CHAPTER : 1

PRELIMINARIES

1.1 INTRODUCTION :

One of the desirable properties of a good estimator is that as the number of observations increases the estimator should come closer and closer in some sense, to the true value of the unknown parameter.

A sequence of estimators $\{T_n\}$ for a parametric function $g(\theta)$ is "Consistent" if T_n converges to $g(\theta)$ in some appropriate sense. As T_n is a random variable, one of the possible way to explain the above property of a sequence of estimators is as $n \longrightarrow \infty$, $|T_n - \theta| \longrightarrow 0$ in some mode of convergence. If this mode of convergence is in probability then T_n is said to be weak consistent, that is let $X_1, X_2, \ldots X_n$ be a random sample from a distribution $F(\cdot, \theta)$, $\theta \in \Theta$ (θ un-known) then an estimator $T_n = T(X_1, X_2, \ldots, X_n)$ of θ is called consistent if for every $\epsilon > 0$,

$$\lim_{n \to \infty} P_{\theta, n} \left\{ |T_n - \theta| \geq \varepsilon \right\} = 0, \ \forall \ \theta \in \Theta \qquad (1.1.1)$$

Example : 1.1.1 : Let $X_1, X_2, \ldots, X_n$ be independent identicaly distributed (i.i.d.) random variables from normal mean θ and variance one, $T_n = \bar{X}_n$.

Note that $\bar{X} \longrightarrow N(\theta, \frac{1}{n})$ hence

$$P_{\theta,n}\left[\,|\,\bar{X}_n - \theta\,|\, \geq \varepsilon\,\right] = P_{\theta,n}\left[\,\left|\frac{\bar{X}_n - \theta}{1/\sqrt{n}}\right| \geq \sqrt{n}\,\varepsilon\,\right]$$

$$= 2\left[\,1 - \Phi(\sqrt{n}\,\varepsilon)\,\right] \longrightarrow 0$$

as $n \longrightarrow \infty$. Hence T_n is consistent estimator for θ.

Example : 1.1.2 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.'s with uniform distribution on $(0,\theta)$. We shall see consistency of T_n where $T_n = \max(X_1, X_2, \ldots, X_n)$. We know here that the probability density function (p.d.f.) of T_n is

$$f_n(t) = \frac{n!}{(n-1)!(n-n)!}\,(t/\theta)^{n-1}\,(1 - t/\theta)^{n-n} \times f_n(t)\frac{1}{\theta}$$

$$= \frac{n\,t^{n-1}}{\theta^n}\,, \qquad 0<t<\theta.$$

Consider $P\left[\,|\,T_n - \theta\,|\, \geq \varepsilon\,\right] = P_\theta\left[\,T_n < \theta - \varepsilon\,\right]$

$$= P_\theta\left[\,X_{(n)} < \theta - \varepsilon\,\right] = 0,$$

if $\varepsilon > \theta$.

For $\varepsilon \leq \theta$; $P_\theta\left[\,X_{(n)} < \theta - \varepsilon\,\right]$

$$= \int_0^{\theta-\varepsilon} \frac{n\,t^{n-1}}{\theta^n}\,dt$$

$$= \frac{n}{\theta^n}\left[\,\frac{t}{n}\,\right]_0^{\theta-\varepsilon}$$

$$= \left(1 - \frac{\varepsilon}{\theta}\right)^n \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty.$$

Thus T_n is consistent for θ.

Example : 1.1.3 : Let $\{X_1, X_2, \ldots, X_n\}$ be a sequence of random variable with marginal distribution function $F(x,\theta)$ and

$P\left[X_i = X_1 \right] = 1 \ , \quad \forall \ \ i = 1,2,3,\ldots..$

In this case note that for any sample point $(X_1, X_2, \ldots, X_n)$,

$T_n \ (X_1, X_2, \ldots, X_n) = S(X_1)$ then as we know that

$$P_\theta \left[\ | \ T_n - \theta \ | \geq \epsilon \ \right]$$

$$= P_\theta \left[| \ S(X_1) - \theta \ | \geq \epsilon \ \right] \not\longrightarrow 0,$$

$$\text{as } n \longrightarrow \infty \ \forall \ \theta \in \Theta \ ,$$

provided $S(X_1)$ is not equal to zero with probability one. Thus $T_n = S(X_1)$ is not consistent estimator for θ.

