CHAPTER : II

METHODS OF OBTAINING CONSISTENT ESTIMATORS

2.1 INTRODUCTION

In the first section of this chapter we state definitions of some terms which are useful for further discussion.

In the second section we discuss mathods of obtaining consistent estimators. Consistent estimators can be obtained by satisfying necessary and sufficient conditions, by weak law of large numbers, via order statistics, with the help of maximum likelihood estimators. Further we discuss inconsistency of maximum likelihood estimators.

In the third section we define order of consistency and discuss lemma on it.

For ready references the following terms are defined.

2.1.1 : Bias of an estimator :

$T(X_1, X_2, \ldots X_n)$ is said to be biased for θ if $E_\theta(T) \neq \theta$ for some $\theta \in \Theta$. The quantity $b(T,\theta) = E_\theta\, T(X_1, X_2, \ldots, X_n) - \theta$ is called the bias of T

2.1.2 : The mean square error (MSE) of an estimator.

The MSE of T is defined to be

$$MSE_\theta(T) = E_\theta\left[\, T(X_1, X_2, \ldots, X_n) - \theta \,\right]^2$$

2.1.3 : Chebyshev's Inequality :

Suppose that $EX^2 < \infty$ and $EX = \mu$, $Var(X) = \sigma^2$, $0<\sigma^2<\infty$.

Then for every $\epsilon > 0$, $P\left[\mid X - E(X) \mid > \epsilon \right] \le \frac{\sigma^2}{\epsilon^2}$

2.1.4 : Probability Integral Transformation :

If X is a random variable with continuous cumulative distribution function $F_X(x)$, then $U = F_X(x)$ is uniformly distributed over the interval (0,1). U is called the probability integral transformation to X further if U is uniformly distributed over the interval (0,1) then $X = F_X^{-1}(U)$ has cumulative distribution function $F_X(\cdot)$.

2.1.5 : Complex function :

A real valued function f is said to be complex if inequality $f\left(\frac{x_1+x_2}{2} \right) \le \frac{1}{2} \left[f(x_1) + f(x_2) \right]$ holds for all values of x_1 and x_2.

2.1.6 : The Jensen's Inequality :

If $g(X)$ is a convex function of X and $E\left\{ \mid g(X) \mid \right\} < \infty$ then $E\left\{ g(X) \right\} \ge g\left\{ E(X) \right\}$.

2.1.7 : Absolute Continuous Measure :

Let $(\mathcal{X}, \mathcal{B}, P)$ and $(\mathcal{X}, \mathcal{B}, Q)$ are probability measure. If $P(A) = 0$ whenever $Q(A) = 0$ then P is said to be absolute continuous measure with respect to Q and denoted by $P<<Q$.

2.1.8 : Radon – Nikodyn Derivative :

Let P and Q be defined on ($\mathcal{X}$, $\mathcal{B}$) and P<<Q.

Then there exist a function f such that

$$P(A) = \int_A f(X)dQ(x) , \qquad \forall \ A \in \mathcal{B}.$$

f is said to be R.N. derivative of P with respect to Q and is denoted by ($\frac{dP}{dQ}$).

2.1.9 : Strong law of large number :

We say that the sequence $\{X_n\}$ obyes the strong law of large numbers (SLLN) with respect to the norming constant $\{B_n\}$ if there exist a sequence of (centring) constants $\{A_n\}$ such that $B_n^{-1} (S_n - A_n) \xrightarrow{a.s.} 0$ as $n \longrightarrow \infty$. Here $B_n > 0$ and $B_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

2.2 METHODS OF OBTAINING CONSISTENT ESTIMATORS :

In this section we discuss the various methods of obtaining consistent estimators.

2.2.1 : Let T_n be an estimator of a parametric function $g(\theta)$ such that it's bias and variance tends to zero as $n \longrightarrow \infty$, then T_n is consistent estimator of θ. The proof is direct consequence of Chebyshev's inequality and fact that $MSE(T_n) = E(T_n - \theta)^2 \longrightarrow 0$, as $n \longrightarrow \infty$.

<u>Example</u> : <u>2.2.1</u> : Let $X_1, X_2, \ldots, X_n$ be i.i.d.r.v.'s with $U(0, \theta)$ $\theta \in (0, \infty)$ define $T_1 = X_{(n)}$. We shall show that $X_{(n)}$ is consistent for θ. Note that

$$f_{T_1}(t, \theta) = \frac{n.t^{n-1}}{\theta^n}, \quad 0<t<\infty, \quad n\geq 1.$$

$$\text{Here } E_\theta(T_1) = \int_0^\theta t.\ f_{T_1}(t, \theta)dt = \frac{n}{n+1}\theta$$

$$E_\theta(T_1^2) = \int_0^\theta t^2.\ f_{T_1}(t, \theta)dt = \frac{n}{n+2}\theta^2$$

$$\text{So Bias in } T_1 = E_\theta(T_1 - \theta) = \left(\frac{n}{n-1} - 1\right)\theta$$

$$= \frac{-\theta}{n+1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

$$\text{and } MSE(T_1) = E(T_1 - \theta)^2$$

$$= \theta^2 \left(\frac{n}{n+2} + 1 - \frac{2n}{n+1}\right)$$

$$= \theta^2 \left(\frac{1}{1+1/n} + 1 - \frac{2}{1+1/n}\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty$$

Hence $T_1 = X_{(n)}$ is consistent for θ.

