CHAPTER 2

ADMISSIBILITY OF ESTIMATORS FOR EXPONENTIAL FAMILIES WITH QUADRATIC LOSS

2.0 Introduction:

In dealing with estimation of a single unknown parameter the criteria employed in evaluating the worth of given estimates is to make comparisons of the expected square deviation (say) of the estimates from the true value. Suppose on the basis of an observation x (or series of observations) on a distribution $P(x,\theta)$ depending on an unknown parameter θ, it is desired to estimate some function $g(\theta)$. The quantity $p(x,\theta)$ may be regarded as the density of $P(x,\theta)$ with respect to measure μ. This measure μ dominates P_θ for $\theta \in \Theta$. A non-randomized estimate of $g(\theta)$ is described by $a(x)$, a function of the observations, and when the error of an estimate is evaluated in terms of quadratic loss, the risk for the estimate $a(x)$ when the true parameter value is θ is calculated by means of the formula,

$$R(\theta, a) = \int [a(x) - g(\theta)]^2 \, p(x, \theta) \, d\mu(x) \qquad (1)$$

The object is to select estimate 'a' which minimizes (1) in some sense.

The quadratic loss as a measure of the discrepancy of an estimate is derived from the following two characteristics.

i) in the case where a(x) represents an unbiased estimate of $g(\Theta)$, and the equation (1) is represented as the variance of $a(x)$; and

ii) from a technical and mathematical viewpoint square error leads itself most easily to manipulation and computation.

Different optimizations criteria are : the minimax criteria, Bayes procedures, uniformly minimum variance unbiased estimates etc. Another desirable property of a statistical procedure is the 'admissibility'. An estimate 'a' is said to be admissible if there exists no other estimate a* such that

$$R(\Theta, a^*) \leq R(\Theta, a)$$

with strict inequality for some Θ. In other words estimating procedure is admissible if it cannot be uniformly improved interms of risk by any other procedure.

Certainly an estimator σ_1 should not be preferred if there exists an estimator σ_2 which is better than σ_1, for every value of $\Theta \in Ⓗ$.

The general question of resolving admissibility of all estimates measured with respect to the quadratic loss function is difficult, it seems worth while to concentrate on the investigation of whether some of the most commonly employed classical estimates are admissible.

In this chapeter we study the problem of admissibility of certain estimators for exponential family with density of the form

$$P(x, \Theta) = \beta(\Theta)\, e^{\Theta T(x)}$$

2.1 On a theorem of Karlin :

In this section the random variable X is assumed to have the density,

$$P(x,\Theta) = [\beta(\Theta)]^{-1} e^{\Theta x}$$

w.r. to a σ - finite measure μ (The above density is of the form of the density given in (1.4.1). Here for convenience β(Θ) is taken as the deuisor instead of the multiplier as taken in (1.4.1)) defined on the real line, and Θ the unknown state of nature, belongs to the set

$$\textcircled{H} = \left\{ \Theta \,/\, \int_{-\infty}^{\infty} e^{\Theta x} \, d\mu(x) < \infty \right\}$$

which is an interval of the real line. Let $\bar{\Theta}$ and $\underline{\Theta}$ be the upper and lower end points of Ⓗ respectively, $\bar{\Theta}$ and $\underline{\Theta}$ may or may not belong to Ⓗ, $\underline{\Theta}$ and $\bar{\Theta}$ may be equal to $-\infty$ and ∞ respectively.

The problem for consideration is the estimation of the quantity $g(\Theta) = E_{\Theta}(x) = \frac{\beta'(\Theta)}{\beta(\Theta)}$ based on a random sample $X_1, X_2, \ldots, X_n$ of size n. There is no less of generality in restricating on attention to the case of a single observation for, as is well known, a sufficient statistic for n observations from an exponential distribution is the sum of the observations whose distribution is also a member of the exponential family.

From theorem 1.4.1 the estimate a X (by taking b=0) is inadmissible if a < 0 or a > 1. Hence the admissibility of aX for g(Θ) is to be disscussed only for $0 \leq a \leq 1$.

For convenience $a = \frac{1}{1+u}$ and the admissibility of $\frac{X}{1+u}$ is to be discussed only for $u \geq 0$.

Karlin has considered the admissibility of linear estimates $\frac{x}{1+u}$ where $u \geq 0$, for $E_\Theta(X)$ and has proved the following results.

Theorem 2.1.1 :

If Θ' being any arbitrary interior point of Ⓗ, and $u \geq 0$

a) $$\int_{\underline{\Theta}}^{\Theta'} \beta^u(\Theta)\, d\Theta = \infty \quad \text{and}$$

b) $$\int_{\Theta'}^{\bar{\Theta}} \beta^u(\Theta)\, d\Theta = \infty$$

then the estimate $\frac{x}{1+u}$ is admissible for estimating $E_\Theta(X)$.

The proof of this directly follows from theorem 1.4.2 with $\gamma = 0$.

Karlin has conjectured that the conditions in theorem 2.1.1 are not merely sufficient, but are necessary also for the admissibility of $\frac{x}{1+u}$. It can be shown that $\frac{x}{1+u}$ is inadmissible for $g(\Theta)$ for certain values of u. In this respect we have the following result.

Lemma (2.1.1):

The estimate $\frac{x}{1+u}$ is inadmissible for $g(\Theta) = \frac{\beta'(\Theta)}{\beta(\Theta)}$ if $u < L_1$ or $u > L_2$, where L_1 and L_2 are the infimum and supremum respectively as Θ varies over Ⓗ, of

$$I^2(\Theta) = -\frac{d}{d\Theta}\left[\frac{\beta(\Theta)}{\beta'(\Theta)}\right] = \frac{\beta(\Theta)\beta''(\Theta) - (\beta'\Theta))^2}{(\beta'(\Theta))^2}$$

Where as the criteria for admissibility in theorem 2.1.1 depend on the behaviour of $\beta(\theta)$, only near the end points of Ⓗ, $\underline{\theta}$ and $\bar{\theta}$. The criteria for inadmissibility in lemma 2.1.1 depend on the variation of $\beta(\theta)$ over the whole interval Ⓗ. It is therefore of interest to obtain criteria for inadmissibility for the estimate $\frac{x}{1+u}$, which depend on the behaviour of $\beta(\theta)$ only at the end points of Ⓗ.

For all $\theta \in$ Ⓗ,

$$\beta(\theta) = \int_{-\infty}^{\infty} e^{\theta x}\, d\mu\,(x) > 0$$

$$\beta'(\theta) = \int_{-\infty}^{\infty} x.\, e^{\theta x}\, d\mu\,(x),$$

$$\beta''(\theta) = \int_{-\infty}^{\infty} x^2.\, e^{\theta x}\, d\mu\,(x) > 0$$ which implies $\beta(\theta)$ is a convex function over the domain Ⓗ. Therefore we have three possible cases :-

(I) $\beta'(\theta)$ is always positive.

In this case $\beta(\theta)$ is always increasing in Ⓗ, it then follows from the definition of Ⓗ that $\underline{\theta}=-\infty$, and further $\mu\,(x < 0) = 0$ as otherwise $\beta(\theta)$ will $\to \infty$ as $\theta \to -\infty$.

(II) $\beta'(\theta)$ is always negative.

In this case $\beta(\theta)$ is always decreasing so that by the definition of Ⓗ, $\bar{\theta} = +\infty$ and further $\mu(x > 0) = 0$.

(III) $\beta'(\theta)$ is negative initially but increases to a positive value.

In this case $\beta(\theta)$ is a decreasing function at first until it reaches a minimum value and increases thereafter, $\bar{\theta}$ and $\underline{\theta}$ may be finite or infinite and $\mu(x<0)>0$ and also $\mu(x>0)>0$. Otherwise let $\mu(x<0)=0$ then we have for every $\theta \in (H)$,

$$\int_0^\infty e^{\theta x}\, d\mu(x) = \beta(\theta)$$

i.e.

$$\beta'(\theta) = \int_0^\infty x e^{\theta x}\, d\mu(x) > 0 \text{ for every } \theta$$

which is a contradiction.

We give the only statement of the improved criteria for inadmissibility of the estimate $\frac{x}{1+u}$, in the form of lemma. For the proof we refer to Joshi (1969).

<u>Lemma</u> (2.1.2) :

The estimate $x/1+u$ is inadmissible for

$$g(\theta) = E_\theta(X) = \frac{\beta'(\theta)}{\beta(\theta)},$$

(I) When $\beta'(\theta)$ is positive for all $\theta \in (H)$, if,

i) $u > \lim \operatorname{Sup}_{\theta \to -\infty} I^2(\theta)$ or

ii) $u < \lim \inf_{\theta \to \bar{\theta}} I^2(\theta)$

(II) When $\beta'(\theta)$ is negative for all $\theta \in (H)$, if

i) $u > \lim \operatorname{Sup}_{\theta \to +\infty} I^2(\theta)$ or

ii) $u < \lim \inf_{\theta \to \underline{\theta}} I^2(\theta)$

(III) When $\beta'(\theta)$ assumes both positive and negative values if,

i) $u < \liminf_{\theta \to \underline{\theta}} I^2(\theta)$ or

ii) $u < \liminf_{\theta \to \bar{\theta}} I^2(\theta)$ □

Necessary conditions for the convergence or divergence of the integrals in Karlin's theorem (2.1.1) are easily obtained. Now using the improved criteria in lemma 2.1.2 the range of values of u for which Karlin's conjectures remain open, is narrowed down.

Karlin's integrals

Let I_1 denote the integral in condition (a) and I_2 the integral in condition (b) of theorem 2.1.1 that is,

$$I_1 = \int_{\underline{\theta}}^{\theta'} \beta^u(\theta)\, d\theta \quad \text{and}$$

$$I_2 = \int_{\theta'}^{\bar{\theta}} \beta^u(\theta)\, d\theta$$

We have seen that there are three possible cases. As the same method is applicable to all the cases, we shall consider only the case (I) that is $\beta'(\theta)$ always positive and prove the following :

Lemma (2.1.3) :

If $\beta'(\theta) > 0$ for all $\theta \in \Theta$, then the integral I_1,

i) converges if $u > \limsup_{\theta \to -\infty} I^2(\theta)$ and

ii) diverges if $u < \liminf_{\theta \to -\infty} I^2(\theta)$. Similarly the integral I_2

iii) converges if $u < \liminf_{\theta \to \bar{\theta}} I^2(\theta)$ and

iv) diverges if $u > \limsup_{\theta \to \bar{\theta}} I^2(\theta)$

Proof :

$$I^2(\theta) = -\frac{d}{d\theta}\left[\frac{\beta(\theta)}{\beta'(\theta)}\right] \qquad (1)$$

Let $$k_1 = \liminf_{\theta \to -\infty} I^2(\theta) \qquad (2)$$

Suppose $k_1 > 0$, and let u be any number less than k_1.

$$\text{Let} \quad u = k_1 - \epsilon, \quad \epsilon > 0 \qquad (3)$$

(2) implies that we can obtain θ_o such that

$$I^2(\theta) \geq k_1 - \frac{\epsilon}{2} \quad \text{for all } \theta \leq \theta_o \qquad (4)$$

Hence by (1),

$$-\frac{d}{d\theta}\left[\frac{\beta(\theta)}{\beta'(\theta)}\right] \geq u + \frac{\epsilon}{2} \quad \text{for} \quad \theta \leq \theta_o \qquad (5)$$

Integrating both sides of (5) with respect to θ from an arbitrary point θ up to the point θ_o, we have

$$-\int_{\theta}^{\theta_o} \frac{d}{d\theta}\left[\frac{\beta(\theta)}{\beta'(\theta)}\right] d\theta \geq \int_{\theta}^{\theta_o} \left(u + \frac{\epsilon}{2}\right) d\theta$$

$$-\left[\frac{\beta(\theta)}{\beta'(\theta)}\right]_{\theta}^{\theta_o} \geq \left(u + \frac{\epsilon}{2}\right)[\theta]_{\theta}^{\theta_o}$$

$$\frac{\beta(\theta)}{\beta'(\theta)} - \frac{\beta(\theta_o)}{\beta'(\theta_o)} \geq \left(u + \frac{\epsilon}{2}\right)(\theta_o - \theta),$$

putting $C_o = \dfrac{\beta(\theta_o)}{\beta'(\theta_o)} > 0$, therefore

$$\frac{\beta(\theta)}{\beta'(\theta)} \geq C_o + \left(u + \frac{\epsilon}{2}\right)(\theta_o - \theta), \text{ for all } \theta \leq \theta_o$$

and hence,

$$\frac{\beta'(\theta)}{\beta(\theta)} \leq \left[C_o + \left(u + \frac{\epsilon}{2}\right)(\theta_o - \theta)\right]^{-1} \text{ for all } \theta \leq \theta_o \qquad (6)$$