1.2 PROPERTIES OF CONSISTENT ESTIMATORS :

In the following we discuss some properties of consistent estimators

1.2.1 : Consistent estimator need not be unique. Note that if T_n is consistent for $g(\theta)$ then $T_n + \frac{1}{n}$ is also consistent for $g(\theta)$.

Example : 1.2.1 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v's with $U\ (0,\theta)$, $\theta \in (0,\infty)$. We define $T_1 = X_{(n)}$ and $T_2 = 2\tilde{X}_m$ where $\tilde{X}_m$ = median, are consistent for θ.

consider

$$P_\theta\left[\ |\ T_2 - \theta\ |\ > \varepsilon\ \right] = P_\theta\left[\ |\ 2\tilde{X}_m - \theta\ |\ > \varepsilon\ \right]$$

$$= 1 - P_\theta\left[\ \frac{\theta-\varepsilon}{2} \le \tilde{X}_m \le \frac{\theta+\varepsilon}{2}\ \right]$$

$$= 1 - \int_{\frac{\theta-\varepsilon}{2}}^{\frac{\theta+\varepsilon}{2}} \frac{n!}{(\frac{n-1}{2})!(\frac{n-1}{2})!} \frac{1}{\theta^n} (X)^{\frac{n-1}{2}} (x-\theta)^{\frac{n-1}{2}}\, dx$$

$$= 2 \int_0^{\frac{\theta-\varepsilon}{2}} \frac{1}{\beta(\frac{n+1}{2},\frac{n+1}{2})} \frac{1}{\theta^n} x^{\frac{n-1}{2}} (x-\theta)^{\frac{n-1}{2}}\, dx.$$

$$\le \frac{2}{\theta^n} \left(\frac{\theta-\varepsilon}{2}\right) \left(\frac{\theta-\varepsilon}{2}\right)^{\frac{n-1}{2}} \left(\frac{\theta+\varepsilon}{2}\right)^{\frac{n-1}{2}} \frac{1}{\beta(\frac{n+1}{2},\frac{n+1}{2})}$$

$$= \left(\frac{\theta-\varepsilon}{\theta}\right) \left(\frac{1}{2}\right)^{n-1} \left(\frac{\theta^2-\varepsilon^2}{\theta^2}\right)^{\frac{n-1}{2}} \frac{1}{\beta(\frac{n+1}{2},\frac{n+1}{2})} \qquad (1.2.1)$$

Now consider $\beta\left(\frac{n+1}{2}, \frac{n+1}{2}\right) = \dfrac{\Gamma\frac{n+1}{2}\ \Gamma\frac{n+1}{2}}{\Gamma n+1}$

$$= \frac{(\frac{n-1}{2})!(\frac{n-1}{2})!}{n!}$$

By using stirling's approximation we have

$$\beta\left(\frac{n+1}{2}, \frac{n+1}{2}\right) = \frac{e^{-(n-1)} \left[(\frac{n-1}{2})^{\frac{n-1}{2}+\frac{1}{2}}\right]^2 \sqrt{2\pi}}{e^{-n}\ n^{n+1/2}}.$$

$$= \frac{\sqrt{2\pi}}{e} \left(\frac{n-1}{2}\right)^n \frac{1}{n^{n+1/2}}$$

$$= \frac{\sqrt{2\pi}}{e} \cdot \frac{1}{2^n} \cdot \left(\frac{n-1}{n}\right)^n \cdot \frac{1}{\sqrt{n}}$$

$$= \frac{\sqrt{2\pi}}{\theta} \cdot \frac{1}{2^n} \cdot (1 - \frac{1}{n})^n \cdot \frac{1}{\sqrt{n}}$$

$$= \frac{\sqrt{2\pi}}{\theta} \frac{1}{2^n \sqrt{n}} \qquad (1.2.2)$$

from (1.2.1) and (1.2.2) we have

$$P_\theta \left[|2 \tilde{X}_m - \theta| > \varepsilon \right] \leq (\frac{\theta - \varepsilon}{\theta}) \frac{2 \cdot \theta^2}{\sqrt{2 \cdot \pi}} (1 - \frac{\varepsilon^2}{\theta^2})^{\frac{n-1}{2}} \sqrt{n}$$

$$\longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Hence $P_\theta \left[|2 \tilde{X}_m - \theta| > \varepsilon \right] \longrightarrow 0$, as $n \longrightarrow \infty$.