<u>2.2.2</u> : Necessary conditions for consistency of estimators.

(P.22 Akahira Takeuehi (1981)).

For any two points θ_1 and θ_2 in Θ there exist σ finite measure μ_n such that $P_{\theta_{1,n}}$ and $P_{\theta_{2,n}}$ are absolutely continuous with respect to μ_n. Then for any two points θ_1 and θ_2 in Θ we define

$$d_n(\theta_1, \theta_2) = \int_{\mathfrak{X}_n} \left| \frac{dP_{\theta_1,n}}{d\mu_n} - \frac{dP_{\theta_2,n}}{d\mu_n} \right| d\mu_n.$$

$$= 2 \sup_{B \in \mathcal{B}(n)} \left| P_{\theta_1,n}(B) - P_{\theta_2,n}(B) \right| \qquad (2.2.2)$$

for every n, dn is metric on Θ which is independent of μ_n.

2.2.3 : Theorem

If there exist a consistent estimator then for any two disjoint points θ_1 and θ_2 in Θ

$$\lim_{n\to\infty} d_n(\theta_1, \theta_2) = 2.$$

Proof : If we denote a consistent estimator by T_n, then for every $\varepsilon > 0$, $\lim_{n\to\infty} P_{\theta,n} \left\{ \| T_n - \theta \| > \varepsilon \right\} = 0$

Let $\varepsilon = \| \theta_1 - \theta_2 \| / 2$, then for any $\delta > 0$ there exist a n_0 such that, every $n \geq n_0$

$$P_{\theta_1,n} \left\{ \| T_n - \theta_1 \| \leq \varepsilon \right\} \geq 1 - \frac{\delta}{4}$$

$$P_{\theta_2,n} \left\{ \| T_n - \theta_2 \| \leq \varepsilon \right\} \geq 1 - \frac{\delta}{4}$$

Then put $A = \left\{ \| T_n - \theta_1 \| \leq \| T_n - \theta_2 \| \right\}$

and $A^c = \left\{ \| T_n - \theta_2 \| \leq \| T_n - \theta_1 \| \right\}$

so $P_{\theta_1,n} \left\{ \| T_n - \theta_1 \| \leq \| T_n - \theta_2 \| \right\} \geq 1 - \frac{\delta}{4}$. This implies that $P_{\theta_1,n}(A) \geq 1 - \frac{\delta}{4}$ (2.2.3)

Similarly $P_{\theta_2,n}(A^c) \geq 1 - \frac{\delta}{4}$.

From (2.2.2) and (2.2.3) we can obtain

$$d_n(\theta_1, \theta_2) = \int_{\mathscr{X}_n} \left| \frac{dP_{\theta_1,n}}{d\mu_n} - \frac{dP_{\theta_2,n}}{d\mu_n} \right| d\mu_n.$$

$$\geq \int_A \frac{dP_{\theta_1,n}}{d\mu_n} d\mu_n + \int_{A^c} \frac{dP_{\theta_2,n}}{d\mu_n} d\mu_n - \int_A \frac{dP_{\theta_2,n}}{d\mu_n} d\mu_n - \int_A \frac{dP_{\theta_2,n}}{d\mu_n} d\mu_n$$

$$+ \int_A \frac{dP_{\theta_1,n}}{d\mu_n} d\mu_n + \int_{A^c} \frac{dP_{\theta_1,n}}{d\mu_n} d\mu_n - \int_{A^c} \frac{dP_{\theta_1,n}}{d\mu_n} d\mu_n$$

$$- \int_{A^c} \frac{dP_{\theta_1,n}}{d\mu_n} d\mu_n.$$

$$= 1 - 2 P_{\theta_2,n}(A) + 1 - 2 P_{\theta_1,n}(A^c)$$

$$= 2 - 2 P_{\theta_1,n}(A^c) - 2 P_{\theta_2,n}(A).$$

$$\geq 2 - \delta.$$

letting $\delta \longrightarrow 0$ we have $\underline{\lim}_{n\to\infty} d_n(\theta_1, \theta_2) \geq 2$

But from (2.2.2) we have

$$\overline{\lim}_{n\to\infty} d_n(\theta_1, \theta_2) \leq 2$$

Hence $\lim_{n\to\infty} d_n(\theta_1, \theta_2) = 2.$

<u>Example</u> : <u>2.2.3</u> : If $X_1, X_2, \ldots, X_n$ be i.i.d. random variable

with $$f(x,\theta) = \begin{cases} \theta \cdot e^{-\theta x} & x \geq 0, \quad \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

then $\lim\limits_{n\to\infty} d_n(\theta_1, \theta_2) = 2$, for $\theta_1 = 1$ $\theta_2 = 2$.