Again integrating both sides of (6) with respect to θ, from an arbitrary point $\theta < \theta_o$ upto θ_o,

$$\int_{\theta}^{\theta_o} \frac{\beta'(\theta)}{\beta(\theta)} d\theta \leq \int_{\theta}^{\theta_o} [C_o + (u + \frac{\epsilon}{2})(\theta_o - \theta)]^{-1} d\theta$$

$$[\log \beta(\theta)]_{\theta}^{\theta_o} \leq \int_{\theta}^{\theta_o} \frac{1}{C_o + (u + \frac{\epsilon}{2})(\theta_o - \theta)} d\theta$$

$$[\log \beta(\theta)]_{\theta}^{\theta_o} \leq \int_{u'}^{C_o} \frac{1}{u'} \frac{du}{-(u + \frac{\epsilon}{2})}$$

where $C_o + (u + \frac{\epsilon}{2})(\theta_o - \theta) = u'$

$$= -(u + \frac{\epsilon}{2})^{-1} \int_{u'}^{C_o} \frac{1}{u'} du'$$

$$= -(u + \frac{\epsilon}{2})^{-1} [\log u']_{u'}^{C_o}$$

$$= -(u + \frac{\epsilon}{2})^{-1} [\log C_o - \log u']$$

$$= -(u + \frac{\epsilon}{2})^{-1} \log (\frac{C_o}{u'})$$

$$= -(u + \frac{\epsilon}{2})^{-1} \log (\frac{C_o}{C_o + (u + \frac{\epsilon}{2})(\theta_o - \theta)})$$

$$= (u + \frac{\epsilon}{2})^{-1} \log (\frac{C_o + (u + \epsilon/2)(\theta_o - \theta)}{C_o})$$

$$= (u + \frac{\epsilon}{2})^{-1} \log [1 + (u + \frac{\epsilon}{2}) C_o^{-1} (\theta_o - \theta)]$$

$$\log \beta(\theta_o) - \log \beta(\theta) \leq (u + \frac{\epsilon}{2})^{-1} \log [1 + (u + \frac{\epsilon}{2}) C_o^{-1} (\theta_o - \theta)]$$

$$\log \frac{\beta(\theta_o)}{\beta(\theta)} \leq (u + \frac{\epsilon}{2})^{-1} \log [1 + (u + \frac{\epsilon}{2}) C_o^{-1} (\theta_o - \theta)]$$

putting $\beta(\theta_o) = \beta_o$, Therefore

$$\log \frac{\beta_o}{\beta} \leq (u + \frac{\epsilon}{2})^{-1} \log [1 + (u + \frac{\epsilon}{2}) C_o^{-1} (\theta_o - \theta)]$$

it implies that

$$\frac{\beta_o}{\beta} \leq [1 + (u + \frac{\epsilon}{2}) C_o^{-1} (\theta_o - \theta)]^{\frac{1}{u + \epsilon/2}}$$

$$\therefore \frac{\beta}{\beta_o} \geq \left[1 + \left(u + \frac{\epsilon}{2}\right) C_o^{-1}(\theta_o - \theta) \right]^{-\frac{1}{u+\epsilon/2}}$$

and hence

$$\beta^u \geq \beta_o^u \left[1 + \left(u + \frac{\epsilon}{2}\right) C_o^{-1} (\theta_o - \theta)\right]^{-\frac{u}{u+\epsilon/2}}$$

for all $\theta \leq \theta_o$ (7)

Integrating both sides of (7) with respect to θ from $-\infty$ to θ_o, we get

$$\int_{-\infty}^{\theta_o} \beta^u(\theta)\, d\theta \geq \int_{-\infty}^{\theta_o} \beta_o^u \left[1 + \left(u + \frac{\epsilon}{2}\right) C_o^{-1}(\theta_o - \theta)\right]^{-\frac{u}{u+\epsilon/2}} d\theta \quad (8)$$

Now consider R.H.S. of (8)

$$\int_{-\infty}^{\theta_o} \beta_o^u \left[1 + \left(u + \frac{\epsilon}{2}\right) C_o^{-1} (\theta_o - \theta)\right]^{-\frac{u}{u+\epsilon/2}} d\theta$$

$$= -\beta_o^u \int_{\infty}^{1} u'^{-\frac{u}{u+\epsilon/2}} \frac{du'}{\left(u + \frac{\epsilon}{2}\right) C_o^{-1}}$$

where $1 + \left(u + \frac{\epsilon}{2}\right) C_o^{-1}(\theta_o - \theta) = u'$

$$= C_o \beta_o^u \left[u + \frac{\epsilon}{2}\right]^{-1} \int_1^{\infty} u'^{-\frac{u}{u+\frac{\epsilon}{2}}} du'$$

$$= C_o \beta_o^u \left(u + \frac{\epsilon}{2}\right)^{-1} \left[\frac{u'^{-\frac{u}{u+\epsilon/2}+1}}{1 - \frac{u}{u+\frac{\epsilon}{2}}}\right]_1^{\infty} = \infty$$

Thus as the integral of right hand of (8) over $(-\infty, \theta_o)$ diverges, the integral of the left hand side that is I_1 must also diverge. Thus I_1 diverges if $u < \liminf_{\theta \to -\infty} I^2(\theta)$. This proves (ii) of the lemma 2.1.3 .

Let $k_2 = \limsup_{\theta \to -\infty} I^2(\theta)$ (9)

Suppose $k_2 > 0$, and let u be any number greater than k_2.

Let $u = k_2 + \epsilon$, $\epsilon > 0$ (10)

(9) implies that we can obtain θ_o such that

$$I^2(\theta) \leq k_2 + \frac{\epsilon}{2} \quad \text{for all } \theta \leq \theta_o \tag{11}$$

Hence by (1),

$$-\frac{d}{d\theta}\left[\frac{\beta(\theta)}{\beta'(\theta)}\right] \leq u - \frac{\epsilon}{2} \quad \text{for } \theta \leq \theta_o . \tag{12}$$

Integrating both sides of (12) with respect to θ from an arbitrary point θ upto the point θ_o.

We have

$$\int_{\theta}^{\theta_o} -\frac{d}{d\theta}\left[\frac{\beta(\theta)}{\beta'(\theta)}\right] d\theta \leq \int_{\theta}^{\theta_o} (u - \frac{\epsilon}{2})\, d\theta$$

$$\therefore \quad \left[-\frac{\beta(\theta)}{\beta'(\theta)}\right]_{\theta}^{\theta_o} \leq (u - \frac{\epsilon}{2})(\theta_o - \theta)$$

$$\frac{\beta(\theta)}{\beta'(\theta)} - \frac{\beta(\theta_o)}{\beta'(\theta_o)} \leq (u - \frac{\epsilon}{2})(\theta_o - \theta)$$

putting $C_o = \frac{\beta(\theta_o)}{\beta'(\theta_o)} > 0$, therefore

$$\frac{\beta(\theta)}{\beta'(\theta)} \leq C_o + (u - \frac{\epsilon}{2})\,(\theta_o - \theta), \text{ for all } \theta \leq \theta_o$$

it implies that,

$$\frac{\beta'(\theta)}{\beta(\theta)} \geq \left[C_o + (u - \frac{\epsilon}{2})\,(\theta_o - \theta)\right]^{-1} \quad \text{for all } \theta \leq \theta_o \tag{13}$$

Again integrating both sides of (13) with respect to θ, from an arbitrary point $\theta < \theta_o$ upto θ_o,

therefore

$$\int_{\theta}^{\theta_o} \frac{\beta'(\theta)}{\beta(\theta)}\, d\theta \geq \int_{\theta}^{\theta_o} \left[C_o + (u - \frac{\epsilon}{2})(\theta_o - \theta)\right]^{-1} d\theta$$

$$\therefore [\log \beta(\theta)]_{\theta}^{\theta_o} \geq \int_{\theta}^{\theta_o} [C_o + (u - \tfrac{\epsilon}{2})(\theta_o - \theta)]^{-1} d\theta \qquad (14)$$

consider,

$$\int_{\theta}^{\theta_o} [\, C_o + (u - \tfrac{\epsilon}{2})\, (\theta_o - \theta)\,]^{-1} d\theta$$

$$= \int_{u'}^{C_o} \frac{1}{u'} \, \frac{du'}{-(u - \frac{\epsilon}{2})},$$

where $\quad C_o + (u - \frac{\epsilon}{2})\, (\theta_o - \theta) = u'$

$$= -\, (u - \tfrac{\epsilon}{2})^{-1} \int_{u'}^{C_o} \frac{1}{u'}\, du$$

$$= -\, (u - \tfrac{\epsilon}{2})^{-1} \, [\, \log u]_{u'}^{C_o}$$

$$= -\, (u - \tfrac{\epsilon}{2})^{-1} \log \left(\frac{C_o}{u'} \right)$$

$$= -\, (u - \tfrac{\epsilon}{2})^{-1} \log \left(\frac{u'}{C_o} \right)^{-1}$$

$$= -\, (u - \tfrac{\epsilon}{2})^{-1} \log \left[\frac{C_o + (u - \frac{\epsilon}{2})\, (\theta_o - \theta)}{C_o} \right]^{-1}$$

$$= -\, (u - \tfrac{\epsilon}{2})^{-1} \log [\, 1 + (u - \tfrac{\epsilon}{2})\, C_o^{-1}\, (\theta_o - \theta)]^{-1}$$

$$= -\, \frac{1}{(u - \frac{\epsilon}{2})} \log [\, 1 + (u - \tfrac{\epsilon}{2})\, C_o^{-1}\, (\theta_o - \theta)\,]^{-1}$$

$$= \log [\, 1 + (\, u - \tfrac{\epsilon}{2})\, C_o^{-1}\, (\theta_o - \theta)\,]^{\frac{1}{u - \frac{\epsilon}{2}}}$$

Thus (14) gives us

$$\log\left[\frac{\beta(\theta_o)}{\beta(\theta)} \right] \geq \log [\, 1 + (u - \tfrac{\epsilon}{2})\, C_o^{-1}\, (\theta_o - \theta)\,]^{\frac{1}{u - \frac{\epsilon}{2}}}$$

therefore,

$$\frac{\beta(\theta_o)}{\beta(\theta)} \geq [\, 1 + (u - \tfrac{\epsilon}{2})\, C_o^{-1}\, (\theta_o - \theta)\,]^{\frac{1}{u - \frac{\epsilon}{2}}}$$

it implies that

$$\frac{\beta(\theta)}{\beta(\theta_o)} \leq \left[1 + \left(u - \frac{\epsilon}{2}\right) C_o^{-1} (\theta_o - \theta)\right]^{-\frac{1}{u-\frac{\epsilon}{2}}}$$

putting $\beta(\theta_o) = \beta_o$, therefore

$$\beta^{u}(\theta) \leq \beta_o^{u} \left[1 + \left(u - \frac{\epsilon}{2}\right) C_o^{-1}(\theta_o - \theta)\right]^{-\frac{u}{u-\epsilon/2}} \text{ for all } \theta \leq \theta_o \quad (15)$$

Integrating both sides of (15) with respect to θ from $-\infty$ to θ_o, we have

$$\int_{-\infty}^{\theta_o} \beta^{u}(\theta)d\theta \leq \beta_o^{u} \int_{-\infty}^{\theta_o} \left[1 + \left(u - \frac{\epsilon}{2}\right) C_o^{-1}(\theta_o-\theta)\right]^{-\frac{u}{u-\epsilon/2}} d\theta \quad (16)$$

Consider the integral right hand side of (16),

$$\int_{-\infty}^{\theta_o} \left[1 + \left(u - \frac{\epsilon}{2}\right) C_o^{-1} (\theta_o-\theta)\right]^{-\frac{u}{u-\epsilon/2}} d\theta$$

$$= \int_{\infty}^{1} u'^{-\frac{u}{u-\epsilon/2}} \frac{du'}{-C_o^{-1}(u - \epsilon/2)},$$

where

$$u' = 1 + \left(u - \frac{\epsilon}{2}\right) C_o^{-1} (\theta_o - \theta)$$

$$= \frac{C_o}{\left(u - \frac{\epsilon}{2}\right)} \int_{1}^{\infty} u'^{-\frac{u}{u-\frac{\epsilon}{2}}} du'$$

$$= \frac{C_o}{\left(u - \frac{\epsilon}{2}\right)} \left[\frac{u'^{1-\frac{u}{u-\epsilon/2}}}{1 - \frac{u}{u-\epsilon/2}} \right]_{1}^{\infty} < \infty$$

As the integral of the right hand side of (16) over $(-\infty, \theta_o)$ converges, the integral of the left hand side of (16) is also converges, that is I_1 must also converge.

Thus I_1 converges if $u > \lim \text{Sup}_{\theta \to -\infty} I^2(\theta)$.
This proves (i) of the lemma 2.1.3 .

Case (iii):

$\bar{\theta}$ may be $+\infty$ or may be finite. If $\bar{\theta}$ is $+\infty$ the proof is exactly similar to that given above.

Next suppose that $\bar{\theta}$ is finite. As $\theta \to \bar{\theta}$, $\beta(\theta)$ being non-decreasing may either

(A) diverge to infinity or

(B) converge to a finite limit.

In case (A), log β also tends to ∞ as θ tends to $\bar{\theta}$ and hence as $\bar{\theta}$ is finite, $\frac{d}{d\theta} \log \beta = \frac{\beta'(\theta)}{\beta(\theta)}$ must diverge to ∞ as $\theta \to \bar{\theta}$. In case (B), by the definition of the set (H) implies that

$$\beta(\theta) = \infty \text{ for any } \theta > \bar{\theta},$$

hence $\beta'(\theta)$ must tends to ∞ as θ tends to $\bar{\theta}$. As $\beta(\theta)$ converges to a finite limit $\frac{\beta'(\theta)}{\beta(\theta)}$ tends to ∞ as θ tends to $\bar{\theta}$.

Thus in both cases (A) and (B) as θ tends to $\bar{\theta}$, $\frac{\beta'(\theta)}{\beta(\theta)}$ tends to ∞ and hence

$$\frac{\beta(\theta)}{\beta'(\theta)} \longrightarrow 0 \tag{17}$$

Let:

$$K_1 = \lim \inf_{\theta \to \bar{\theta}} I^2(\theta) \tag{18}$$

and let u be any number $< K_1$. Put

$$u = K_1 - \epsilon \;, \epsilon > 0 \tag{19}$$

(18) implies that we can find $\theta_o < \bar{\theta}$, such that for all θ, $\theta_o \le \theta < \bar{\theta}$

$$I^2(\theta) \ge K_1 - \frac{\epsilon}{2} \tag{20}$$

Hence by (1),

$$-\frac{d}{d\theta}\left[\frac{\beta(\theta)}{\beta'(\theta)}\right] \ge u + \frac{\epsilon}{2} \;, \; \theta_o \le \theta < \bar{\theta} \tag{21}$$

Integrating both sides of (21) with respect to θ, from an arbitrary point θ ($< \bar{\theta}$) to the point $\bar{\theta}$,

we have,

$$\int_{\theta}^{\bar{\theta}} -\frac{d}{d\theta}\left[\frac{\beta(\theta)}{\beta'(\theta)}\right] d\theta \ge \left(u + \frac{\epsilon}{2}\right) \int_{\theta}^{\bar{\theta}} d\theta$$

$$\therefore \left[-\frac{\beta(\theta)}{\beta'(\theta)}\right]_{\theta}^{\bar{\theta}} \ge \left(u + \frac{\epsilon}{2}\right)(\bar{\theta} - \theta)$$

From (17) $\frac{\beta(\theta)}{\beta'(\theta)} \longrightarrow 0$ as $\theta \longrightarrow \bar{\theta}$.