Thus $2 \tilde{X}_m$ is consistent for θ, similarly from example 1.1.2.

$T_1 = X_{(n)}$ is consistent for θ.

1.2.2 : Unbiased estimator need not be consistent and viceversa.

Example : 1.2.2 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v'.s with $N(\theta, 1)$. Define $T_n = X_1$, $\forall$ n.

Here $E_\theta(T_n) = E_\theta(X_1) = \theta$, $\forall \theta \in \Theta$

Consider $P_\theta \left[| T_n - \theta | \geq \varepsilon \right] = P_\theta \left[| X_1 - \theta | \geq \varepsilon \right] \not\longrightarrow 0$

This implies that T_n is not consistent.

Example : 1.2.3 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v's with $U(0, \theta)$. Here we know that $T_n = X_{(n)}$ is consistent estimator for θ but $E_\theta(T_n) = \frac{n}{n+1} \theta$.

Thus T_n is not unbiased. Hence consistent estimator need not be unbiased.

1.2.3 : Sample mean is consistent for population mean.

For reference we define weak law of large numbers (WLLN) (Bhat (1985) P.193)

Let X_n be a sequence of random variables and let

$S_n = \sum_{k=1}^{n} X_k$; $n = 1, 2, \ldots\ldots$ we say that $\{X_n\}$ obyes the WLLN with respect to the sequence of constants $\{B_n\}$; $B_n > 0$, $B_n \uparrow \infty$, if there exist a sequence of real constants A_n such that

$$B_n^{-1} (S_n - A_n) \xrightarrow{P} 0, \text{ as } n \longrightarrow \infty$$

A_n is called centring constants and B_n is norming constants.

Let $X_1, X_2, \ldots, X_n$ be a randam sample from $f(\cdot)$.

If $S_n = \sum_{i=1}^{n} X_i$, $A_n = ES_n = nEX_1$ and $B_n = n$ then by WLLN

$n^{-1}(S_n - nE(X_1)) \xrightarrow{P} 0$, as $n \longrightarrow \infty$.

This can be proved by Theorem 1 (pp 257 Rohatgi(1986)).

Thus $\frac{S_n}{n} \xrightarrow{P} E(X_1)$, that is sample mean is consistent for population mean.

1.2.4 : If T_{1n} is consistent for $g_1(\theta)$ and T_{2n} is consistent for $g_2(\theta)$ then,

a) $(T_{1n} \pm T_{2n})$ is consistent for $(g_1(\theta) \pm g_2(\theta))$

b) $(T_{1n}\ T_{2n})$ is consistent for $(g_1(\theta)\ g_2(\theta))$

c) $\frac{T_{1n}}{T_{2n}}$ is consistent for $\frac{g_1(\theta)}{g_2(\theta)}$ provided that T_{2n} is not zero for every n and $g_2(\theta)$ is also not zero.

Above properties of two consistent estimators can be proved by Theorem 6.1 (P.108 Bhat (1985)).

1.2.5 : For reference the definition of central limit theorem is given below. (P.69 Zacks (1981)).

If $\{X_n\}$ is a sequence of i.i.d. random variables having a finite variance σ^2, $0<\sigma^2<\infty$ and if

$E(X) = \mu$ then,

$$\lim_{n\to\infty} P\left[\sqrt{n}\ (\bar{X}_n - \mu) \leq \sigma\ .\ \varepsilon\right] = \Phi(\varepsilon)$$

where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

1.2.6 : Consistency is preserved under continuous transformation or function, that is let T_n be consistent estimator for θ and if g is continuous function on Θ, then $g(T_n)$ is consistent for $g(\theta)$.

Proof for this property will be as following.

