Consider the family $\{ d_\alpha \} \subset L^1(\mathbb{R})$ that seeks to approximate identity,

$$\hat{d}_\alpha(\omega) \simeq 1 \quad \omega \in \mathbb{R} \qquad \text{as} \qquad \alpha \longrightarrow 0 \qquad \text{---(1.1.7)}$$

In particular, we may use the normalization

$$\hat{d}_\alpha(0) = \int_{-\infty}^{\infty} d_\alpha(x) \ dx = 1 \qquad \text{---(1.1.8)}$$

An excellent member is the family of Gaussian functions

$$g_\alpha(x) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}} \qquad \alpha > 0 \qquad \text{---(1.1.9)}$$

<u>Theorem</u>(1.2): Let $f \in L^1(\mathbb{R})$. Then

$$\lim_{\alpha \to 0^+} (f * g_\alpha)(x) = f(x) \qquad \text{---(1.1.10)}$$

at every point where f is continuous

Hence $\{ g_\alpha \}$ is an approximation of the convolution identity.

<u>Theorem</u>(1.3): If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the Fourier transform $\hat{f}$ of f is in $L^2(\mathbb{R})$ and satisfies the following "Parseval Identity"

$$\| \hat{f} \|_2^2 = 2\pi \| f \|_2^2 \qquad \text{---(1.1.11)}$$

From above theorem $\mathcal{F}$ may be considered as, bounded linear operator on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with range in $L^2(\mathbb{R})$, that is,

$$\mathcal{F} : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \text{ such that}$$

$$\| \mathcal{F} \| = \sqrt{2\pi}$$

Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, $\mathcal{F}$ has a norm preserving extension to all of $L^2(\mathbb{R})$. More precisely, if $f \in L^2(\mathbb{R})$, then its truncations

$$f_N(x) = \begin{cases} f(x) & \text{for } |x| \le N \\ 0 & \text{for otherwise} \end{cases} \qquad \text{---(1.1.12)}$$

where N may be any positive integer

are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. So that $\hat{f}_N \in L^2(\mathbb{R})$.

Moreover, $\{ \hat{f}_N \}$ is a Cauchy sequence in $L^2(\mathbb{R})$, and by the completeness of $L^2(\mathbb{R})$, there is a function $\hat{f}_\infty \in L^2(\mathbb{R})$ such that

$$\lim_{N \longrightarrow \infty} \| \hat{f}_N - \hat{f}_\infty \|_2 = 0$$

Definition: The Fourier transform $\hat{f}$ of a function $f \in L^2(\mathbb{R})$ is defined to be the Cauchy limit $\hat{f}_\infty$ of $\{ \hat{f}_N \}$, and the notation

$$\hat{f}(\omega) = \underset{N \longrightarrow \infty}{\text{l.i.m.o.t.}} \; \hat{f}_N(\omega)$$

$$= \underset{N \longrightarrow \infty}{\text{l.i.m.o.t.}} \int_{-N}^{N} e^{i\omega x} f(x)\, dx$$

which stands for 'limit in the mean of order two' will be used.

Obviously, the definition of $\hat{f}$ of $f \in L^2(\mathbb{R})$, should be independent of choice of $f_N \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. And thus

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx \quad \text{for} \quad f \in L^2(\mathbb{R})$$

Some important results and theorems in Fourier analysis.