Therefore,

$$\frac{\beta(\theta)}{\beta'(\theta)} \ge \left(u + \frac{\epsilon}{2}\right)(\bar{\theta} - \theta), \quad \theta_o \le \theta < \bar{\theta}$$

it implies that

$$\frac{\beta'}{\beta} \le \left[\left(u + \frac{\epsilon}{2}\right)(\bar{\theta} - \theta)\right]^{-1}, \quad \theta_o \le \theta < \bar{\theta} \tag{22}$$

Integrating both sides of (22) with respect to θ, from θ_o to a point $\theta < \bar{\theta}$, we have

$$\int_{\theta_0}^{\theta} \frac{\beta'(\theta)}{\beta(\theta)} d\theta \leq \int_{\theta_0}^{\theta} [(u + \frac{\epsilon}{2}) (\bar{\theta} - \theta)]^{-1} d\theta$$

$$\therefore \quad [\log \beta(\theta)]_{\theta_0}^{\theta} \leq (u + \frac{\epsilon}{2})^{-1} \int_{\theta_0}^{\theta} \frac{1}{\bar{\theta} - \theta} d\theta$$

$$\log[\frac{\beta(\theta)}{\beta(\theta_0)}] \leq -(u + \frac{\epsilon}{2})^{-1} \int_{\bar{\theta}-\theta_0}^{\bar{\theta}-\theta} \frac{1}{u'} du'$$

where $u' = \bar{\theta} - \theta$

$$= - (u + \frac{\epsilon}{2})^{-1} [\log u']_{\bar{\theta}-\theta_0}^{\bar{\theta}-\theta}$$

$$= - (u + \frac{\epsilon}{2})^{-1} \log (\frac{\bar{\theta} - \theta}{\bar{\theta} - \theta_0})$$

hence

$$\log [\frac{\beta(\theta)}{\beta(\theta_0)}] \leq - (u + \frac{\epsilon}{2})^{-1} \log (\frac{\bar{\theta} - \theta}{\bar{\theta} - \theta_0})$$

$$= \log (\frac{\bar{\theta} - \theta}{\bar{\theta} - \theta_0})^{-\frac{1}{u+\frac{\epsilon}{2}}}$$

Therefore

$$\frac{\beta(\theta)}{\beta(\theta_0)} \leq (\frac{\bar{\theta} - \theta}{\bar{\theta} - \theta_0})^{-\frac{1}{u+\frac{\epsilon}{2}}}$$

i.e. $\beta(\theta) \leq \beta(\theta_0) (\bar{\theta} - \theta)^{-\frac{1}{u+\frac{\epsilon}{2}}} \cdot (\bar{\theta} - \theta_0)^{\frac{1}{u+\frac{\epsilon}{2}}}$

it implies that

$$\beta^u (\theta) \leq \beta^u(\theta_0)(\bar{\theta} - \theta)^{-\frac{u}{u+\frac{\epsilon}{2}}} \cdot (\bar{\theta} - \theta_0)^{\frac{u}{u+\frac{\epsilon}{2}}} \qquad (23)$$

Integrating on both sides of (23) with respect to θ from θ_0 to $\bar{\theta}$,

$$\int_{\theta_0}^{\bar{\theta}} \beta^u(\theta)\, d\theta \le \beta^u(\theta_0) \int_{\theta_0}^{\bar{\theta}} (\bar{\theta}-\theta)^{-\frac{u}{u+\epsilon}/2} (\bar{\theta}-\theta_0)^{\frac{u}{u+\epsilon}/2}\, d\theta$$

$$= \beta^u(\theta_0)(\bar{\theta}-\theta_0)^{\frac{u}{u+\epsilon}/2} \int_{\theta_0}^{\bar{\theta}} (\bar{\theta}-\theta)^{-\frac{u}{u+\epsilon}/2}\, d\theta$$

$$= -\beta^u(\theta_0)(\bar{\theta}-\theta_0)^{\frac{u}{u+\epsilon}/2} \int_{\bar{\theta}-\theta_0}^{0} u'^{\frac{u}{u+\epsilon}/2}\, du'$$

where $u' = \bar{\theta} - \theta$

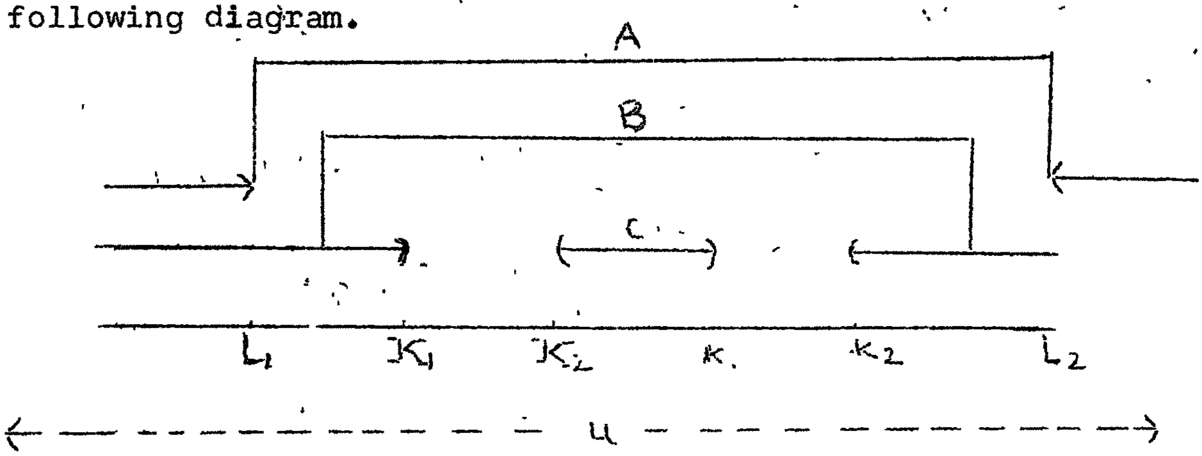
$$= \beta^u(\theta_0)(\bar{\theta}-\theta_0)^{\frac{u}{u+\epsilon}/2} \int_{0}^{\bar{\theta}-\theta_0} u'^{-\frac{u}{u+\epsilon}/2}\, du'$$

$$= \beta^u(\theta_0)(\bar{\theta}-\theta_0)^{\frac{u}{u+\epsilon}/2} \left[\frac{u'^{1-\frac{u}{u+\epsilon}/2}}{1-\frac{u}{u+\epsilon}/2} \right]_0^{\bar{\theta}-\theta_0} < \infty$$

As the integral of right hand side of (23) converges so that the integral of left hand side is also converges, that is I_2 converges. Thus I_2 converges if $u < \liminf_{\theta \to \bar{\theta}} I^2(\theta)$. This proves (iii) of the lemma (2.1.3).

Case (iv) is exactly similar. □

On combining lemma 2.1.2 and 2.1.3, the range of values of u for which Karlin's conjectures remain open is narrowed down; which can be seen from the following diagram.



Values of u indicating the ranges for which Karlin's conjecture to be solved.

A : range of u for which $\frac{x}{1+u}$ is inadmissible according to lemma 2.1.1 .

B : range of u for which $-\frac{x}{1+u}$ is inadmissible according to lemma (2.1.2).

C : range of u for which $\frac{u}{1+u}$ is admissible according to lemma 2.1.3 .

Note that C may be empty i.e. $k_1 < K_2$.

A further discussion on Karlin's conjecture remain open is discuss in Joshi (1969).

Example (2.1.1):

Let the measure be such that

$$d\mu(x) = 0, \qquad x < 0$$

$$= \left[\frac{e^{-ax} - e^{-bx}}{b-a}\right] dx, \quad x \geq 0,\ 0 < a < b.$$

By definition we have,

$$\beta(\theta) = \int_0^\infty e^{\theta x}\, d\mu(x)$$

$$= \int_0^\infty e^{\theta x}\left(\frac{e^{-ax} - e^{-bx}}{b-a}\right) dx$$

$$= \frac{1}{b-a}\left\{\int_0^\infty e^{-x(a-\theta)} dx - \int_0^\infty e^{-x(b-\theta)} dx\right\}$$

$$= \frac{1}{b-a}\left\{\left[\frac{e^{-x(a-\theta)}}{-(a-\theta)}\right]_0^\infty - \left[\frac{e^{-x(b-\theta)}}{-(b-\theta)}\right]_0^\infty\right\}$$

$$\beta(\theta) = \frac{1}{(a-\theta)(b-\theta)}$$

$$\therefore\ \beta(\theta) = \frac{1}{(a-\theta)(b-\theta)}, \quad (\underline{\theta} = -\infty,\ \bar{\theta} = a) \qquad (24)$$

$$\therefore \beta'(\theta) = \frac{a+b-2\theta}{[(a-\theta)(b-\theta)]^2} > 0$$

Therefore,

$$\beta''(\theta) = \frac{2(a+b-2\theta)^2 - 2(a-\theta)(b-\theta)}{[(a-\theta)(b-\theta)]^3}$$

hence

$$I^2(\theta) = -\frac{d}{d\theta}\frac{\beta(\theta)}{\beta'(\theta)} = \frac{\beta(\theta)\,\beta''(\theta) - (\beta'(\theta))^2}{[\beta'(\theta)]^2}$$

$$= 1 - 2\left[\frac{(a-\theta)(b-\theta)}{(a+b-2\theta)^2}\right] \qquad (25)$$

From (24) it is clear that our distribution falls under case (I).

We shall claim that by using Karlin's conjecture the estimate $\frac{x}{1+u}$ is inadmissible by showing either I_1 is finite or I_2 is finite (or both) for every value of u .

To show this first consider I_1 .

$$I_1 = \int_{\underline{\theta}}^{\theta'} \beta^{u}(\theta)\,d\theta$$

$$= \int_{-\infty}^{\theta'} [(a-\theta)(b-\theta)]^{-u}\,d\theta$$

$$\leq \int_{-\infty}^{\theta'} (a-\theta)^{-2u}\,d\theta$$

$$= -\int_{\infty}^{a-\theta'} u'^{-2u}\,du', \quad \text{where } u' = a-\theta$$

$$= \int_{a-\theta'}^{\infty} u'^{-2u}\,du'$$

$$= \left[\frac{u'^{1-2u}}{1-2u}\right]_{a-\theta'}^{\infty}$$

$$= \begin{cases} \infty & \text{if } u \leq \frac{1}{2} \\ \text{finite} & \text{if } u > \frac{1}{2} \end{cases}$$

Therefore I_1 is finite if $u > \frac{1}{2}$, Now consider I_2.

$$I_2 = \int_{\theta'}^{a} \beta u(\theta)\, d\theta$$

$$\therefore\ I_2 = \int_{\theta'}^{a} [(a-\theta)(b-\theta)]^{-u}\, d\theta$$

$$\leq \int_{\theta'}^{a} (a-\theta)^{-2u}\, d\theta$$

$$= -\int_{a-\theta'}^{0} u'^{-2u}\, du', \quad \text{where } u' = a - \theta$$

$$= \int_{0}^{a-\theta'} u'^{-2u}\, du'$$

$$= \left[\frac{u'^{1-2u}}{1-2u} \right]_{0}^{a-\theta'}$$

$$= \begin{cases} \infty & \text{if } u > \frac{1}{2} \\ \text{finite} & \text{if } u \leq \frac{1}{2} \end{cases}$$

Hence I_1 is finite for $u > \frac{1}{2}$ and I_2 is finite for $u \leq \frac{1}{2}$. Hence according to Karlin's conjecture, the estimate $\frac{x}{1+u}$ is inadmissible for every $u \geq 0$.

Now we have to find L_1 and L_2, where

$L_1 = \lim\inf\ I^2(\theta)$ and | as θ traverses the interval $(-\infty, a)$.
$L_2 = \lim\sup\ I^2(\theta)$

For finding L_1 and L_2, it is sufficient to show that $I^2(\theta)$ is increases in θ, then

$$L_1 = \lim\inf_{\theta\to-\infty} I^2(\theta) = \lim_{\theta\to-\infty} I^2(\theta)$$

$$L_2 = \lim\sup_{\theta\to a} I^2(\theta) = \lim_{\theta\to a} I^2(\theta)$$

We have to show $I^2(\theta) \uparrow \theta$.

From (25), we have to show

$$\frac{1-I^2(\theta)}{2} = \frac{(a-\theta)\ (b-\theta)}{(a+b-2\theta)^2} \downarrow \theta$$

that is

$$\frac{y\ (y+b-a)}{(2y + b-a)^2} \uparrow y, \quad y = a - \theta$$

that is

$$\frac{1 + \frac{b-a}{y}}{[2 + (\frac{b-a}{y})]^2} \uparrow y$$

that is

$$\frac{1 + Z}{(2+Z)^2} \downarrow Z, \text{ where } Z = \frac{b-a}{y}$$

that is

$$\frac{1}{2+Z} - \frac{1}{(2+Z)^2} \downarrow Z$$

that is

$u - u^2$ is increasing in u for $0 < u < \frac{1}{2}$

which is obvious.

Hence $I^2(\theta)$ is increasing in θ. Therefore

$$L_1 = \lim_{\theta \to -\infty} I^2(\theta)$$

$$= \lim_{\theta \to -\infty} 1 - \frac{2(a-\theta)(b-\theta)}{(a+b-2\theta)^2}$$

$$= \frac{1}{2}$$

and

$$L_2 = \lim_{\theta \to a} I^2(a)$$

$$= \lim_{\theta \to a} 1 - \frac{2(a-\theta)(b-\theta)}{(a+b-2\theta)^2}$$

$$= 1$$

So from lemma 2.1.1 , the inadmissibility of the estimate $\frac{x}{1+u}$ cannot be derived. It can be shown that $K_1=K_2=1$ and $k_1=k_2=1/2$. Hence by lemma 2.1.2 the estimate $\frac{x}{1+u}$ is inadmissible for $u < K_1 = 1$ or $u > k_2 = 1/2$ and hence inadmissible for every u . ∎

2.2 Truncation and Admissibility :

Let X have the p.d.f. of the form

$$P(x, \theta) = \beta(\theta)\, e^{\theta x}, \qquad (1)$$

where $\int_X e^{x\theta}\, d\mu(x) = \frac{1}{\beta(\theta)} < \infty$.