Fix θ say $\theta = \theta_0$ and g is continuous at θ_0, then $\forall\ \varepsilon > 0$ there exist a $\delta > 0$ such that

$$|\ g(\theta) - g(\theta_0)\ | < \varepsilon \quad \text{whenever } |\ \theta - \theta_0| < \delta.$$

So for every $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$\left[\ |\ \theta - \theta_0\ | < \delta\ \right] \subset \left[\ |\ g(\theta) - g(\theta_0)\ | < \varepsilon\ \right] \qquad (1.2.5)$$

We know that T_n is consistent for θ, then for every

$r > 0$, $P_{\theta_o} \{ | T_n - \theta_o | > r \} \longrightarrow 0$, as $n \longrightarrow \infty$

or $P_{\theta_o} \{ | T_n - \theta_o | \le r \} \longrightarrow 1$, as $n \longrightarrow \infty$. (1.2.6)

Then from (1.2.5) and (1.2.6) we get for every $\varepsilon > 0$,

$$P_{\theta_o} \{ | g(T_n) - g(\theta_o) | \le \varepsilon \} \ge P_{\theta_o} \{ | T_n - \theta_o | \le r \}$$

$$\longrightarrow 1, \text{ as } n \longrightarrow \infty.$$

$$\Rightarrow \lim_{n\to\infty} P_\theta \{ |g(T_n) - g(\theta_o) | \le \varepsilon \} \ge 1.$$

Thus $1 \ge \lim_{n\to\infty} P_{\theta_o}\{ | g(T_n) - g(\theta_o) | \le \varepsilon \} \ge 1$ Hence $g(T_n)$ is consistent for $g(\theta_o)$.

Example : 1.2.5 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with density

$$f_X(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x \ge 0, \quad \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Consider $g(\theta) = \begin{cases} 1 & \text{if } \theta > 2 \\ 0 & \text{otherwise.} \end{cases}$

Here g is not continuous at $\theta = 2$ and let $T_n = \overline{X}$ which is consistent estimator for θ. We shall examine the consistency of $g(T_n)$ for $g(\theta)$.

Let $\theta_o > 0$ be fixed.

<u>Case I</u> : Suppose that $\theta_o > 2$, then $g(\theta)$ is continuous at θ_o.

So $g(\theta_o^+) = \lim_{\substack{n\to\infty \\ h\to 0}} g(\theta_o + h) = 1$

$g(\theta_o^-) = \lim_{\substack{n\to\infty \\ h\to 0}} g(\theta_o - h) = 1$

Note that $\theta_o - h$ satisfies inequality

$$2 < \theta_o - h < \theta_o , \qquad \text{if } h < \theta_o - 2$$

So $$g(\theta_o - h) = \begin{cases} 1 & \text{if } h < \theta_o - 2 \\ 0 & \text{otherwise} \end{cases}$$

Consider now $$P_\theta \left\{ \mid g(T_n) - g(\theta) \mid < \varepsilon \right\}$$

$$= P_\theta \left\{ \mid g(T_n) - 1 \mid < \varepsilon \right\}$$

$$= P_\theta \left\{ T_n > 2 \right\}$$

$$= P_\theta \left\{ (\bar{X}_n - \theta) \sqrt{n} > (2 - \theta) \sqrt{n} \right\}$$

$$\longrightarrow 1, \quad \text{as } n \longrightarrow \infty.$$

Thus $g(T_n) \xrightarrow{P_{\theta_o}} g(\theta_o)$

<u>Case II</u> : Suppose $\theta_o < 2$, g is again continuous at θ_o and

$$g(T_n) \xrightarrow{P_{\theta_o}} g(\theta_o)$$

<u>Case III</u> : $\theta_o = 2$ Consider

$$P_2 \left[\mid g(T_n) - g(2) \mid > \varepsilon \right]$$

$$= P_2 \left[\mid g(T_n) - 0 \mid > \varepsilon \right]$$

$$= P_2 \left[g(\bar{X}) > \varepsilon \right]$$

Let $\varepsilon \geq 1$, then $P_2 \left[g(\bar{X}) > \varepsilon \right] = P(\varphi) = 0.$

Let $0<\varepsilon<1$, then

$$P_2 \left[g(\bar{X}) > \varepsilon \right] = P_2 \left[g(\bar{X}) = 1 \right]$$

$$= P_2 \left[\frac{1}{n} \sum_{i=1}^{n} X_i > 2 \right]$$

$$= P_2 \left[\frac{1}{n} \sum_{i=1}^{n} (X_i - 2) > 0 \right]$$

$$= P_2 \left[\sum_{i=1}^{n} (X_i - 2) > 0 \right]$$

Hence $\lim_{n\to\infty} P_2 \left[\sum_{i=1}^{n} (X_i - 2) > 0 \right]$ will be by dividing both sides by $2. \sqrt{n}$

$$\lim_{n\to\infty} P_2 \left[\frac{1}{2. \sqrt{n}} \sum_{i=1}^{n} (X_i - 2) > 0 \right]$$

$$= \lim_{n\to\infty} P_2 \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - 2}{2} \right) > 0 \right] = 1/2$$

By applying central limit theorem (1.2.5).