We know $\lim\limits_{n\to\infty} d_n(\theta_1, \theta_2)$

$$= \int_{\mathscr{X}^n} \left| \frac{dP_{\theta_{1,n}}}{d\mu_n} - \frac{dP_{\theta_{2,n}}}{d\mu_n} \right| d\mu_n.$$

$$= \int_{\mathscr{X}^n} \left| e^{-\Sigma x_i} - 2^n.\ e^{-2\Sigma x_i} \right| d\underline{x}$$

$$= \int_{\mathscr{X}^n} e^{-\Sigma x_i} \left| 1 - 2^n.\ e^{-\Sigma x_i} \right| d\underline{x}.$$

$$= E_{X_1, X_2, \ldots, X_n} \left| 1 - 2^n.\ e^{-\Sigma x_i} \right|$$

where $X_1, X_2, \ldots, X_n \rightsquigarrow$ i.i.d.

exponential with e^{-x} $x > 0$.

$$= \int_0^\infty \left| 1 - 2^n.e^{-y} \right| f_n(y)dy$$

where $f_n(y)$ is the density of Y.

$$= \int_0^{n\log 2} (\ 2^n.e^{-y} - 1\ )\ f_n(y)dy + \int_{n\log 2}^\infty (\ 1 - 2^n.e^{-y}\ )\ f_n(y)dy.$$

$$= 2\int_0^{n\log 2} (2^n.e^{-y} - 1)\ f_n(y)dy + \int_0^{\infty} (1 - 2^n.e^{-y})\ f_n(y)dy.$$

$$= 2\int_0^{n\log 2} 2^n.e^{-y}.\ \frac{e^{-y}\ y^{n-1}}{\sqrt{n}}\ dy - 2\int_0^{n\log 2} f_n(y)dy.$$

$$= 2\ \left[\int_0^{n\log 2} 2^n.e^{-2y}\ .\ \frac{y^{n-1}}{\sqrt{n}}\ dy - \int_0^{n\log 2} f_n(y)\ dy\right]$$

$$= 2\ \left[\int_0^{2n\log 2} 2^n.e^{-u}\ \frac{(u/2)^{n-1}}{\sqrt{n}}\ \frac{du}{2} - \int_0^{n\log 2} f_n(y)\ dy\right]$$

$$= 2\ \left[\int_0^{2n\log 2.} f_n(y)dy - \int_0^{n\log 2} f_n(y)dy\right]$$

$$= 2\ \left[\int_{n\log 2}^{2n\log 2} f_n(y)dy\right]$$

$$= 2\ P_{X_1, X_2, \ldots, X_n}\ (\ n\log 2 < X < 2_n\ \log 2\ )$$

$$= 2\ P_{X_1, X_2, \ldots, X_n}\ (\ n\log 2 - n < \sum_{i=1}^{n} (\ X_i - 1\ ) < 2\ n\log 2 - n)$$

$$= 2\ P_{X_1, X_2, \ldots, X_n}\ (\ \log 2 - 1 < \sum_{i=1}^{n}\ (\ \frac{X_i - 1}{n}\ ) < 2\ \log 2 - 1\ )$$

$$= 2\ \left[\ G_n\ (2\ \log 2 - 1\ ) - G_n\ (\log 2 - 1)\ \right]$$

where G_n is c.d.f of $\Sigma\ (\ \frac{x_i - 1}{n}\ )$, Here by WLLN

$$G_n(x) \longrightarrow \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Hence $\lim_{n\to\infty}\ d_n(\ \theta_1,\ \theta_2) = 2.$

2.2.4 : Theorem

The following theorem shows that a necessary condition for the existance of consistent estimator is that the limit of the Kull-back leibler information is infinite.