Theorem(1.4):

$$\int_{-\infty}^{\infty} f(x)\, \hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(x)\, g(x)\, dx$$

Parseval Identity:

$$\langle f , g \rangle = \frac{1}{2\pi} \langle \hat{f} , \hat{g} \rangle$$

Where,

$$\langle f , g \rangle = \int_{-\infty}^{\infty} f(x)\, \overline{g(y)}\, dy$$

Theorem(1.5): For fixed real number a and b if $f \in L^1(\mathbb{R})$ then

$$1)\ \mathcal{F}(f(t+a)) = e^{ia\omega}\hat{f}(\omega)$$

$$2)\ \mathcal{F}^{-1}(\hat{f}(\omega+b)) = e^{-ibt} f(t)$$

(1.2) Fourier Series:

A Fourier series is a special case of Fourier transform representing a periodic or repeating function. Let $L^2(0,2\pi)$ denotes the collection of all measurable functions f defined on the interval $(0,2\pi)$ with

$$\int_0^{2\pi} |f(x)|^2\, dx < \infty$$

where, f is a piecewise continuous function. It will always be assumed that functions in $L^2(0,2\pi)$ are extended periodically to the real line.

$\mathbb{R} = (-\infty,\infty)$ namely $f(x) = f(x + 2\pi)$ for all x

Hence the collection $L^2(0,2\pi)$ is often called the space of 2π-periodic square integrable functions. $L^2(0,2\pi)$ is a vector space. Any $f \in L^2(0,2\pi)$ has a Fourier series representation

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \qquad \text{---(1.2.1)}$$

where, the constants C_n, called the Fourier coefficients of f, are defined by,

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx \qquad \text{---(1.2.2)}$$

The convergence of the series in (1.2.1) is in $L^2(0,2\pi)$, meaning that,

$$\lim_{M,N \longrightarrow \infty} \int_0^{2\pi} \left| f(x) - \sum_{n=-M}^{N} C_n \, e^{inx} \right|^2 dx = 0$$

From the equation (1.2.1) we notice that f decomposes into a sum of infinitely many mutually orthogonally components

$g_n(x) = C_n e^{inx}$ where orthogonality means that,

$$\langle g_n, g_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} g_n(x) \cdot g_m(x) \, dx = 0 \quad \text{for all} \quad m \neq n$$

---(1.2.3)

Since $g_n(x) = C_n e^{inx}$ is combination of sinusoidal waves sin(nx) and cos(nx). We may say that Fourier series expansion of f is equivalent to resolving the function in terms of its various frequency components. It may however be noted that this spectrum of f (description of amplitude at various frequency levels) exists only at a discrete values of frequencies. It is therefore, a discrete spectrum some time referred to as line spectrum. It is due to this reason that the Fourier series representation (1.2.1) is called Discrete Fourier Transform.

Analogous to the Hilbert space $L^2(\mathbb{R})$, and space $L^2(0,2\pi)$, the space $\ell^2(\mathbb{Z})$ are also a Hilbert spaces with the inner product

$$\langle \{ a_k \}, \{ b_k \} \rangle = \sum_{k \in \mathbb{Z}} a_k \cdot \overline{b}_k \qquad \text{---(1.2.4)}$$

We know that the Fourier transform of an analog signal f with finite energy describes the spectral behavior. Similarly the 'discrete Fourier transform' $\mathcal{F}^*$ of a digital signal $\{ C_k \} \in \ell^p(\mathbb{Z})$ to describe its spectral behavior as follows,

$$(\mathcal{F}^* \{ C_k\}) (x) = \sum_{n=-\infty}^{\infty} C_k e^{ikx} \qquad ---(1.2.5)$$

The 'discrete Fourier transform' of $\{C_k\}$ is the "Fourier Series" with "Fourier coefficient" given by $\{ C_k \}$. We do not know whether the series (1.2.5) is convergent or not, but, for $\{ C_k \} \in \ell^2(\mathbb{Z})$ series is absolutely and uniformly convergent, therefore we considered this series as a "symbol" of sequence $\{ C_k \}$. Since $e^{ix} = \cos(x)+i\sin(x)$ the series in (1.2.5) can also be written as,

$$f(x) = \sum_{n=-\infty}^{\infty} C_k e^{ikx} = \frac{a_o}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] \qquad ---(1.2.6)$$

where,

$$a_o = \frac{1}{\pi} \int_0^{2\pi} f(t)\, dt$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(kx)\, dt \qquad ---(1.2.7)$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(kx)\, dt$$

The $f(x)$ in (1.2.6) can be used as a notation for Fourier series . Set

$$(S_N f)(x) = \sum_{k=-N}^{N} C_k e^{ikx}$$

$$= \frac{a_o}{2} + \sum_{k=1}^{N} [a_k \cos(kx) + b_k \sin(kx)] \quad \text{---(1.2.8)}$$

where, N is positive integer.