Hence $g(T_n)$ is consistent for $g(\theta)$ at all continuity points of g but not at discrete points.

1.3 : JOINT AND MARGINAL CONSISTENCY

1.3.1 : If we are interested in real valued parametric function $g(\theta)$, then definition (1.1.1) will be

$$\lim_{n\to\infty} P_{\theta,n} \left[\mid T_n - g(\theta) \mid \geq \varepsilon \right] = 0 \qquad (1.3.1)$$

that is $T_n \xrightarrow{P_\theta} g(\theta)$, $\forall\, \theta \in \Theta$.

<u>Example</u> : <u>1.3.1</u> : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.'s with $N(\theta, 1)$ where $\Theta = \{ -1, 1 \}$.

Define $$T_n(C) = \begin{cases} -1 & \bar{X} < C \\ 1 & \text{if } \bar{X} \geq C \end{cases},$$

Note that for $\varepsilon > 2$, $P_\theta \left[\mid T_n(C) - \theta \mid \geq \varepsilon \right] = 0$

But for $\varepsilon \geq 2$, we have

$$P_\theta \left[\mid T_n(C) - \theta \mid \geq \varepsilon \right]$$

$$= P_\theta \left[\mid T_n(C) - \theta / \geq \varepsilon,\ \bar{X} < C \right] + P_\theta \left[\mid T_n(C) - \theta / \geq \varepsilon,\ \bar{X} \geq C \right] \qquad \ldots\ldots (1.3.2)$$

Let $\theta = +1$ so (1.3.2) will be

$$P_1 \left[\mid -1 - 1 \mid \geq \varepsilon,\ \bar{X} < C \right] + P_1 \left[\mid 1 - 1 \mid \geq \varepsilon,\ \bar{X} > C \right]$$

$$= P_1 \left[2 \geq \varepsilon,\ \bar{X} < C \right] + P_1 \left[0 \geq \varepsilon,\ \bar{X} > C \right]$$

$$= P_1 \left[2 \geq \varepsilon,\ \bar{X} < C \right]$$

$$= P_1 \left[\bar{X} < C \right]$$

$$= \Phi \left(\sqrt{n}\ (C - 1) \right) \longrightarrow \begin{cases} 1 & \text{if } C > 1 \\ 0.5 & \text{if } C = 1 \\ 0 & \text{if } C < 1 \end{cases} \qquad (1.3.3)$$

as $n \longrightarrow \infty$

Similarly $\theta = -1$

$$P_\theta \left[\mid T_n (C) - \theta \mid \geq \varepsilon \right] \longrightarrow \begin{cases} 0 & \text{if } C > -1 \\ 0.5 & \text{if } C = -1 \\ 1 & \text{if } C < -1 \end{cases} \qquad (1.3.4)$$

as $n \longrightarrow \infty$.

Hence $T_n(C)$ is consistent for $\theta = -1$ if $C > -1$ and for

$\theta = 1$ if $C < 1$

1.3.2 : If we are interested in veetor valued parameter that is $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ then a veetor valued statistic is given by

$T_n = (T_{n1}, T_{n2}, \ldots, T_{nk})$. The sequence $\{T_n\}$ is said to be marginally consistent for $\underline{\theta}$ if the i^{th} component of T_n, that is T_{ni} is consistent for the i^{th} component of

θ (i.e. θ_i) $\forall\ i = 1, 2, \ldots, K$.

Example : 1.3.2 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v's with $N(\mu, \sigma^2)$ Here both parameter μ and σ^2 are unknown,

$\theta = (\mu, \sigma^2)$.