Let for each n $\{ \bar{X}_n : dP_{\theta,n} / d\mu > 0 \}$ does not depend on θ. If there exist a consistent estimator then the following holds for any two disjoint points θ_1 and θ_2

$$\lim_{n\to\infty} I_n(\theta_1, \theta_2) = \infty.$$

where
$$I_n(\theta_1, \theta_2) = \int_{\mathcal{X}(n)} \frac{dP_{\theta_{1,n}}}{d\mu_n} \log \left(\frac{dP_{\theta_{1,n}}}{dP_{\theta_{2,n}}} \right) d\mu_n$$

Proof : Let $0<\delta<1/2$ and putting $y_n = \dfrac{dP_{\theta_{2,n}}}{dP_{\theta_{1,n}}}$

we have for sufficiently large n

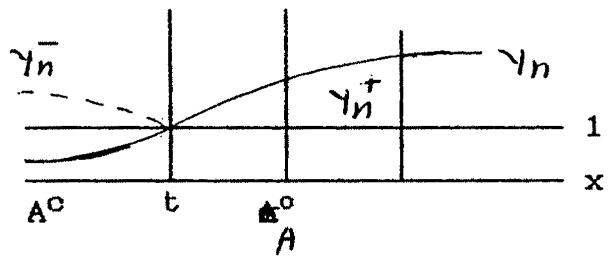
$$E_{\theta_{1,n}} (| y_n - 1 |) = \int_{\mathcal{X}(n)} \left| \frac{dP_{\theta_{2,n}}}{dP_{\theta_{1,n}}} - 1 \right| dP_{\theta_{1,n}}$$

$$= \int_{\mathcal{X}(n)} \left| \frac{dP_{\theta_{2,n}} - dP_{\theta_{1,n}}}{d\mu_n} \right| d\mu_n .$$

$$= d_n(\theta_1, \theta_2) \geq 2 - 2\delta.$$

putting $y_n^+ = \max \{ y_n - 1, 0 \}$ and

$y_n^- = \max \{ 1 - y_n, 0 \}$



$$E_{\theta_{1,n}}(y_n^+) - E_{\theta_{1,n}}(y_n^-) = E_{\theta_{1,n}}(y_n^+ - y_n^-)$$

$$= E_{\theta_{1,n}}(y_n - 1)$$

$$= \int_{\mathscr{X}(n)} \left\{ \frac{dP_{\theta_{2,n}}}{d\mu_n} - \frac{dP_{\theta_{1,n}}}{d\mu_n} \right\} d\mu_n$$

$$= 0$$

For sufficiently large n, $E_{\theta_{1,n}}(y_n^+) + E_{\theta_{1,n}}(y_n^-)$

$$= E_{\theta_{1,n}}(\,|\,y_n - 1\,|\,) \geq 2 - 2\,\varepsilon.$$

that is $E_{\theta_{1,n}}(y_n^+ + y_n^-) = E_{\theta_{1,n}}(\,|\,y_n - 1\,|\,) \geq 2 - 2\,\varepsilon.$

Hence for sufficiently large n we obtain

$E_{\theta_{1,n}}(y_n^+) = E_{\theta_{1,n}}(y_n^-) \geq 1 - \varepsilon.$

Since $0 \leq y_n^- \leq 1$ and above equation holds for sufficiently large n

$$1 - \varepsilon \leq E_{\theta_{1,n}}(y_n^-)$$

$$= \int_{y_n^- \geq 1-2\varepsilon} y_n^- \, dP_{\theta_{1,n}}(\tilde{x}_n) + \int_{y_n^- < 1-2\varepsilon} y_n^- \, dP_{\theta_{1,n}}(\tilde{x}_n)$$

$$\leq P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon) + (1 - 2\varepsilon) P_{\theta_{1,n}}(y_n^- \leq 1 - 2\varepsilon)$$

$$= P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon) + (1 - 2\varepsilon)\{P_{\theta_{1,n}}(y_n^- \leq 1 - 2\varepsilon)$$

$$+ P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon) - P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon)\}$$

$$= P_{\theta_{1,n}}\{y_n^- \geq 1 - 2\varepsilon\} + (1 - 2\varepsilon) - (1 - 2\varepsilon).$$

$$P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon).$$

$$= 2\varepsilon \, P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon) + (1 - 2\varepsilon).$$

that is

$$1 - \varepsilon \leq 2\,\varepsilon\, P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon) + (1 - 2\varepsilon).$$

This implies that $\varepsilon \leq 2\,\varepsilon\, P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon).$

So $\frac{1}{2} \leq P_{\theta_{1,n}}(y_n^- \geq 1 - 2\varepsilon).$

But for sufficiently large n from above

$$I_n(\theta_1, \theta_2) = E_{\theta_{1,n}}^{-}(-\log y_n)$$

From picture we have $y_n = (1 + y_n^+)\ (1 - y_n^-)$

So $I_n\ (\theta_1,\ \theta_2) = E_{\theta_{1,n}}\ (-\log y_n)$

$$= E_{\theta_{1,n}}\left[-\log\left\{(1 + y_n^+)\ (1 - y_n^-)\right\}\right]$$

$$= E_{\theta_{1,n}}\left[-\log\ (1 + y_n^+)\right] - E_{\theta_{1,n}} \log\ (1 - y_n^-)$$