This is called partial sum of the Fourier series f.

The N^{th} degree trigonometric polynomial

$$D_N(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \cos(kx)$$

Since,

$$\frac{1}{2} + \sum_{k=1}^{\infty} \cos(kx) = \frac{\sin(x + \frac{1}{2})}{2 \sin(x/2)} \qquad x \text{ is not multiple of } 2\pi$$

$$D_N(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \cos(k) = \frac{\sin(x + \frac{1}{2})}{2 \sin(x/2)} \quad \text{---(1.2.9)}$$

$D_N(x)$ is called as Dirichlet kernel of degree N. Here x must not multiple of 2π. If it is, i.e., $x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \cdots$ then we interpret $D_n(x) = N + 1/2$, so that D_N will be continuous on $(-\infty, \infty)$

$$(S_N f)(x) = \frac{a_o}{2} + \sum_{k=1}^{N} [a_k \cos(kx) + b_k \sin(kx)]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\cos(kx)\int_0^{2\pi} f(t)\cos(kx)dt + \right.$$

$$\left. + \sin(kx)\int_0^{2\pi} f(t)\sin(kx)dt\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{\infty}\cos k(x-t)\right]dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)\, D_N(x-t)\, dt \qquad \text{---(1.2.10)}$$

This shows that function f can be obtained by the "convolution" of f with the Dirichlet kernel of degree N. The integral (1.2.10) is meaningful if $f \in L^1(0,2\pi)$. On the other hand, if f is any function in $L^p(0,2\pi)$ $1 \le p < \infty$, then we can define the 'Inverse discrete Fourier transform' $\mathcal{F}^{*^{-1}}$ of f by

$$(\mathcal{F}^{*^{-1}} f)(k) = C_k(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\, dx$$

That is, $\mathcal{F}^{*^{-1}}$ takes $f \in L^p(0,2\pi)$ to a bi-infinite sequence $\{ C_k(f) : k \in \mathbb{Z} \}$. This sequence defines the Fourier series

$$\sum_{k\in\mathbb{Z}} C_k(f) \cdot e^{ikx} \qquad \text{---(1.2.11)}$$

and is called sequence of Fourier coefficient.

<u>Theorem</u>(1.6): Let $f \in L^2(0,2\pi)$. Then the sequence $\{ C_k(f) \}$ of Fourier coefficient of f is in $\ell^2(\mathbb{Z})$ and satisfies Bessel's inequality.

$$\sum_{k\in\mathbb{Z}} |C_k(f)|^2 \leq \| f \|^2_{L^2(0,2\pi)} \qquad ---(1.2.12)$$

<u>Theorem</u>(1.7): Let $\{ C_k \} \in \ell^2(\mathbb{Z})$. Then there exists $f \in L^2(0,2\pi)$ such that C_k is the k^{th} Fourier coefficient of f. Further more,

$$\sum_{k\in\mathbb{Z}} |C_k|^2 \leq \| f \|^2_{L^2(0,2\pi)} \qquad ---(1.2.13)$$

From this Theorem, we have, discrete Fourier transform $\mathcal{F}^*$ maps $\ell^2(\mathbb{Z})$ into $f \in L^2(0,2\pi)$.

The Parseval Identity is given as,

$$\sum_{k\in\mathbb{Z}} |C_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} | f(x) |^2 \, dx \quad f \in L^2(0,2\pi) \qquad ---(1.2.14)$$