From the form of the density function, it is clear that x is uniformly minimum variance unbiased estimate (UMVUE) of

$$E(x) = -\frac{\partial \log \beta(\theta)}{\partial \theta} = g(\theta) \text{ (say)}.$$

Further

$$\text{var}(x) = -\frac{\partial^2 \log \beta}{\partial \theta^2} > 0 \text{ for every } \theta \in \Theta,$$

$g(\theta)$ is strictly increasing function of θ. The equation $\frac{\partial \log P}{\partial \theta} = x - g(\theta) = 0$ has exactly one root $\theta(x) = g^{-1}(x)$, which certainly exists in virtue of strictly increasing nature of $g(\theta)$. Further

$$\frac{\partial^2 \log P}{\partial \theta^2} = \frac{\partial^2 \log \beta}{\partial \theta^2} < 0 \text{ for every } x, \text{ and}$$

every $\theta \in \Theta$. Hence at $\hat{\theta}(x)$ there is a unique maximum.

Karlin (1958) has considered the admissibility of the estimates of the type $a_\gamma(x) = \gamma x$, $\gamma > 0$ for estimating $E(x)$ with the squared error loss. That is

$$R(a, \theta) = E\, [a(x) - g(\theta)]^2 \qquad (2)$$

It has been shown that if the exponential family is truncated to a fixed set A, then the truncated family continues to be an exponential.

For let A be such that

$$0 < \int_A P(x, \theta)\, d\mu(x) = b(\theta) \text{ (say)} \qquad (3)$$

Then the truncated family has density function, with respect to σ- finite measure v,

$$P_A(x, \theta) = \frac{\beta(\theta)}{b(\theta)} e^{x\theta}, \; x \in A \qquad (4)$$

and

$$d_v(x) = \phi_A(x)\, d\mu(x)$$

$\phi_A(x)$ being the indicator function of the set A.

From this it immediately follows that x continues to be UMVUE of $E(x / x \in A) = -\frac{\partial}{\partial\theta}\left\{\log \frac{\beta(\theta)}{b(\theta)}\right\} = g_A(\theta)$ (say) and UMVUE may not exist for $g(\theta)$.

Similarly the likelihood equation becomes

$$x - g_A(\theta) = 0 \text{ and } var(x/x \in A) = -\frac{\partial^2}{\partial\theta^2}\left[\log \frac{\beta(\theta)}{b(\theta)}\right] > 0 .$$

We expect that the admissibility of γx may remain invariant under truncation if instead of estimating $g(\theta)$ we estimate $g_A(\theta)$ and the risk function is correspondingly altered to,

$$R_A(a,\theta) = E\left\{[a(x) - g_A(\theta)]^2 / x \in A\right\} \qquad (5)$$

Let Θ_T be the natural range of the parameter, when the distribution is truncated and note that $\Theta_T \supseteq \Theta$.

As the admissibility of an estimate is closely connected with the structure of the natural range of the parameter. The admissibility of an estimators may be destroyed by truncation.

(A) Admissibility is destroyed by truncation:

In this subsection we give two examples, where the admissibility is destroyed by truncation. These two examples have been disscussed by Kale (1964).

Example (2.2.1) :

Let $P(x,\theta) = \frac{1}{2}(1-\theta^2)\, e^{x\theta}$, $\Theta=(-1,1)$, $d\mu(x) = e^{-|x|}dx$ and A be the set $(0, \infty)$. Thus we have

$$b(\theta) = \int_0^\infty P(x,\theta)\, d\mu(x)$$

$$= \int_0^\infty \frac{1}{2}(1-\theta^2)\, e^{x\theta}\, e^{-|x|}\, dx$$

$$= \frac{1}{2}(1-\theta^2) \int_0^\infty e^{-x(1-\theta)}\, dx$$

$$= \frac{1+\theta}{2}$$

Thus

$$P_A(x,\theta) = \begin{cases} (1-\theta)\, e^{x\theta} & , \; x \geq 0 \\ 0 & , \; x < 0 \end{cases}$$

and $dv(x) = \phi_A(x)\, e^{-|x|}\, dx$

here $\Theta_T = (-\infty, 1)$ hence $\Theta_T \supset \Theta$.

Karlin (1958) has shown that all estimates γx, $0<\gamma \leq \frac{1}{2}$ are admissible in the non-truncated case. Now we shall show that no estimate of the type γx is admissible in the truncated case.

From (5) we have,

here $$g_A(\theta) = \frac{-\partial}{\partial\theta}\left[\log \frac{\beta(\theta)}{b(\theta)}\right] = \frac{-\partial}{\partial\theta}\left[\log(1-\theta)\right] = \frac{1}{1-\theta}$$

That is,

$$R_A(\gamma,\theta) = E\left[(\gamma x - \frac{1}{1-\theta})^2 / x \in A\right]$$

$$= \int_0^\infty (\gamma x - \tfrac{1}{1-\theta})^2 \, (1-\theta) \, e^{x\theta} \, e^{-|x|} \, dx$$

$$= \frac{1}{(1-\theta)^2} \int_0^\infty [\gamma x(1-\theta)-1]^2 \, (1-\theta) \, e^{-x(1-\theta)} \, dx$$

$$= \frac{1}{1-\theta} \int_0^\infty (\gamma y-1)^2 \, e^{-y} \, \frac{dy}{1-\theta} \, , \quad \text{where } y = x(1-\theta)$$

$$= \frac{1}{(1-\theta)^2} \left\{ \int_0^\infty \gamma^2 y^2 \, e^{-y} dy + \int_0^\infty e^{-y} dy - 2\gamma \int_0^\infty y \, e^{-y} \, dy \right\}$$

$$= \frac{1}{(1-\theta)^2} \left[\gamma^2 \Gamma 3 \; + 1 \; - 2\gamma \Gamma 2 \right]$$

$$R_A (\gamma, \theta) = \frac{2\gamma^2 - 2\gamma + 1}{(1-\theta)^2} \qquad (6)$$

and $$\frac{\partial \, R_A(\gamma,\theta)}{\partial\gamma} = \frac{2(2\gamma - 1)}{(1-\theta)^2}$$

For $\gamma > \frac{1}{2}$ and for every $\theta \in Ⓗ_T$, $\frac{\partial R_A(\gamma,\theta)}{\partial\gamma} > 0$ and $R_A(\gamma,\theta)$ is strictly increasing function of γ for every $\theta \in Ⓗ_T$. Hence for $\gamma > \frac{1}{2}$ there exists γ' ($\gamma' < \gamma$) such that $R_A(\gamma',\theta) < R_A(\gamma,\theta)$, for all $\theta \in Ⓗ_T$.

Thus for $\gamma > \frac{1}{2}$, γx is inadmissible. For $\gamma < \frac{1}{2}$ and for every $\theta \in Ⓗ_T$, $\frac{\partial R_A(\gamma,\theta)}{\partial\gamma} < 0$ and $R_A(\gamma,\theta)$ is strictly decreasing function of γ for every $\theta \in Ⓗ_T$. Hence for $\gamma < \frac{1}{2}$ there exists γ'' such that

$$R_A (\gamma'', \theta) < R_A(\gamma, \theta) \text{ for all } \theta \in Ⓗ_T .$$

Thus for $\gamma < \frac{1}{2}$, γx is inadmissible. Thus

For $\gamma \neq \frac{1}{2}$, γx is inadmissible. (7)

Consider the case $\gamma = \frac{1}{2}$. Let θ be known to be in the interval $(-1,1)$. We are estimating $g_A(\theta) = \frac{1}{1-\theta}$ and as θ ranges in $(-1,1)$, $g_A(\theta)$ ranges in $(\frac{1}{2}, \infty)$.

If $x < 1$, which occurs with positive probability we would be estimating $\frac{1}{1-\theta}$ by a quantity less than $\frac{1}{2}$. Hence the estimate $\frac{1}{2}x$ can be improved by estimate $a(x)$ defines as

$$a(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \geq 1 \\ \frac{1}{2}, & \text{if } x < 1 \end{cases}$$

consider

$$R_A(a,\theta)-R_A(\tfrac{1}{2},\theta) =$$

$$= \int_0^\infty [a(x)-g_A(\theta)]^2 P_A(x,\theta)\,d_v(x) - \int_0^\infty [\tfrac{1}{2}x-g_A(\theta)]^2 P_A(x,\theta)\,d_v(x)$$

$$= \int_0^\infty [a(x)-\tfrac{1}{1-\theta}]^2 P_A(x,\theta)\,d_v(x) - \int_0^\infty [\tfrac{x}{2}-g_A(\theta)]^2 P_A(x,\theta)\,d_v(x)$$

$$= \int_0^1 (\tfrac{1}{2}-\tfrac{1}{1-\theta})^2 P_A(x,\theta)\,d_v(x) + \int_1^\infty (\tfrac{x}{2}-\tfrac{1}{1-\theta})^2 P_A(x,\theta)\,dv(x) -$$

$$- \int_0^1 (\tfrac{x}{2}-\tfrac{1}{1-\theta})^2 P_A(x,\theta)\,d_v(x) - \int_1^\infty (\tfrac{x}{2}-\tfrac{1}{1-\theta})^2 P_A(x,\theta)\,dv(x)$$

$$= \int_0^1 (\tfrac{1}{2}-\tfrac{1}{1-\theta})^2 P_A(x,\theta)\,d_v(x) - \int_0^1 (\tfrac{x}{2}-\tfrac{1}{1-\theta})^2 P_A(x,\theta)\,d_v(x)$$

$$= \int_0^1 [(\tfrac{1}{2}-\tfrac{1}{1-\theta})^2 - (\tfrac{x}{2}-\tfrac{1}{1-\theta})^2]\, P_A(x,\theta)\,dv(x)$$

$$= \int_0^1 \tfrac{1}{2}(1-x)\,(\tfrac{1}{2}+\tfrac{x}{2}-\tfrac{2}{1-\theta})\,P_A(x,\theta)\,d_v(x)$$

Now as $x \in (0,1)$ then $(1-x) > 0$

$$\max_{x \in (0,1)} (\tfrac{1}{2}+\tfrac{x}{2}) = 1$$

$$\min_{\theta \in (-1,1)} \frac{2}{1-\theta} = 1$$

Hence for $x \in (0,1)$ and $\theta \in (-1,1)$

$$\tfrac{1}{2}(1-x)\,(\tfrac{1}{2}+\tfrac{x}{2}-\tfrac{2}{1-\theta}) < 0$$

Therefore,

$$R_A(a, \theta) - R_A(\tfrac{1}{2}, \theta) < 0, \text{ for every } \theta \in \textcircled{H}$$

that is

$$R_A(a, \theta) < R_A(\tfrac{1}{2}, \theta), \text{ for every } \theta \in \textcircled{H} .$$

Which proves that $\frac{1}{2}x$ is inadmissible.

Thus all estimates of the type γx, $0 < \gamma \leq \frac{1}{2}$ are inadmissible in the truncated case; when the parameter is known to belong to $(-1, 1)$.

It is very interesting that if the truncated distribution is treated on its own merit with $\textcircled{H}_T = (-\infty, 1)$ then $\frac{1}{2}x$ is the only admissible estimate of $\frac{1}{1-\theta}$.

Lemma (2.2.1) :

In the truncated case $\frac{1}{2}x$ is the only admissible estimator of its expected value, $\theta \in (-\infty, 1)$.

Proof :

$$P(x, \theta) = (1 - \theta)\, e^{x\theta}, \quad x \geq 0, \ \theta \in (-\infty, 1)$$

Here $\beta(\theta) = 1-\theta$, $\beta'(\theta) = -1$

$$\therefore \quad E_\theta(x) = -\frac{\beta'(\theta)}{\beta(\theta)} = \frac{1}{1-\theta}$$

According to theorem 2.1.1 , the estimate $\frac{x}{1+u}$ is admissible if,

$$I_1 = \int_{-\infty}^{\theta'} (\beta(\theta))^{-u}\, d\theta = \infty$$

and

$$I_2 = \int_{\theta'}^{1} (\beta(\theta))^{-u}\, d\theta = \infty$$

are satisfied.

In this case we have $u = 1$.

Now, $I_1 = \int_{-\infty}^{\theta'} (1-\theta)^{-1}\, d\theta = [-\log(1-\theta)]_{-\infty}^{\theta'} = \infty$

Similarly,

$$I_2 = \int_{\theta'}^{1} (1-\theta)^{-1}\, d\theta = [-\log(1-\theta)]_{\theta'}^{1} = \infty$$

Thus I_1 and I_2 are diverges if $u = 1$, and hence $\frac{x}{1+u} = \frac{1}{2}x$ is an admissible estimator of $(1-\theta)^{-1}$, and from (7) it follows that $\frac{1}{2}x$ is the only admissible estimator. □

The following is an example to illustrate that, an admissible estimate may become inadmissible after truncation eventhough $\Theta_I = \Theta$. This example have been discussed by Kale (1964).

Example (2.2.2) :

Let $d\mu(x) = dx$,

$$P(x, \theta) = \begin{cases} -\theta\, e^{\theta x} & , \; x \geq 0, \; \Theta = (-\infty, 0) \\ 0 & , \; x < 0 \end{cases}$$

Here $\beta(\theta) = -\theta$ and $E_\theta(x) = -\frac{1}{\theta}$

According to theorem 2.1.1 , if

$$I_1 = \int_{-\infty}^{\theta'} (\beta(\theta))^{-u}\, d\theta = \infty \text{ and}$$

$$I_2 = \int_{\theta'}^{0} (\beta(\theta))^{-u}\, d\theta = \infty$$

then the estimate $\frac{x}{1+u}$ is admissible. If $u = 1$, then

$$I_1 = \int_{-\infty}^{\theta'} (-\theta)^{-1}\, d\theta = [-\log \theta]_{-\infty}^{\theta'} = \infty$$

and $I_2 = \int_{\theta'}^{0} (-\theta)^{-1}\, d\theta = [-\log \theta]_{\theta'}^{0} = \infty$

Thus for $u = 1$, I_1 and I_2 diverges. Therefore, $\frac{x}{1+u} = \frac{1}{2}x$ is an admissible estimate of $-\frac{1}{\theta}$.

Now we shall do the truncation and check for admissibility.