Consider firstly $E_{\theta_{1,n}}\left[-\log\ (1 + y_n^+)\right] \geq - E_{\theta_{1,n}}\ (y_n^+)$ and

secondly $B = \left\{ x\ :\ 1 - y_n^- \leq 2\delta\right\}$

$A = \left\{ x\ :\ y_n \geq 1 \right\},\quad A^c = \left\{ x\ :\ y_n < 1 \right\}.$

then $E_{\theta_{1,n}}\ \left\{ \log\ (1 - y_n^-) \right\}$

$$= \int_{A \cup A^c} \log\ (1 - y_n^-)\ f_1(x)dx.$$

$$= \int_{A^c} \log\ (1 - y_n^-)\ f_1(x)dx.$$

$$= \int_{A^c.B} \log\ (1 - y_n^-)\ f_1(x)dx + \int_{A^c.B^c} \log\ (1 - y_n)\ f_1(x)dx.$$

$$\leq \int_{A^c.B^c} \log\ (1 - y_n^-)\ f_1(x)dx$$

$$\leq \log\ 2\ \delta.\ \left[\ \int_{A^c.B^c} f_1(x)dx\ \right]$$

$$\leq \log\ 2\ \delta\ P_1(B^c)$$

$$\leq \frac{1}{2} \log 2\,\delta.$$

And $$E(y_n^+) = \int_A y_n^+ f_1(x)dx = \int_A (1 - y_n^+ - 1)\, f_1(x)dx.$$

$$= \int_A \left(\frac{dP_{\theta_{2,n}}}{dP_{\theta_{1,n}}} - 1 \right) dP_{\theta_{1,n}}$$

$$= \int_A (dP_{\theta_{2,n}} - dP_{\theta_{1,n}})\, dx < 1.$$

Hence $$I_n(\theta_1, \theta_2) = E_{\theta_{1,n}} \left[- \log (1 + y_n^+)\right] - E_{\theta_{1,n}} \log (1 - y_n^-)$$

$$\geq - E_{\theta_{1,n}} (y_n^+) - \frac{1}{2} \log 2\,\delta.$$

$$\geq - 1 - \frac{1}{2} \log \delta.$$

$$= \infty. \qquad \text{as } \delta \longrightarrow 0.$$

<u>Example</u> : <u>2.2.4</u> : We shall show that Kullback leibler information for exponential distribution at $\theta_1 = 1$ and $\theta_2 = 2$ is infinity.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variable with

$$f(x, \theta) = \begin{cases} \theta . e^{-\theta . x} & x \geq 0, \quad \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here $\prod_{i=1}^{n} f(x_i, \theta_1) = e^{-\Sigma x_i}$ and $\prod_{i=1}^{n} f(x_i, \theta_2) = 2^n . e^{-2\Sigma x_i}$

$$I_n(\theta_1, \theta_2) = \int_{\mathcal{X}(n)} f(x_i, \theta_1) \log \frac{f(x_i, \theta_1)}{f(x_i, \theta_2)}\, d\underline{x}$$

$$= \int_{\mathscr{X}(n)} e^{-\Sigma x_i} \log \left(\frac{e^{-\Sigma x_i}}{2^n \cdot e^{-2\Sigma x_i}} \right) d\underline{x}$$

$$= \int_{\mathscr{X}(n)} e^{-\Sigma x_i} (- \Sigma x_i + 2 \Sigma x_i - n\log 2) d\underline{x}$$

$$= \int_{\mathscr{X}(n)} e^{-\Sigma x_i} (\Sigma x_i - n\log 2) d\underline{x}$$

$$= \int_{\mathscr{X}(n)} e^{-\Sigma x_i} \Sigma x_i \, d\underline{x} - \int_{\mathscr{X}(n)} n\log 2 \cdot e^{-\Sigma x_i} \, d\underline{x}.$$

$$= \sum_{i=1}^{n} \int_0^{\infty} \cdots \int_0^{\infty} x_i \, e^{-\Sigma x_i} \, d\underline{x} - \int_0^{\infty} \cdots \int_0^{\infty} \log 2 \, e^{-\Sigma x_i} \, d\underline{x}$$

$$= \sum_{i=1} \overline{|2} - \log 2 = n - \log 2 \longrightarrow \infty , \quad \text{as } n \longrightarrow \infty.$$

2.2.5 : Another method of obtaining consistent estimator is by the use of weak law of large numbers (WLLN).

Let us consider a suitable function $U(X)$, such that

$E_\theta\left[U(X)\right] = g(\theta)$ is such that g^{-1} is continuous function. Then WLLN is applied to $U(X_i)$ $i = 1,2,\ldots,n$ gives

$$T_n = \frac{1}{n} \sum_{i=1}^{n} U(X_i) \xrightarrow{P} g(\theta)$$

since g^{-1} is continuous it follows that $g^{-1}(T_n)$ is consistent for $g^{-1}\left[g(\theta)\right] = \theta$.