We know that $\overline{X} \longrightarrow \mu$ in probability,

$s^2 = \frac{1}{n} \Sigma\, X_i - \overline{X})^2 \longrightarrow \sigma^2$ in probability so the statistic

$T_n = (\overline{X}, s^2)$ is marginally consistent for

$\theta = (\mu, \sigma^2)$

1.3.3 : For any two points $U = (U_1, U_2, \ldots, U_k)$ and $V = (V_1, V_2, \ldots, V_k)$ in $\mathbb{R}^k$ the euclidian distance between U and V is given by

$$\| U - V \| = \sqrt{\sum_{i=1}^{k} (U_k - V_k)^2}$$

consider $K = 1$, then $\| U - V \| = | U_1 - V_1 |$

A sequence of T_n is said to be jointly consistent for θ if $\| T_n - \theta \| \longrightarrow 0$ in probability for every $\theta \in \Theta$, that is for every $\varepsilon > 0$,

$$P_\theta \left[\| T_n - \theta \| \geq \varepsilon \right] \longrightarrow 0, \text{ as } n \longrightarrow \infty, \ \forall \ \theta \in \Theta \qquad (1.3.5)$$

Example : 1.3.3 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.'s with $N(\mu, \sigma^2)$ both unknown, that is $\theta = (\mu, \sigma^2)$. Here $T_n = (\bar{X}, s^2)$ is jointly consistent for θ.

Consider $P_\theta \left[\| T_n - \theta \| \geq \varepsilon \right]$

$$= P \left[\| (\bar{X}, s^2) - (\mu, \sigma^2) \| \geq \varepsilon \right]$$

$$= P \left[| \sqrt{(\bar{X} - \mu)^2 + (s^2 - \sigma^2)^2} | \geq \varepsilon \right]$$

$$= P \left[| (\bar{X} - \mu)^2 + (s^2 - \sigma^2)^2 \geq \varepsilon^2 \right]$$

$$\leq P \left[(\bar{X} - \mu)^2 \geq \frac{\varepsilon^2}{2} \text{ or } (s^2 - \sigma^2) \geq \frac{\varepsilon^2}{2} \right]$$

$$= P \left[\left\{ (\bar{X} - \mu)^2 \geq \frac{\varepsilon^2}{2} \right\} \cup \left\{ (s^2 - \sigma^2)^2 \geq \frac{\varepsilon^2}{2} \right\} \right]$$

$$\leq P\left[(\bar{X}-\mu)^2 \geq \frac{\varepsilon^2}{2}\right] + P\left[(s^2-\sigma^2)^2 \geq \frac{\varepsilon^2}{2}\right]$$

$$= P\left[(\bar{X}-\mu) \geq \frac{\varepsilon}{\sqrt{2}}\right] + P\left[(s^2-\sigma^2) \geq \frac{\varepsilon}{\sqrt{2}}\right]$$

$$\longrightarrow 0, \text{ as } n \longrightarrow \infty$$

As we know that $\bar{X}$ is consistent for μ and s^2 is consistent for σ^2 . Hence $(\bar{X}, s^2)$ is consistent for (μ, σ^2).

1.3.4 : Joint consistency implies marginal consistency and vice-versa.

This property can be proved as follows:

Suppose that $\{T_n\}$ is jointly consistent for θ. That is

$\forall\ \varepsilon > 0$, and fixed $\theta_o \in \Theta$ we have

$$P_{\theta_o}\left\{ \| T_n - \theta_o \| \geq \varepsilon \right\} \longrightarrow 0, \text{ as } n \longrightarrow \infty \qquad (1.3.6)$$

Note that $| T_{ni} - \theta_{oi}| \leq \| T_n - \theta_o \| \quad \forall \quad i = 1, 2, \ldots, K$

Hence $P_{\theta_o}\left\{ | T_{ni} - \theta_{oi})| \geq \varepsilon \right\} \leq P_{\theta_o}\left\{ \| T_n - \theta_o \| \geq \varepsilon \right\} \longrightarrow 0$,

as $n \longrightarrow \infty$ from (1.3.6)

Thus $T_{ni} \longrightarrow \theta_{oi}$ in P_{θ_o} probability for $i = 1, 2, \ldots, K$.

Hence joint consistency implies marginal consistency.

Similarly suppose that $\left\{T_{ni}\right\}$ is marginaly consistent for θ_i

then for $P_{\theta i}\left\{ | T_{ni} - \theta i | \geq \varepsilon \right\} \longrightarrow 0$,

as $n \longrightarrow \infty \quad i = 1, 2, \ldots, K$.

Note that $P\left[\| T_n - \theta \| \geq \varepsilon\right] \leq \sum_{i=1}^{k} P_{\theta i}\left\{| T_{ni} - \theta_i |^2 \geq \frac{\varepsilon^2}{k}\right\}$

$\longrightarrow 0$, as $n \longrightarrow \infty$. Hence the result.