Let the set A be (c,∞), $c > o$ then

$$b(\theta) = \int_c^\infty P(x, \theta)\, dx$$

$$= \int_c^\infty -\theta\, e^{\theta x}\, dx$$

$$b(\theta) = e^{c\theta} \quad \text{as } \theta \in (-\infty, 0)$$

Therefore,

$$P_A(x, \theta) = \begin{cases} -\theta\, e^{\theta(x-c)} & , \; x \geq c \\ 0 & , \; x < c \end{cases}$$

Here $Ⓗ_T = (-\infty, 0)$. Thus $Ⓗ_T = Ⓗ = (-\infty, 0)$

$$R_A(\gamma, \theta) = E\left[[\gamma x - g_A(\theta)]^2 / x \in A\right]$$

where $g_A(\theta) = \frac{c\theta - 1}{\theta}$

Now, we shall show that no estimate of the type γx is admissible for $c\theta - 1/\theta$.

$$R_A(\gamma,\theta) = E\left[[\gamma x - (\tfrac{c\theta-1}{\theta})]^2 / x \in A\right]$$

$$= \frac{1}{\theta^2}\; E\left[[\gamma\theta - (c\theta-1)]^2 / x \in A\right]$$

$$= \frac{1}{\theta^2} \int_c^\infty [\gamma\theta x - (c\theta-1)]^2\, P_A(x,\theta)\, dx$$

$$= \frac{1}{\theta^2} \int_c^\infty [\gamma^2\theta^2 x^2 + (c\theta-1)^2 - 2(c\theta-1)\gamma\theta x]\, P_A(x,\theta)\, dx$$

$$= \frac{1}{\theta^2} \int_c^\infty \gamma^2\theta^2 x^2\, (-\theta)\, e^{\theta(x-c)} dx + \frac{(c\theta-1)^2}{\theta^2} \int_c^\infty -\theta\, e^{\theta(x-c)}\, dx$$

$$- \frac{2(c\theta-1)}{\theta^2} \int_c^\infty \gamma\theta x(-\theta)\, e^{\theta(x-c)}\, dx$$

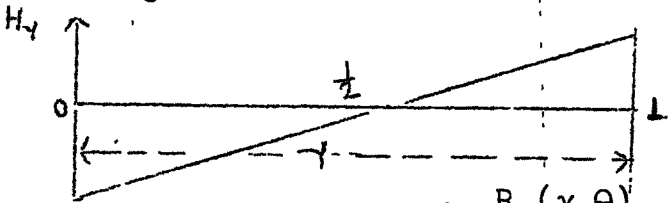
$$= c^2\gamma^2 - \frac{2c\gamma^2}{\theta} + \frac{2\gamma^2}{\theta^2} + \frac{(c\theta-1)^2}{\theta^2} - \frac{2\gamma(c\theta-1)^2}{\theta^2}$$

$$= \gamma^2\left[\frac{(1-c\theta)^2+1}{\theta^2}\right] - 2\gamma\,\frac{(1-c\theta)^2}{\theta^2} + \frac{(1-c\theta)^2}{\theta^2}$$

Thus

$$\frac{\partial R_A(\gamma,\theta)}{\partial\gamma} = \frac{2\gamma}{\theta^2}\,[\,1+(1-c\theta)^2] - \frac{2}{\theta^2}\,(1-c\theta)^2$$

$$= \frac{2}{\theta^2}\left[(1-c\theta)^2(\gamma-1)+\gamma\right] = H_\gamma \text{ (say)}$$



For $\gamma < \frac{1}{2}$ and for every $\theta \in Ⓗ$, $\frac{\partial}{\partial\gamma} R_A(\gamma,\theta) < 0$ and $R_A(\gamma,\theta)$ is strictly decreasing function of γ for every $\theta \in Ⓗ$.

Hence for $\gamma < \frac{1}{2}$ there exists γ'' such that

$$R_A(\gamma'',\theta) < R_A(\gamma,\theta) \quad \text{for every } \theta \in Ⓗ .$$

Hence γx is inadmissible for $0 < \gamma < \frac{1}{2}$ (8)

Consider $\gamma_o x$, $\gamma_o \in [\frac{1}{2}, 1]$,

construct $a_o(x)$ where

$$a_o(x) = \begin{cases} \gamma_o x & , \ x \geq \frac{c}{\gamma_o} \\ c & , \ c \leq x \leq \frac{c}{\gamma_o} \end{cases}$$

Now,

$$R_A(a_o,\theta) - R_A(\gamma_o,\theta) =$$

$$= \int_c^\infty [a_o(x) - (\tfrac{c\theta-1}{\theta})^2]\, P_A(x,\theta)\,dx - \int_c^\infty [\gamma_o x - (\tfrac{c\theta-1}{\theta})^2] P_A(x,\theta)\,dx$$

$$= \int_c^{c/\gamma_o} [c - (\tfrac{c\theta-1}{\theta})^2] P_A(x,\theta)\,dx + \int_{c/\gamma_o}^{\infty} [\gamma_o x - (\tfrac{c\theta-1}{\theta})^2] P_A(x,\theta)\,dx$$

$$-\int_{c}^{c/\gamma_o}[\gamma_o x-(\tfrac{c\theta-1}{\theta})^2]\,P_A(x,\theta)dx-\int_{c/\gamma_o}^{\infty}[\gamma_o x-(\tfrac{c\theta-1}{\theta})^2]\,P_A(x,\theta)dx$$

$$=\int_{c}^{c/\gamma_o}[c-(\tfrac{c\theta-1}{\theta})^2]-[\gamma_o x-(\tfrac{c\theta-1}{\theta})]^2\,P_A(x,\theta)\,dx$$

$$=\int_{c}^{c/\gamma_o}(c-\gamma_o x)\,[c+\gamma_o x-2(\tfrac{c\theta-1}{\theta})]\,P_A(x,\theta)\,dx$$

$$x\in(c,\tfrac{c}{\gamma_o})\implies c-\gamma_o x>0$$

$$\max_{x\in(c,\frac{c}{\gamma o})}(c+\gamma ox)=c+c=2c$$

$$\min_{\theta\in\Theta}(\tfrac{c\theta-1}{\theta})=\min_{\theta\in\Theta}(c-\tfrac{1}{\theta})=a$$

Thus as $x\in(c,\frac{c}{\gamma o})$ and $\theta\in(-\infty,0)$ then

$$(c-\gamma ox)\,[c+\gamma ox-2(\tfrac{c\theta-1}{\theta})]<0$$

Therefore

$$R_A(a_o,\theta)-R_A(\gamma_o,\theta)<0$$

That is

$$R_A(a_o,\theta)<R_A(\gamma_o,\theta)\quad\text{for every }\theta\in\Theta.$$

Hence $\gamma_O x$ is inadmissible, for $\frac{1}{2}\leq\gamma_o<1$ (9)

From (8) and (9), γx is inadmissible for every γ in $(0, 1)$. □

(B) Admissibility or inadmissibility is preserved in truncation:

In this subsection it will be shown that, an inadmissible procedure continue to be inadmissible even after truncation for the exponential model discussed above.

Further an example is given to show that an admissible estimator continue to be admissible one even after truncation.

Lemma (2.2.2):

For any mode of truncation all the estimated γx, $\gamma>1$ continue to remain inadmissible.

Proof:

$$P(x, \Theta) = \beta(\Theta)\, e^{\Theta x} ; \text{Ⓗ} = (-\infty, \infty)$$

we have seen that $E(X) = -\beta'(\Theta)/\beta(\Theta) = g(\Theta)$ (say), Therefore,

$$\beta(\Theta) \int x\, e^{\Theta x}\, d\mu(x) = -\frac{\beta'(\Theta)}{\beta(\Theta)}$$

Similarly,

$$\beta(\Theta) \int x^2\, e^{\Theta x}\, d\mu(x) = \frac{2\beta'^2(\Theta) - \beta(\Theta)\,\beta''(\Theta)}{\beta^2(\Theta)}$$

We shall consider only the estimates of the form $a_\gamma(x)=\gamma x$ where γ is positive constant, The value $\gamma=1$ provides the unique unbiased estimate of $E(x)$ within this family $a_\gamma(x)$.

Let γx be the estimator of $g(\Theta)$ then its risk function is given by

$$\begin{aligned} R(\gamma x,\Theta) &= E[\gamma x-g(\Theta)]^2 \\ &= \int [\gamma x-g(\Theta)]^2\, P(x,\Theta)\, d\mu(x) \\ &= \int [\gamma^2 x^2+g^2(\Theta) - 2\gamma x g(\Theta)]\, P(x,\Theta)\, d\mu\,(x) \end{aligned}$$

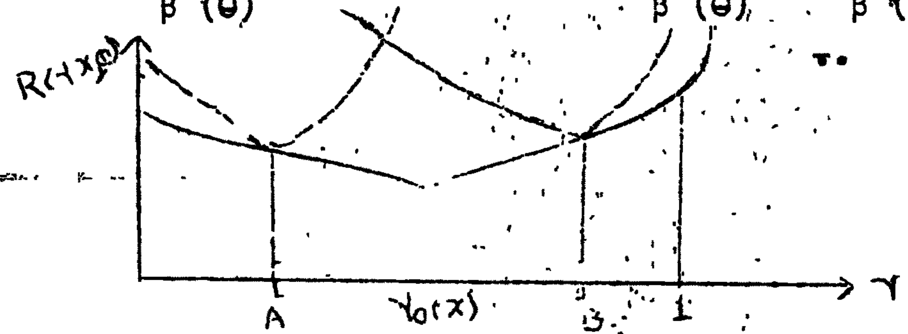
$$= \gamma^2 \int x^2 P(x,\Theta)\, d\mu(x) + g^2(\Theta) \int P(x,\Theta)\, d\mu(x) - 2\gamma g(\Theta) \int x\, P(x,\Theta)\, d\mu(x)$$

$$= \gamma^2 \int x^2\, \beta(\Theta)\, e^{\Theta x} d\mu(x) + g^2(\Theta) \int \beta(\Theta)\, e^{\Theta x} d\mu(x) - 2\gamma\, g(\Theta).$$

$$\int x\, \beta(\Theta)\, e^{\Theta x}\, d\mu(x).$$

$$= \gamma^2 \int x^2 \beta(\theta)\, e^{\theta x} d\mu(x) + \left[\frac{-\beta'(\theta)}{\beta(\theta)} \right]^2 \int \beta(\theta)\, e^{\theta x}\, d\mu(x) - 2\gamma \left(\frac{-\beta'(\theta)}{\beta(\theta)} \right) \int x\, \beta(\theta)\, e^{\theta x}\, d\mu(x).$$

$$= \gamma^2 \left[\frac{2\beta'^2(\theta) - \beta(\theta)\beta''(\theta)}{\beta^2(\theta)} \right] - 2\gamma \frac{\beta'^2(\theta)}{\beta^2(\theta)} + \frac{\beta'^2(\theta)}{\beta^2(\theta)} \quad \text{.. (10)}$$



For given $\theta \in \Theta$ one can obtain $\gamma_0(\theta)$ the point of minima at which $R(\gamma x, \theta)$ is minimum. By equating the derivative of (10) to zero we have

$$\gamma_0(\theta) = \frac{1}{1 + \dfrac{\beta'^2(\theta) - \beta(\theta)\beta''(\theta)}{\beta^2(\theta)}} \quad (11)$$

$$= \frac{1}{1 + \dfrac{\beta^2(\theta)}{\beta'^2(\theta)} \operatorname{var}(x)} < 1 \quad (12)$$

$$= \left[1 + \operatorname{var}(x) \left[\frac{\partial}{\partial\theta} \log \beta(\theta) \right]^{-2} \right]^{-1}, \text{ for all } \theta \in \Theta \quad \text{.. (13)}$$

Thus for $\gamma > 1$, $R(\gamma x, \theta) > R(x, \theta)$ and hence for $\gamma > 1$, γx is inadmissible.

Let

$$A = \inf_{\theta \in \Theta} \gamma_0(\theta) = \gamma_0(\underline{\theta})$$

$$B = \sup_{\theta \in \Theta} \gamma_0(\theta) = \gamma_0(\bar{\theta})$$

Hence the possible admissible estimates lie in the interval mapped by

$$\gamma_\Theta = [\,1 + \mathrm{var}(x)[\,\tfrac{\partial}{\partial\Theta} \log \beta(\Theta)\,]^{-2}\ \,]^{-1}$$

say [A, B] (see figure)

Let the distribution is truncated to the set A, then

$b(\Theta) = \int_A \beta(\Theta)\, e^{\Theta x}\, d\mu(x)$. Then the truncated p.d.f. is $P_A\,(x,\Theta) = \frac{\beta(\Theta)}{b(\Theta)}\, e^{\Theta x}$. In this case

$$R_A(\gamma x,\, \Theta) = E\,[\,[\gamma x - g_A(\Theta)]^2 / x \in A\,]$$

Proceeding on the similar lines as given above, we get

$$\gamma_0(\Theta) = [\,1 + \mathrm{var}(x\ /x \in A)[\tfrac{\partial}{\partial\Theta} \log \tfrac{\beta(\Theta)}{b(\Theta)}]^{-2}\ ]^{-1} \leq 1 \qquad \ldots \quad (14)$$

The possible admissible estimates lie in the interval mapped by (14) say $[A_T,\, B_T]$.

Thus for $\gamma > 1$

$$R\,(\gamma x,\, \Theta) > R(x,\, \Theta) \qquad \text{for all } \Theta$$

hence γx is inadmissible.

Thus for any mode of truncation all the estimates γx, $\gamma > 1$ continue to remain inadmissible. □

Example (2.2.3):

Let $P(x,p) = n_{cx}\, P^x\, (1 - P)^{n-x}$, $x = 0,1,\ldots n$, $0<P<1$

Taking dominating measure is n_{c_x}, then

$P(x,\, p)$ can be written as,

$$P(x,\, p) = p^x\, (1-p)^{n-x}$$

$$= \left(\frac{p}{1-p} \right)^x (1-p)^n$$

$$= (1-p)^n \; e^{x \log \frac{p}{1-p}}$$

$$= (1 + e^{\Theta})^{-n} e^{x\Theta}, \text{ where } \Theta = \log \frac{p}{1-p}$$

$$= p(x, \Theta) \quad (\text{say}), \quad -\infty < \Theta < \infty$$

Now we shall show that admissibility is preserved in truncation.