Example : 2.2.5 : Let $X_1, X_2, \dots, X_n$ be distributed according

to $$f(x, \theta) = \begin{cases} \theta . \ x^{\theta-1} & \text{if } 0<x<1, \ \theta > 0 \\ 0 & \text{otherwise} \end{cases}$$

Here $$E_\theta(X) = \frac{\theta}{\theta+1}$$ and by WLLN

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} \frac{\theta}{\theta+1} = g(\theta).$$

This implies that $g^{-1}(y) = \frac{y}{1-y}$, $0<y<1.$,

which is continuous.

Hence $$g^{-1}(\bar{X}_n) = \frac{\bar{X}_n}{1 - \bar{X}_n} \xrightarrow{P} g^{-1}\left[g(\theta)\right] = \theta.$$

Example : 2.2.6 :

If X_i be i.i.d. with $U(0, \theta)$ then $U(X_i) = -\log X_i$

$i = 1,2,\dots,n$ are i.i.d. exponential with mean $\frac{1}{\theta}$.

Thus $$\frac{1}{n} \sum_{i=1}^{n} (-\log X_i) \xrightarrow{P} \frac{1}{\theta} = g(\theta)$$

so $$\frac{-n}{\Sigma \log X_i} \xrightarrow{P} \theta.$$

2.2.6 : Assume that $E(X_i^r) = \mu_r'$ exists then

$$\frac{1}{n} \sum_{i=1}^{n} X_i^r \xrightarrow{P} \mu_r' \quad \text{for } r = 0,1,2,\dots,K.$$

If follows from marginal and joint consistency for g

continuous, $g(m_1', m_2', \dots, m_k')$ is consistent for

$g(\mu_1', \mu_2', \ldots, \mu_k')$. This method of obtaining consistent estimator is called method of moments.

For example $X_1, X_2, \ldots, X_n$ be i.i.d. with $N(\mu, \sigma^2)$. Here

$$\bar{X} = m_1' \xrightarrow{P} \mu \quad \text{and} \quad m_2' \xrightarrow{P} \mu^2 + \sigma^2.$$

Thus (a) $m_2' - (m_i')^2 = \frac{1}{n} \Sigma (X_i - \bar{X})^2 \xrightarrow{P} \sigma^2$

(b) $\bar{X} + K \sqrt{S_n^2} \xrightarrow{P} \mu + K.\sigma$

(c) $\Phi\left(\frac{a - \bar{X}}{\sqrt{S_n^2}} \right) \xrightarrow{P} \Phi\left(\frac{a - \mu}{\sigma} \right).$

2.2.7 : Consistent estimators can be obtained by using the order statistics.

For $0<\lambda<1$, let ξ_λ denote the λ^{th} percentile that is a solution of the equation $F(x, \theta) = \lambda$. We assume that F is continuous and strictly increasing, so that $\xi_\lambda(\theta) = F^{-1}(\lambda)$ is uniquely defined for each fixed $\theta \in \Theta$. This would be the case for example in Cauchy, Laplace, Weibul and other continuous distribution.

Let $r = [n_\lambda] + 1$ then we define X(r) as the corrsponding sample aquantile. By using the probability integral transformation for each fixed $\theta \in \Theta$, $F(X_{(r)}) = U_{(r)}$ has the same distribution as the r^{th} order statistics of a sample of size n from U(0, 1). Note that $E(U_{(r)}) = \frac{r}{n+1}$ and

$E(U^2_{(r)}) = \frac{r(r+1)}{(n+1)(n+2)}$. Then we can check that

$$E\left[U_{(r)} - \lambda\right]^2 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Thus $U_{(r)} \xrightarrow{P} \lambda$ from which it follows that

$$F^{-1}(U_{(r)}) \xrightarrow{P} \xi_\lambda.$$

<u>Example</u> : <u>2.2.7</u> : Let us check above criteria for Cauchy distribution, $f(x) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2} \qquad -\infty<x<\infty$

then $\qquad F(x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+(t-\theta)^2}\, dt.$

$$= \frac{1}{\pi} \left[\tan^{-1} (x - \theta) + \frac{\pi}{2} \right]$$

so $\quad P\left[X \le \xi_\lambda\right] = \lambda = F(\xi_\lambda, \theta) = \frac{1}{\pi} \left[\tan^{-1} (\xi_\lambda - \theta) + \frac{\pi}{2}\right]$

This implies that $\pi (\lambda - 1/2) = \tan^{-1} (\xi_\lambda - \theta)$.