1.4 : UNIFORMLY CONSISTENT ESTIMATOR

1.4.1 : The consistency of T_n for θ implies that, agiven $\varepsilon > 0$ and $\delta > 0$ there exist $n_o(\varepsilon, \delta, \theta)$ such that

$$P_\theta \left\{ | T_n - \theta | < \varepsilon \right\} \geq 1 - \delta, \ \forall \ n \geq n_o$$

Let $\sup_{\theta \in \Theta} n_o \left\{ \varepsilon, \delta, \theta \right\} = N_o$. If T_n converges uniformly to θ then based on a sample of size $N_o < \infty$ it is possible to estimate θ so that the probability that the error is at most ε is at leat $1 - \delta$.

Let X_i be i.i.d. r.v.'s with $N(\theta,1)$ distribution and define

$T_n = \bar{X}$. Here observe that for

$$n \geq \left\{ \frac{1}{\varepsilon} \Phi^{-1} \left(1 - \frac{\delta}{2}\right) \right\}^2$$

$P_\theta = \left\{ | T_n - \theta | < \varepsilon \right\} \geq 1 - \delta, \ \forall \ \theta \in \Theta$. That is

$$N_o = \left\{ \frac{1}{\varepsilon} \Phi^{-1} \left(1 - \frac{\delta}{2}\right) \right\}^2 + 1 \qquad \text{(say)}$$

For $\varepsilon = 0.001$ and $\delta = 0.01$ $N_o = (3330)^2 + 1$.

Similarly consider X_i be i.i.d. r.v.'s with $B(1,\theta)$ distribution and define $T_n = \bar{X}_n$.

By using Normal approximation for large n we observe that for

$n \geq \left\{ \frac{\sqrt{\theta(1-\theta)}}{\varepsilon} \Phi^{-1} \left(1 - \frac{\delta}{2}\right) \right\}^2$

$P_\theta \left[\,|\, \bar{X}_n - \theta \,|\, < \varepsilon \right] \geq 1 - \delta, \ \forall \ \theta \in \Theta.$

That is $N_o = \left[\frac{\sqrt{\theta(1-\theta)}}{\varepsilon} \Phi^{-1} \left(1 - \frac{\delta}{2}\right) \right]^2 + 1$ (say)

For $\varepsilon = 0.001$ and $\delta = 0.01$

$N_o = 66410 + 1.$

<u>1.5</u> : <u>ALTERNATIVE DEFINITIONS OF WEAK CONSISTENCY</u>

<u>1.5.1</u> : If $O_p(n^\alpha)$ is a random variable Z_n such that,

$\delta > 0 \quad \lim_{n\to\infty} P \left[n^{-\alpha} \,|Z_n|\, > \delta \right] = 0$

T_n is said to be weakly consistent for θ, if $T_n = \theta + O_p(1)$

where $O_p(1)$ is a random variable Z such that $\lim_{n\to\infty} P\{|Z|>\delta\} = 0$

<u>Example</u> : <u>1.5.1</u> : Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.'s with $N(\theta,1)$. Here $\bar{X}_n$ is consistent estimator for θ and we can write $\bar{X}_n = \theta + (\bar{X}_n - \theta)$

since $\lim_{n\to\infty} P_\theta \left\{ \,|\, \bar{X}_n - \theta \,|\, > \delta \right\} = 0$

we have $\bar{X}_n = \theta + O_p(1)$. Hence $\bar{X}_n$ is consistent estimator for θ.

<u>1.5.2</u> : Let T_n be a statistic based on n i.i.d. observations adrawn from the $F(x,\theta)$. The estimate T_n has the form

$T_n = t(F_n(x))$, where $F_n(x)$ be the empirical distribution function and let $t(F(x,\theta)) = \theta$. Then T_n is said to be Fisher consistent for θ. (pp.287 Cox and Hinkley (1979))

<u>Example</u> : <u>1.5.2</u> : Sample mean $\bar{X}_n$ is Fisher consistent for population mean. Let θ be the mean of the distribution, then provided estimator $\theta = \int xdF(x)$. Let X_i's be i.i.d. from $F(\cdot)$.

Further $t(F_n(x)) = \int_{-\infty}^{\infty} x.\ dF_n(x) = \bar{X}_n$.

We know that from WLLN $\bar{X}_n \xrightarrow{P} \theta$.