Here $\beta(\Theta) = (1 + e^{\Theta})^{-n}$ then $E(x) = \dfrac{e^{\Theta}}{1 + e^{\Theta}}$

According to theorem 2.1.1, if $u = 0$, then

$$I_1 = \int_{-\infty}^{\Theta'} (\beta(\Theta))^{-u} d\Theta = \int_{-\infty}^{\Theta'} d\Theta = \infty \quad \text{and}$$

$$I_2 = \int_{\Theta'}^{\infty} (\beta(\Theta))^{-u} d\Theta = \int_{\Theta'}^{\infty} d\Theta = \infty$$

the estimate $\frac{x}{1+u} = x$ is admissible for $\dfrac{e^{\Theta}}{e^{\Theta}+1}$. Hence x is admissible for p.

Let distribution be truncated at the point x = 0, then its pdf is

$$P(x, p) = \frac{{}^{n}c_{x}\, p^x (1-p)^{n-x}}{1 - (1-p)^n}, \quad x = 1, \ldots, n \quad 0<P<1 .$$

After reparamatrization,

$$P(x, \Theta) = \frac{(1 + e^{\Theta})^{-n}}{1 - (1 + e^{\Theta})^{-n}} e^{x\Theta}, \; Ⓗ_T = (-\infty, \infty) = Ⓗ .$$

Here

$$\beta(\Theta) = \frac{(1 + e^{\Theta})^{-n}}{1 - (1 + e^{\Theta})^{-n}}$$

According to theorem 2.1.1, if $u = 0$ then

$$I_1 = \int_{-\infty}^{\Theta'} d\Theta = \infty \quad \text{and}$$

$$I_2 = \int_{\Theta'}^{\infty} d\Theta = \infty$$

Thus X is admissible for E(X). Hence the admissibility is preserved in truncation. ☐

Remark :

Whatever be the form of truncation, x continues to be admissible. This follows since I_1 and I_2 are diverges for $u = 0$ even after truncation.

(C) Karlin's Conjecture:

In this subsection assume that $\Theta = R$ and therefore we assume that $\Theta = \Theta_T$.

Karlin's sufficinet conditions for admissibility of $\frac{x}{1+u}$, $u \geq 0$ are discussed in theorem 2.1.1 .

Karlin has conjectured that '' these conditions are also necessary ''.

We have shown (Lemma 2.2.2) that all estimates γx, $\gamma > 1$ continue to remain inadmissible after truncation, hence we will concentrate on γ, $0 < \gamma \leq 1$. Let $\gamma = (1+u)^{-1}$, $u \geq 0$.

Considering the negation of the statement of the theorem 2.1.1 , we have the following result.

If $\frac{x}{1+u}$ is inadmissible then atleast one of the conditions (a) and (b) is not satisfied. That is if $\frac{x}{1+u}$ is inadmissible then one of the integrals must be convergent. If Karlin's conjecture be true then the convergence of at least one of the integrals implies the inadmissibility of $\frac{x}{1+u}$.

Lemma (2.2.3)

If Karlin's conjecture be true then an inadmissible estimate $\frac{x}{1+u}$, $u \geq 0$ continues to remain inadmissible after truncation.

Proof:

As $\frac{x}{1+u}$ is inadmissible, at least one of the integrals in condition (a) and condition (b) is convergent, say, $\int_{\Theta'}^{\bar{\Theta}} (\beta(\Theta))^{-u} d\Theta < \infty$

Now for the truncated distribution $\beta(\Theta)$ is to be replaced by $\beta(\Theta)/b(\Theta)$ and we consider

$$\int_{\Theta'}^{\bar{\Theta}} \left[\frac{\beta(\Theta)}{b(\Theta)} \right]^{-u} d\Theta$$

Now

$$\left[\frac{\beta(\Theta)}{b(\Theta)} \right]^{-u} = b^{u}(\Theta)\,(\beta(\Theta))^{-u}$$

$$\leq [\beta(\Theta)]^{-u}\text{, for all } u \geq 0 \text{ and as } 0<b(\Theta)<1 \text{ for all } \Theta \in \text{Ⓗ} .$$

Therefore,

$$\int_{\Theta'}^{\bar{\Theta}} \left[\frac{\beta(\Theta)}{b(\Theta)} \right]^{-u} d\Theta \leq \int_{\Theta'}^{\bar{\Theta}} \beta(\Theta)^{-u} d\Theta < \infty$$

it implies that

$$\int_{\Theta'}^{\bar{\Theta}} \left(\frac{\beta(\Theta)}{b(\Theta)} \right)^{-u} d\Theta < \infty \qquad (15)$$

(15) shows that if Karlin's conjecture is true and $\frac{x}{1+u}$ is inadmissible then it continues to remain inadmissible even after truncation. ∎

(2.3) Admissibility of scale parameter :

Brown (1966) and Farrell (1964) have given sufficient conditions for the admissibility of the estimators of the location parameter. By making log transformation one can obtain the corresponding results for the scale parameter. Zidek (1969) has shown that when the estimation problem is invariant under a group of transformations G and the induced group $\bar{G}$ acts transitively on the parameter space, the best invariant estimator is formal Bayes. Portnoy (1971) has given sufficient conditions for the admissibility of a formal Bayes estimator, when the loss is quadratic. We apply Portnoy (1971) result for estimating a power of the scale parameter by the best scale invariant estimator. So to begin with we give Portnoy (1971) result which is useful later for determining the sufficient condition for admissibility.

Let $\mathcal{X}$, Θ and $\mathcal{A}$ is the real line R. Consider the loss function L : $\Theta \times \mathcal{A} \times \mathcal{X} \quad [0, \infty)$ of the form,

$$L(\theta, a, x) = V(\theta)\ (a - g(\theta))^2$$

where $\quad V : \Theta \times \mathcal{X} \longrightarrow (0, \infty)$ and

$\quad g : \Theta \longrightarrow R$,

are measurable functions.

Let $p(x, \theta)$ be the density function with respect to a σ- finite measure μ ; assume that

$$P(x, \theta) > 0, \text{ for all } x \in \mathcal{X}, \quad \theta \in \Theta \tag{1}$$

The non-randomized decision rules, which are measurable functions $\phi : \mathcal{X} \to \mathbb{A}$, and define the risk of ϕ to be

$$R(\phi, \theta) = \int L(\theta, \phi(x)) \; P(x, \theta) \; d\mu(x)$$

The formal Bayes rule ϕ_π is given by,

$$\phi_\pi(x) = \frac{\int g(\theta) \; v(\theta, x), \; P(x, \theta) \; \pi(\theta) \; d\theta}{\int v(\theta, x) \; P(x, \theta) \; \pi(\theta) \; d\theta} \tag{2}$$

where $\pi(\theta)$ is the prior distribution of θ, $\theta \in (\underline{\theta}, \bar{\theta}) = \Theta$, $(\underline{\theta}, \bar{\theta})$ is an interval in the real line.

Define, for $\theta \in (\underline{\theta}, \bar{\theta})$ and $x \in \mathcal{X}$

$$h_1(\theta, x) = \int_{\theta}^{\bar{\theta}} [\phi_\pi(x) - g(\theta')] \; P(x, \theta') \; v(\theta', x) \pi(\theta') \; d\theta' \tag{3}$$

$$h_2(\theta, x) = P(x, \theta) \; v(\theta x) \; \pi(\theta) \tag{4}$$

and

$$\lambda(\theta) = E_\theta\left[\frac{h_1(\theta, x)}{h_2(\theta, x)} \right]^2 . \pi(\theta) \; v(\theta, x) \tag{5}$$

note that $h_2(\theta, x) > 0$, for all $\theta \in (\underline{\theta}, \bar{\theta})$ and $x \in \mathcal{X}$.

We give below only the statement of the theorem (Portnoy 1971 pp 1382) in which the formal Bayes estimator ϕ_π is admissible under certain conditions.

<u>Theorem (2.3.1)</u> :

Consider the statistical decision theory problem described above. Suppose $\lambda(\theta)$ is a continuous function of θ on $(\underline{\theta}, \bar{\theta})$, and suppose further that for every compact (closed finite) sub-interval $[a_0, b_0] \subset (\underline{\theta}, \bar{\theta})$

$$\int_{a_o}^{b_o} R(\phi_\pi, \theta)\, \pi(\theta)\, d\theta < \infty \tag{6}$$

Suppose also that for every $C \in (\underline{\theta}, \bar{\theta})$ conditions (A) and (B) hold.

(A) $$\int_C^{\bar{\theta}} R(\phi_\pi,\theta)\, \pi(\theta)\, d\theta = \infty \Rightarrow \int_C^{\bar{\theta}} \frac{d\theta}{\lambda(\theta)} = \infty \tag{7}$$

(B) $$\int_{\underline{\theta}}^{C} R(\phi_\pi,\theta)\, \pi(\theta)\, d\theta = \infty \Rightarrow \int_{\underline{\theta}}^{C} \frac{d\theta}{\lambda(\theta)} = \infty \tag{8}$$

then ϕ_π is admissible.

(I) A Sufficient Condition for Admissibility :

In this and the next sub sections we discuss the results of Divakar Sharma (1973).

Let X have the probability density $\theta.\ P(\theta x)\ I_{(0,\infty)}(x)$ with respect to Lebesque measure, where θ is positive on $(0,\infty)$ and $I_A(x)$ is 1 if $x \in A$, 0 otherwise, Let the loss in estimating θ^m by d, where m is a real number, be $(d-\theta^m)^2\, \theta^{-2m}$.

The prior density of θ w.r.to Lebesque measure is θ^{-1} i.e. $\pi(\theta) = \frac{1}{\theta}$, $0<\theta<\infty$. Using (2), the formal Bayes rule is,

$$\phi_\pi(x) = \frac{\int \theta^m \theta^{-2m}\, P(x,\theta)\, \frac{1}{\theta}\, d\theta}{\int \theta^{-2m}\, P(x,\theta)\, \frac{1}{\theta}\, d\theta}$$

$$= \frac{\int_0^\infty \theta^{-m-1}\, \theta\, P(\theta x)\, d\theta}{\int_0^\infty \theta^{-2m}\, P(\theta x)\, d\theta} = \frac{\int_0^\infty \theta^{-m}\, p(\theta x)\, d\theta}{\int_0^\infty \theta^{-2m}\, p(\theta x)\, d\theta}$$

$$= \frac{\int_0^\infty t^{-m}\, x^m\, P(t)\, dt/x}{\int_0^\infty t^{-2m}\, x^{2m}\, P(t)\, dt/x} \quad \text{where } t = \theta x$$

$$= x^{-m} \frac{\int_0^\infty t^{-m} p(t)\, dt}{\int_0^\infty t^{-2m} p(t) dt}$$

$$= x^{-m} [\int_0^\infty x^{-m} p(x)dx \,/ \int_0^\infty x^{-2m} P(x)\, dx]$$

$\phi_\pi(x) = b_m x^{-m}$, where $b_m = [\int_0^\infty x^{-m} p(x)\, dx / \int_0^\infty x^{-2m} p(x)\, dx]$

Thus the best scale invariant estimator of Θ^m is the formal Bayes estimator $b_m x^{-m}$.

Theorem (2.3.2):

The estimator $b_m X^{-m}$ of Θ^m is admissible for quadratic loss if,

$$\int_0^\infty \frac{x^{2(m-1)}}{p(x)} [\int_x^\infty (b_m - t^m)\, t^{-2m}\, p(t)dt]^2 dx < \infty.$$

Proof:

By using equations (3), (4) we shall find $h_1(\Theta,x)$.

$h_1(\Theta,x) = \int_\Theta^{\bar{\Theta}} [\, \phi_\pi(x) - g(\Theta')]\, P(x,\Theta')V(\Theta')\, \pi(\Theta')\, d\Theta'$

where $\phi_\pi(x)$ is the formal Bayes estimator of Θ^m is $bm\, X^{-m}$ with respect to the prior distribution.

$\pi(\Theta') = \frac{1}{\Theta'}$, $0 < \Theta' < \infty$, Therefore,

$$h_1(\Theta,x) = \int_\Theta^\infty [\, bm\, x^{-m} - \Theta'^m]\, \Theta'\, P(\Theta' x)\, \Theta'^{-2m} (\Theta')^{-1} d\Theta'$$

$$= \int_\Theta^\infty [bm - (\Theta' x)^m]\, \Theta'^{-2m}\, x^{-m}\, P(\Theta' x)\, d\Theta'$$

$$= \int_{\Theta x}^\infty (bm - t^m)\, t^{-2m}\, x^{2m}\, x^{-m} P(t)dt/x, \text{ where } t = \Theta' x$$

$$h_1(\Theta,x) = x^{m-1} \int_{\Theta x}^\infty (bm - t^m)\, t^{-2m}\, P(t)\, dt$$

and

$$h_2(\theta,x) = P(x,\theta)\ V(\theta,x)\ \pi(\theta)$$

$$= \theta\ P(\theta x)\ \theta^{-2m}\ \frac{1}{\theta} = P(\theta x)\ \theta^{-2m}$$

Then

$$E_\theta\left[\frac{h_1(\theta,x)}{h_2(\theta,x)}\right]^2 = \int_0^\infty \left[\frac{h_1(\theta,x)}{h_2(\theta,x)}\right]^2 .\ \theta\ P(\theta x)\ dx$$

$$= \int_0^\infty \frac{x^{2(m-1)}\left[\int_{\theta x}^\infty (bm-t^m)\, t^{-2m} P(t)\,dt\right]^2}{(\theta^{-2m})^2\ [P(\theta x)]^2}\ .\ \theta\ P(\theta x) .\ dx$$

$$E_\theta\left[\frac{h_1(\theta,X)}{h_2(\theta,X)}\right]^2 = \int_0^\infty \frac{x^{2(m-1)}}{\theta^{-4m-1} P(\theta x)}\left[\int_{\theta x}^\infty (bm-t^m)\ t^{-2m} P(t)\,dt\right]^2 .dx$$

Further from (5)

$$\lambda(\theta) = E_\theta\left[\frac{h_1(\theta,X)}{h_2(\theta,X)}\right]^2 \pi\ (\theta)\ V(\theta)$$

$$= \int_0^\infty \frac{x^{2(m-1)}}{\theta^{-4m-1} P(\theta x)}\left[\int_{\theta x}^\infty (bm-t^m)\ t^{-2m}\ P(t)\ dt\right]^2 \frac{1}{\theta}\ \theta^{-2m} .\ dx$$

$$= \int_0^\infty \frac{x^{2(m-1)}}{\theta^{-2m} P(\theta x)}\left[\int_{\theta x}^\infty (bm-t^m)\ t^{-2m} P(t)\ dt\right]^2 dx$$

$$= \int_0^\infty \frac{(y/\theta)^{2m-2}}{\theta^{-2m}\ P(y)}\left[\int_y^\infty (bm-t^m)\ t^{-2m}\ P(t)\ dt\right]^2 .\ \frac{1}{\theta}\ dy, \text{ where } y=\theta x$$

$$= \theta \int_0^\infty \frac{y^{2(m-1)}}{P(y)}\left[\int_y^\infty (bm-t^m)\ t^{-2m}\ P(t)\ dt\right]^2 dy$$

That is,

$$\lambda(\theta) = \theta \int_0^\infty \frac{x^{2(m-1)}}{P(x)}\left[\int_x^\infty (bm-t^m)\ t^{-2m}\ P(t)\,dt\right]^2 .\ dx \quad ..(9)$$

$= \theta .\ D$, where

$$D = \int_0^\infty \frac{x^{2(m-1)}}{P(x)}\left[\int_x^\infty (bm-t^m)\ t^{-2m}\ P(t)\ dt\right]^2 dx$$

It is given that D is finite, under this condition we shall prove that all the conditions of Portnoy (1971) theorem 2.3.1 are satisfied.