So, $(\xi_\lambda - \theta) = \tan \pi (\lambda - 1/2) = F^{-1}(\lambda)$.

$X_{(r)}$ is the λ^{th} sample quantile where $r = [n_\lambda] + 1$.

$X_1, X_2, \ldots, X_n \longrightarrow f(x, \theta)$ i.e. Cauchy

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r} f(x)$$

$$= \frac{n!}{(r-1)!(n-r)!} \left[\frac{1}{\pi} \tan^{-1} (x-\theta) + \frac{1}{2} \right]^{r-1} \times$$

$$\left[\frac{1}{2} - \frac{1}{\pi} \tan^{-1} (x - \theta)\right] \frac{1}{\pi} \frac{1}{1+(x-\theta)^2}$$

By using probability integral transformation for every $\theta \in \Theta$ fixed $F(X_{(r)}) = U_{(r)}$ has the same distribution of r^{th} otrder statistics of a sample of size n from U(0, 1) so

$F(X_{(r)}) = U_{(r)}$.

$$g_{U_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} x^{r-1} (1-x)^{n-r}.$$

$$= \frac{\Gamma n+1}{\Gamma r \; \Gamma n-r+1} x^{r-1} (1-x)^{n-r}.$$

$$= \frac{x^{r-1} (1-x)^{n-r}}{B(r, n-r+1)}$$

Hence $E(U_{(r)}) = \frac{r}{n+1}$ and $E(U^2_{(r)}) = \frac{r(r+1)}{(n+1)(n+2)}$

so $\mathrm{Var}(U_{(r)}) = \frac{r(r+1)}{(n+1)(n+2)} - \frac{r^2}{(n+1)^2} \longrightarrow 0$, as $n \longrightarrow \infty$.

This implies that $U_{(r)} \xrightarrow{P} \lambda$

that is $U_{(r)} \xrightarrow{P} F(\xi_\lambda, \theta)$

Hence $F^{-1}(U_{(r)}) \xrightarrow{P} \xi_\lambda$.

<u>Example</u> : <u>2.2.8</u> : For $C(\mu, 1)$ the median $\xi_{1/2} = \mu$ and thus $X([\frac{n}{2}] + 1)$ the sample median is consistent for μ.

Here $r = [\frac{n}{2}] + 1$ that is median order

$F(X_{(r)}) = U_{(r)} \longrightarrow U(0, 1)$

so $E(U_{(r)}) = \frac{r}{n+1}$ and $E(U^2_{(r)}) = \frac{r(r+1)}{(n+1)(n+2)}$

consider $P\left[|U_{(r)} - \xi_{1/2}| > \varepsilon\right]$

$$= P\left[|U_{(r)} - \frac{1}{2}| > \varepsilon\right]$$

$$< \text{M.S.E } U_{(r)} / \varepsilon^2$$

$$\leq \frac{\text{Var }(U_{(r)})}{\varepsilon^2}$$

$$= \frac{\frac{r(r+1)}{n(n+1)} - \frac{r^2}{(r+1)}}{\varepsilon^2} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty$$

This implies that $U_{(r)} \xrightarrow{P} \frac{1}{2}$

so $F^{-1}(U_{(r)}) \xrightarrow{P} F^{-1}(\frac{1}{2})$

that is $X_{(r)} \xrightarrow{P} \mu$

Hence $X_{([\frac{n}{2}]+1)} \xrightarrow{P} \mu$.

<u>2.2.8</u> : In the following method, maximum likelihood estimator is used to obtain the consistent esatimator.

Assume that X is either discrete with probability mass function $f(x, \theta)$ or it's absolutely continuous with respect to lebesgue measure with probability density function $f(x, \theta)$. We define the likelihood of the sample $(X_1, X_2, \ldots, X_n)$ as the function $l(\theta \mid \underline{x}) = \prod_{i=1}^{n} f(x_i, \theta)$ where $(X_1, X_2, \ldots, X_n)$ are fixed and θ varies over Θ. The method of maximum likelihood estimation consists of choosing (as an estimator of θ) a value $\hat{\theta}(X_1, X_2, \ldots, X_n)$ that maximizes $l(\theta \mid x)$ as a function of θ for given $\underline{X}$.

Under certain regularity conditions it can be shown that maximum likelihood estimator (mle) is consistent.

For ready reference we state the following regularity conditions (Zacks(1981)).

I) For almost all X in an interval Θ of θ including the true value θ_0, the derivatives

$$\frac{\partial}{\partial\theta} \log f_\theta(x), \quad \frac{\partial^2}{\partial\theta^2} \log f_\theta(x) \text{ and } \frac{\partial^3}{\partial\theta^3} \log f_\theta(x) \text{ exists.}$$

II) $\frac{\partial}{\partial\theta} \log f_\theta(x)$, $\frac{\partial^2}{\partial\theta^2} \log f_\theta(x)$ and $\left[\frac{\partial}{\partial\theta} \log f_\theta(x)\right]^2$ are dominated by integrable functions.