(i) $\lambda(\theta)$ is continuous function of θ on $(0,\infty)$, it follows from (9) since D is finite,

Consider,

$$R(\phi_\pi,\theta) = \int_0^\infty (bm\, x^{-m} - \theta^m)^2\, \theta^{-2m}\, \theta\, P(\theta x)\, dx$$

$$= \int_0^\infty [bm-(\theta x)^m]^2\, \theta^{-2m}\, \theta\, P(\theta x)\, dx$$

$$= \int_0^\infty (bm - t^m)^2\, t^{-2m}\, \theta\, P(t) \cdot \frac{1}{\theta}\, dt, \text{ where } t = \theta x$$

$$= \int_0^\infty (bm - t^m)^2\, t^{-2m}\, P(t)\, dt$$

However, we assume $R(\phi_\pi,\theta) < \infty$. That is

$$\int_0^\infty (bm - t^m)^2\, t^{-2m}\, P(t)\, dt < \infty \tag{10}$$

In the following we shall show that conditions (A) and (B) are satisfied.

For the L.H.S. of the condition (A), consider

$$\int_C^{\bar{\theta}} R(\phi_\pi,\theta)\, \pi(\theta)\, d\theta = \infty$$

$$\int_C^\infty \left[\int_0^\infty (bm-t^m)^2\, t^{-2m}\, P(t)dt\right]\frac{1}{\theta}\, d\theta$$

since $0 < \int_0^\infty (bm - t^m)^2\, t^{-2m}\, P(t)\, dt < \infty$, therefore,

it is enought to prove

$$\int_C^\infty \frac{1}{\theta}\, d\theta = \infty, \text{ and which is obvious.}$$

Hence under (10) L.H.S. of (A) holds.

Now R.H.S. of the condition (A) is

$$\int_C^{\bar{\theta}} \frac{d\theta}{\lambda(\theta)} = \int_C^{\infty} \frac{d\theta}{\theta . D}$$

Further as D is finite, to prove $\int_C^{\infty} \frac{d\theta}{\lambda(\theta)} = \infty$ it is enough to prove $\int_C^{\infty} \frac{d\theta}{\theta} = \infty$, and which is obvious. Hence the R.H.S. of (A) holds under D is finite.

Similarly, for the L.H.S. of the condition (B) consider,

$$\int_{\underline{\theta}}^{C} R(\phi_\pi, \theta)\ \pi\ (\theta)\ d\theta = \infty$$

$$= \int_0^C \left[\int_0^{\infty} (bm - t^m)^2\ t^{-2m}\ P(t)\ dt \right] \frac{1}{\theta}\ d\theta$$

since $0 < \int_0^{\infty} (bm - t^m)^2\ t^{-2m}\ P(t)\ dt < \infty$, therefore it is enough to prove

$\int_0^C \frac{1}{\theta}\ d\theta = \infty$, which is obvious. Now the R.H.S. of the condition (B) is

$$\int_{\underline{\theta}}^{C} \frac{d\theta}{\lambda(\theta)} = \int_0^C \frac{d\theta}{\theta . \bar{D}}$$

Further D is finite to prove $\int_0^C \frac{d\theta}{\lambda(\theta)} = \infty$ it is enough to prove $\int_0^C \frac{1}{\theta}\ d\theta = \infty$, which is obvious. Therefore R.H.S. of (B) holds under D is finite. Thus the conditions (A) and (B) are satisfied when D is finite.

Hence,

If $R(\phi_\pi, \theta) < \infty$ and D is finite then the formal Bayes estimator $bm\ X^{-m}$ is admissible for θ^m. □

<u>Example (2.3.1)</u>:

Let Z have the Probability density $\theta .\ e^{-\theta Z}$; $Z>0$ with respect to Lebseque measure, and $Z_1, Z_2, \ldots, Z_n$ be independent and identically distributed as Z. Then $\sum_{i=1}^{n} Z_i = X$ have a p.d.f., $P(x,\theta) = \frac{1}{\Gamma n} (n\theta)^n e^{-n\theta x} x^{n-1}$, $x > 0$.

This density can be written in the form.

$$P(x, \theta) = \theta \cdot p(\theta x)$$
$$= \theta \cdot \frac{1}{\Gamma n} n^n \theta^{n-1} e^{-n\theta x} x^{n-1}, \; x > 0$$

it implies

$$P(x) = \frac{n^n x^{n-1}}{\Gamma n} e^{-nx}, \; x > 0$$

Therefore, the formal Bayes estimator is,

$$bm = \int_0^\infty x^{-m} p(x)\, dx \Big/ \int_0^\infty x^{-2m} p(x)\, dx$$

$$= \int_0^\infty x^{-m} \frac{n^n}{\Gamma n} x^{n-1} e^{-nx}\, dx \Big/ \int_0^\infty x^{-2m} \frac{n^n}{\Gamma n} x^{n-1} e^{-nx}\, dx$$

$$= \int_0^\infty x^{n-m-1} e^{-nx}\, dx \int_0^\infty x^{n-2m-1} e^{-nx}\, dx$$

$$= \frac{\Gamma n - m}{n^{n-m}} \Big/ \frac{\Gamma n - 2m}{n^{n-2m}}$$

$$\text{i.e } b_m = \frac{\Gamma n - m}{\Gamma n - 2m} \cdot n^{-m}$$

That is

$$bm\, x^{-m} = \frac{n - m}{n - 2m} (nx)^{-m}$$

is the formal Bayes estimator of θ^m with respect to the prior distribution $\frac{1}{\theta}$, $0<\theta<\infty$. From theorem 2.3.2 for $bm\, x^{-m}$ to be admissible, it is enough to show that

$$\int_0^\infty \frac{x^{2(m-1)}}{p(x)} \left[\int_x^\infty (bm - t^m)\, t^{-2m} p(t)\, dt \right]^2 dx < \infty \quad ..(11)$$

Now L.H.S. of (11), that is

$$D = \int_0^\infty \frac{x^{2(m-1)}}{p(x)} \left\{ \int_x^\infty \underset{I}{bm\ t^{-2m}\, p(t)dt} - \int_x^\infty \underset{II}{t^{-m}\, p(t)\, dt} \right\}^2 dx$$

<u>Part-I</u>

$$= bm \int_x^\infty t^{-2m}\, p(t)\ dt$$

$$= bm \frac{n^n}{\Gamma n} \int_x^\infty t^{n-2m-1}\, e^{-nt}\, dt$$

$$= bm \frac{n^n}{\Gamma n} \int_{nx}^\infty e^{-y} \left(\frac{y}{n}\right)^{n-2m-1} \frac{dy}{n}, \text{ where } y = nt$$

$$= bm \frac{n^{2m}}{\Gamma n} \int_{nx}^\infty e^{-y}\, y^{n-2m-1}\, dy$$

$$= bm \frac{n^{2m}}{\Gamma n} (n-2m-1)! \sum_{r=o}^{n-2m-1} \frac{(nx)^r\, e^{-nx}}{r!},$$

$$\left(\text{since } \int_x^\infty e^{-t}\, t^k\, dt = k! \sum_{r=o}^{k} \frac{e^{-x}\, x^r}{r!}\right)$$

$$= \frac{\Gamma n-m}{\Gamma n-2m}\ n^{-m}\ \frac{n^{2m}}{n}\ \Gamma n-2m\ e^{-xn} \sum_{r=o}^{n-2m-1} \frac{(nx)^r}{r!}$$

$$= \frac{n^m}{\Gamma n}\ \Gamma n-m\ e^{-nx} \sum_{r=o}^{n-2m-1} \frac{(nx)^r}{r!}$$

similarly

$$\underline{\text{Part-II}} = \frac{n^m}{\Gamma n}\ e^{-nx}\ \Gamma n-m \sum_{r=o}^{n-m-1} \frac{(nx)^r}{r!}$$

Hence, L.H.S. of (11) =

$$= \int_0^\infty \frac{x^{2(m-1)}}{p(x)} \left\{ \frac{n^m\, e^{-nx}\, \Gamma n-m}{\Gamma n} \sum_{r=o}^{n-2m-1} \frac{(nx)^r}{r!} - \frac{n^m\, e^{-nx}\, \Gamma n-m}{\Gamma n} \sum_{r=o}^{n-m-1} \frac{(nx)^r}{r!} \right\}^2 . dx$$

$$= \frac{n^{2m}\, (\Gamma n-m)^2}{(\Gamma n)^2} \int_0^\infty \frac{x^{2(m-1)}}{\frac{n^n\, x^{n-1}\, e^{-nx}}{\Gamma n}}\ e^{-2nx}\ K^2\, dx$$

where $K = \sum_{r=0}^{n-2m-1} \frac{(nx)^r}{r!} - \sum_{r=0}^{n-m-1} \frac{(nx)^r}{r!}$

$$= \frac{n^{2m-n}(\Gamma(n-m))^2}{\Gamma n} \int_0^\infty x^{2m-n-1} e^{-nx} K^2 \, dx$$

<u>Case (i)</u> :

if $m = 0$, $D < \infty$.

<u>Case (ii)</u>:

if $m > 0$ then

$$D = \frac{n^{2m-n}(\Gamma(n-m))^2}{\Gamma n} \int_0^\infty x^{2m-n-1} e^{-nx} \Big[\sum_{r=n-2m}^{n-m-1} \frac{(nx)^r}{r!}\Big]^2 dx \quad .. (12)$$

Since $\Big[\sum_{r=n-2m}^{n-m-1} \frac{(nx)^r}{r!}\Big]^2 = \sum_{r=2(n-2m)}^{2(n-m-1)} a_r x^r$ for

suitable coefficients $a_{2n-4m}, \ldots, a_{2(n-m-1)}$

From (12),

$$D = \frac{n^{2m-n}}{\Gamma n} (\Gamma(n-m))^2 \sum_{r=2(n-2m)}^{2(n-m-1)} \int_0^\infty a_r x^{r+2m-n-1} e^{-nx} dx < \infty$$

provided $n > 2m$.

<u>Case (iii)</u>:

if $m < 0$ then

$$D = \frac{n^{2m-n}}{\Gamma n} (\Gamma(n-m))^2 \int_0^\infty x^{2m-n-1} e^{-nx} \Big[\sum_{s=n-m}^{n-2m-1} \frac{(nx)^s}{s!}\Big]^2 . dx$$

Arguing as above we will have

$$D = \frac{2^{2m-n}}{\Gamma n} (\Gamma(n-m))^2 \sum_{s=2(n-m)}^{2(n-2m-1)} \int_0^\infty x^{2m-n-1+s} e^{-nx} dx$$

$< \infty$ provided $n > 2m$.

Thus $D < \infty$ for all $n > 2m$

Hence $\frac{\Gamma(n-m)}{\Gamma(n-2m)} (nx)^{-m}$ is admissible for θ^m if $n > 2m$. □

(II) An Admissible estimator in an exponential family :

In this subsection the following theorem gives us an admissible estimator of an integral power of the natural parameter in an exponential family, when the loss is quadratic.

It may be noted that if the range of the variable is $(0,\infty)$ then the natural parameter is the reciprocal of the scale parameter.

Theorem (2.3.3):

Let the random variable z have the probability density $\beta(\theta)\ e^{-\theta z}\ r(z)$ with respect to Lebesque measure, where r(z) is a probability density on (0,b) with b possibly infinite. Let the parameter space under consideration be $(0,\infty)$, Also let

$$\int_0^{nb} r_n(x)\ x^{-2m}\ dx < \infty \tag{13}$$

where r_n is the n-fold convolution of r, and with $0<C\leq bn$, $m^* = \max(0,m)$. Let

$$\lim_{\theta\to\infty} \quad \theta^{n-j-1} \int_0^{\theta C} \exp(-x)\ x^{j-2m^*} r_n(x)\ dx < \infty \quad .. (14)$$

$$\text{for } 0\leq\ j\leq\ 2(2m^*-m-1).$$

Then, for $n > 2m$, $\frac{\Gamma n-m}{\Gamma n-2m}\ (n\bar{Z})^{-m}$ is an admissible estimator of θ^m with an integral m and quadratic loss .

Proof:

The probability density of $X = \sum_{i=1}^{n} Z_i$ can be seen to be $\beta^n(\theta)\ e^{-\theta x} r_n(x)$.

Let the loss in estimating θ^m by d be $(d-\theta^m)\ \theta^{-2m}$

and the prior density for Θ be $\Theta^{n-1}/\beta^n(\Theta)$. Then the formal Bayes estimator of Θ^m is $\frac{\Gamma(n-m)}{\Gamma(n-2m)} X^{-m}$; that is

$$\phi_\pi(x) = \frac{\Gamma(n-m)}{\Gamma(n-2m)} X^{-m}$$

The risk of this estimator is

$$R(\phi_\pi(x), \Theta) = E[\Theta^{-2m}(d - \Theta^m)^2]$$

$$= \int_0^\infty \Theta^{-2m} \left(\frac{\Gamma(n-m)}{\Gamma(n-2m)} x^{-m} - \Theta^m\right)^2 P(x, \Theta)\, dx$$

$$= \int_0^\infty \beta^n(\Theta) \exp(-\Theta x)\, r_n(x)\, \Theta^{-2m}\left(\Theta^m - \frac{\Gamma(n-m)}{\Gamma(n-2m)} x^{-m}\right)^2 . dx$$

Now $r_n(x)$ is a probability density and (13) holds, there--fore the risk function is continuous in Θ, $\Theta \in (0,\infty)$.

Now we use theorem 2.3.1

$$h_2(\Theta,x) = P(x,\Theta)\, V(\Theta,x)\, \pi(\Theta)$$

$$= \beta^n(\Theta) \exp(-\Theta x) r_n(x) .\ \Theta^{-2m} \frac{\Theta^{n-1}}{\beta^n(\Theta)}$$

$$= \exp(-\Theta x)\, r_n(x)\, \Theta^{n-2m-1} \qquad (15)$$

and

$$h_1(\Theta,x) = \int_\Theta^{\bar{\Theta}} [\phi_\pi(x) - g(\Theta')] P(x,\Theta') V(\Theta',x) \pi(\Theta')\, d\Theta'$$

$$= \int_\Theta^\infty \left(\frac{\Gamma(n-m)}{\Gamma(n-2m)} x^{-m} - \Theta'^m\right) \beta^n(\Theta') e^{-\Theta' x} r_n(x)\, \Theta'^{-2m} \frac{\Theta'^{n-1}}{\beta^n(\Theta')}\, d\Theta'$$

$$= \int_\Theta^\infty \frac{\Gamma(n-m)}{\Gamma(n-2m)} x^{-m} e^{-\Theta' x} r_n(x)\, \Theta'^{n-2m-1}\, d\Theta' -$$

$$- \int_\Theta^\infty e^{-\Theta' x} r_n(x)\, \Theta'^{n-m-1}\, d\Theta'$$

$$= \frac{\Gamma(n-m)}{\Gamma(n-2m)} x^{-m} r_n(x) \int_{\theta}^{\infty} e^{-\theta' x} \theta'^{\,n-2m-1} d\theta' -$$

$$- r_n(x) \int_{\theta}^{\infty} e^{-\theta' x} \theta'^{\,n-m-1} d\theta'$$

$$h_1(\theta,x) = \frac{\Gamma(n-m)}{\Gamma(n-2m)} x^{-m} r_n(x)\, I_1 - r_n(x)\, I_2 \text{ (say)} \qquad (16)$$

Where

$$I_1 = \int_{\theta}^{\infty} e^{-\theta' x} \theta'^{\,n-2m-1} d\theta'$$

$$= \frac{(n-2m-1)!}{x^{n-2m}} \sum_{j=0}^{n-2m-1} \frac{e^{-\theta x} (\theta x)^j}{j!}$$

(since $\int_{x}^{\infty} e^{-t} t^k dt = k! \sum_{j=0}^{k} \frac{e^{-x} x^j}{j!}$)

similarly,

$$I_2 = \int_{\theta}^{\infty} e^{-\theta' x} \theta'^{\,n-m-1} d\theta'$$

$$= \frac{(n-m-1)!}{x^{n-m}} \sum_{j=0}^{n-m-1} \frac{e^{-\theta x} (\theta x)^j}{j!}$$

Therefore,

$$h_1(\theta,x) = \frac{\Gamma(n-m)}{\Gamma(n-2m)} x^{-m} r_n(x) \frac{(n-2m-1)!}{x^{n-2m}} \sum_{j=0}^{n-2m-1} \frac{e^{-\theta x} (\theta x)^j}{j!}$$

$$- r_n(x) \frac{(n-m-1)!}{x^{n-m}} \sum_{j=0}^{n-m-1} \frac{e^{-\theta x} (\theta x)^j}{j!}$$

$$= r_n(x)\, \Gamma(n-m)\, x^{m-n} \sum_{j=0}^{n-2m-1} \frac{e^{-\theta x} (\theta x)^j}{j!} -$$

$$- r_n(x)\, \Gamma(n-m)\, x^{m-n} \sum_{j=0}^{n-m-1} \frac{e^{-\theta x} (\theta x)^j}{j!}$$

$$h_1(\Theta,x) = e^{-\Theta x} r_n(x) \,\Gamma(n-m)\, x^{m-n} \left[\sum_{j=0}^{n-2m-1} \frac{(\Theta x)^j}{j!} - \sum_{j=0}^{n-m-1} \frac{(\Theta x)^j}{j!} \right]$$

If $m > 0$ then

$$h_1(\Theta,x) = - r_n(x) \,\Gamma(n-m)\, x^{m-n} e^{-\Theta x} \sum_{j=n-2m}^{n-m-1} \frac{(\Theta x)^j}{j!}$$

If $m < 0$ then

$$h_1(\Theta,x) = r_n(x) \,\Gamma(n-m)\, x^{m-n} e^{-\Theta x} \sum_{j=n-m}^{n-2m-1} \frac{(\Theta x)^j}{j!}$$

Let $m^* = \max(0,m)$ then

$$h_1(\Theta,x) = -(\mathrm{Sgn}\ m)\, r_n(x)\, x^{m-n} \,\Gamma(n-m)\, e^{-\Theta x} \sum_{j=n-m-m^*}^{n-2m-1+m^*} \frac{(\Theta x)^j}{j!}$$

where (Sgn m) denotes the sign of m.

Hence

$$\lambda(\Theta) = E_\Theta \left[\frac{h_1(\Theta,X)}{h_2(\Theta,X)} \right]^2 \pi(\Theta)\, V(\Theta)$$

$$= E_\Theta \left\{ \frac{-\mathrm{Sgn}(m)\, r_n(x)\, x^{m-n} \,\Gamma(n-m)\, e^{-\Theta x} \sum_{j=n-m-m^*}^{n-2m-1+m^*} (\Theta x)^j/j!}{e^{-\Theta x}\, r_n(x)\, \Theta^{n-2m-1}} \right\}^2$$

$$\times \quad \frac{\Theta^{n-1}}{\beta^n(\Theta)} \quad \Theta^{-2m}$$

$$= \frac{(\Gamma(n-m))^2}{\Theta^{2n-2m-2}}\, \Theta^{n-1} \int_0^{nb} x^{2(m-n)}\, e^{-\Theta x}\, r_n(x) \left[\sum_{j=n-m-m^*}^{n-2m-1-m^*} (\Theta x)^j/j! \right]^2 dx$$

$$= (\Gamma(n-m))^2\, \Theta^{n+1-2m^*} \int_0^{nb} (\Theta x)^{2(m-n+m^*)}\, x^{-2m^*}\, e^{-\Theta x}\, r_n(x) \,.$$

$$\left[\sum_{j=n-m-m^*}^{n-2m-1+m^*} (\Theta x)^j/j! \right]^2 dx$$

Therefore,

$$\lambda(\theta) = \theta^{n+1-2m^*}(\sqrt{n}-m)^2 \int_0^{nb} e^{-\theta x}\, r_n(x)\, x^{-2m^*} \times \text{a polynomial in } \theta x \text{ of degree } 2(2m^*-m-1)\; dx \qquad ..(17)$$

Because of (13), except for a constant

$$\int_0^{nb} \exp(-\theta x)\, r_n(x)\, x^{j-2m^*}\, dx, \quad 0 \leq j \leq 2(2m^*-m-1) \qquad ..(18)$$

is a mgf (for a non-negative θ), hence $\lambda(\theta)$ is a continuous function of θ on $(0,\infty)$. Thus for admissibility we have only to check Portnoy conditions A and B, Since,

$$\lim_{\theta \to 0} \int_0^{nb} e^{-\theta x}\, r_n(x)\, x^{j-2m^*}\, dx =$$

$$= \int_0^{nb} r_n(x)\, x^{j-2m}\, dx < \infty, \quad 0 \leq j \leq 2(2m^*-m-1)$$

By taking a typical term with j^{th} $[0 \leq j \leq 2(2m^*-m-1)]$ power of x from the polynomial. A typical term of $\frac{\lambda(\theta)}{\theta}$ will be

$$\theta^{n-2m^*}(\sqrt{n}-m)^2 \int_0^{nb} r_n(x)\, x^{j-2m^*}\, dx.$$

Hence

$$\lim_{\theta \to 0} \frac{\lambda(\theta)}{\theta} = 0 \text{ provided } n > 2m.$$

Thus for sufficiently small θ say $0<\theta<C'$,

$\lambda(\theta) < \theta$. Therefore,

$$\int_0^{C'} \frac{1}{\lambda(\theta)}\, d\theta > \int_0^{C'} \frac{1}{\theta}\, d\theta = \infty$$

Hence for n > 2m, then R.H.S. of (B) is satisfied. Hence the condition (B) is satisfied.

For condition (A) we write (18) as

$$\int_0^C e^{-\theta x}\, r_n(x)\, x^{j-2m^*}\, dx + \int_C^{nb} e^{-\theta x}\, r_n(x)\, x^{j-2m^*}\, dx, \quad 0<C\leq nb$$

Then

$$\lim_{\theta\to\infty} \theta^k \int_C^{nb} e^{-\theta x}\, r_n(x)\, x^{j-2m^*}\, dx$$

$$\leq \lim_{\theta\to\infty} \theta^k\, e^{-\theta C} \int_C^{nb} r_n(x)\, x^{j-2m^*}\, dx$$

$$= 0, \quad \text{for every real } k \text{ and } 0\leq j\leq 2(2m^*-m-1)$$

Thus

$$\lambda(\theta) = \theta^{n+1-2m^*} (\sqrt{n}-m)^2 \int_0^C e^{-\theta x}\, r_n(x)\, x^{j-2m^*}$$

$$= \theta^{n+1-2m^*} (\sqrt{n}-m)^2 \int_0^{\theta C} e^{-t}\, r_n(\tfrac{t}{\theta})(\tfrac{t}{\theta})^{j-2m^*}\, \frac{dt}{\theta}$$, where [illegible]

$$= \theta^{n-j} (\sqrt{n}-m)^2 \int_0^{\theta C} e^{-x}\, r_n(\tfrac{x}{\theta})\, x^{j-2m^*}\, dx$$

Therefore,

$$\lim_{\theta\to\infty} \frac{\lambda(\theta)}{\theta} = \lim_{\theta\to\infty} \theta^{n-j-1}(\sqrt{n}-m)^2 \int_0^{\theta C} e^{-x} r_n(\tfrac{x}{\theta})\, x^{j-2m^*}\, dx$$

$$< \infty, \quad \text{from (14)} \qquad ..(19)$$

From (19) the nature of the two integrals

$$\int_C^{\infty} \frac{1}{\lambda(\theta)}\, d\theta \quad \text{and} \quad \int_C^{\infty} \frac{1}{\theta}\, d\theta \quad \text{is the same.}$$

Hence as

$$\int_C^{\infty} \frac{1}{\theta}\, d\theta = \infty \quad \text{we have.} \quad \int_C^{\infty} \frac{1}{\lambda(\theta)}\, d\theta = \infty$$

Thus under the condition (14), the R.H.S. of the condition (A) holds.

Thus under the conditions $n > 2m$ and condition (14), the Portnoy conditions (A) and (B) are satisfied.

Hence $\frac{\Gamma n-m}{\Gamma n-2m}(nZ)^{-m}$ is an admissible estimator of Θ^m with an integral m and quadratic loss for $n > 2m$.

□

Example (2.3.2):

$$r(z) = \frac{1}{\Gamma k} z^{k-1} e^{-z},\quad z > 0,\ k > 0$$
$$= 0, \qquad \text{otherwise.}$$

then
$$r_n(x) = \frac{1}{\Gamma nk} x^{nk-1} e^{-x},\quad x > 0$$

The probability density of $X = \sum_{i=1}^{n} Z_i$ is

$$P(x,\Theta) = \beta^n(\Theta)\, e^{-\Theta x} \frac{1}{\Gamma nk} x^{nk-1} e^{-x},\quad x > 0.$$

Let the loss in estimating Θ^m by d be $(d-\Theta^m)^2\, \Theta^{-2m}$ and the prior density for Θ be $\Theta^{n-1}/\beta^n(\Theta)$. Then the formal Bayes estimator of Θ^m is $\frac{\Gamma n-m}{\Gamma n-2m}(n\bar{Z})^{-m}$.

In order to show that the formal Bayes estimator is admissible for estimating Θ^m, we shall verify conditions (13) and (14) given in theorem 2.3.3 . Therefore,

$$\int_0^\infty r_n(x)\, x^{-2m}\, dx = \int_0^\infty \frac{1}{\Gamma nk} x^{nk-1} e^{-x} x^{-2m}\, dx$$
$$= \frac{1}{\Gamma nk} \int_0^\infty x^{nk-2m-1} e^{-x}\, dx$$
$$= \frac{\Gamma nk-2m}{\Gamma nk}$$
$$< \infty \text{ provided } nk > 2m.$$

and

$$\lim_{\theta \to \infty} \theta^{n-j-1} \int_0^{\theta C} e^{-x} x^{j-2m^*} r_n(\tfrac{x}{\theta}) \, dx$$

$$= \lim_{\theta \to \infty} \theta^{n-j-1} \int_0^{\theta C} e^{-x(1+\frac{1}{\theta})} x^{j-2m^*-nK-1} \frac{\theta^{1-nk}}{\Gamma nk} \, dx$$

$$= \frac{1}{\Gamma nk} \lim_{\theta \to \infty} \theta^{n-j-nk} \int_0^{\theta C} e^{-x(1+\frac{1}{\theta})} x^{j-2m^*+nk-1} \, dx$$

$$= \frac{1}{\Gamma nk} \lim_{\theta \to \infty} \theta^{n-j-nk} \cdot \frac{\Gamma j-2m^*+nk}{(1+\frac{1}{\theta})^{j-2m^*+nk}} , \quad 0 \le j \le 2(2m^*-m-1).$$

$< \infty$, only if $nk \geq n$.

Thus conditions (13) and (14) holds if $nk \geq n > 2m$.

Hence $\frac{\Gamma n-m}{\Gamma n-2m} (n\bar{Z})^{-m}$ is an admissible estimator of θ^m □