III) For every $\theta \in \Theta$, $\left| \frac{\partial^3}{\partial\theta^3} \log f_\theta(x) \right| < H(x)$.

and $E_\theta \left[H(x)\right] < K$, where K is independent of θ and is positive.

Under these regularity conditions any consistent solution of the likeliihood equation $\Sigma \frac{\partial}{\partial\theta} \log f_i(x,\theta) = 0$, provides a maximum of the likelihood with probability tending to unity as the sample size tends to infinity.

2.2.9 : The consistency of maximum likelihood est mators.

Consider n independent observations from a distribution $f(x|\theta)$ and for each n we choose the maximum likelihood estinmator $\hat{\theta}$ so that

$$\log L(x|\hat{\theta}) \geq \log L(x|\theta) \qquad (2.2.9)$$

We denote the true value of θ by θ_o. Consider the random variable $\frac{L(x|\theta)}{L(x|\theta_o)}$. By Jensen's inequality (2.1.6) for

$\theta^* \neq \theta_o$

$$E_{\theta_o}\left\{ \log \frac{L(x|\theta^*)}{L(x|\theta_o)} \right\} < \log E_{\theta_o}\left\{ \frac{L(x|\theta^*)}{L(x|\theta_o)} \right\} \qquad (2.2.10)$$

Hence $\quad E_{\theta_o}\left\{ \log \frac{L(x|\theta^*)}{L(x|\theta_o)} \right\} < 0.$

This implies that

$$E_{\theta_o}\left\{ \log L(x|\theta^*) - \log L(x|\theta_o) \right\} < 0.$$

So $E_{\theta_o}\left\{\frac{1}{n} \log L(x|\theta^*)\right\} < E_{\theta_o}\left\{\frac{1}{n} \log L(x|\theta_o)\right\} \qquad (2.2.11)$

Provided that the expectation on the R.H.S. exists. Now for any value of θ,

$$\frac{1}{n} \log L(x|\theta) = \frac{1}{n} \sum_{i=1}^{n} \log f(x_i|\theta) \text{ is the mean of a}$$

set of n independent identical random variables with expectation.

$$E_{\theta_o}\left[\log f(x|\theta) \right] = E_{\theta_o}\left[\frac{1}{n} \log L(x|\theta) \right]$$

By SLLN $\frac{1}{n} \log L(x|\theta)$ converges with probability unity to it's expectation as n increases. Thus for large n we have from (2.2.11) with probability one.

$$\frac{1}{n} \log L(x|\theta^*) < \frac{1}{n} \log L(x|\theta_0)$$

Or

$$\lim_{n\to\infty} \text{Prob.}\left\{ \log L(x|\theta^*) < \log L(x|\theta_0) \right\} = 1 \tag{2.2.12}$$

On the other hand equation (2.2.9) with $\theta = \theta_0$ given

$$\log L(x|\hat{\theta}) \geq \log L(x|\theta_0) \tag{2.2.13}$$

Since (2.2.12) and (2.2.13) are contradiction, it follows that as $n \longrightarrow \infty$, $\hat{\theta}$ does not converges to $\theta^* \neq \theta_0$ but it convonverges to θ_0 in probability. This establishes the consistency of the maximum likelihood estimators.

For example in $N(\theta, \sigma^2)$ we can show that the mle(θ, σ^2) is $(\bar{X}, \frac{S^2}{n})$. Similarly for $f(x, \mu, \sigma) = \frac{1}{\sigma} \exp\left\{1 - (x-\mu) / \sigma\right\}$ the m.l.e of (μ, σ) is $\left\{x_{(1)}, \sum_{i=1}^{n} \frac{X_{(i)} - X_{(1)}}{n-1}\right\}$.

However consider the Laplace or double exponential distribution with unknown location μ and $\sigma = 1$. This has probability density function $f(x, \sigma) = \frac{1}{2} \exp\left\{- | x - \mu|\right\}$,

$$\log l(\mu \mid \underline{x}) = - n\log 2 - \sum_{i=1}^{n} | x_i - \mu | .$$

If $n = 2m + 1$ then $\log l(\mu \mid \underline{x})$ is maximized at $\hat{\mu} = X_{(m+1)}$ but if $n = 2m$ then $\log l(\mu \mid \underline{x})$ is maximized for any value of $\mu \in (X_{(m)}, X_{(m+1)})$ and $\hat{\mu}$ is not defined uniquely. By convention we can define $\hat{\mu} = X_{([\frac{n}{2}]+1)}$ the sample median.

Similarly if we consider $b(1, \theta)$ model with $\theta \in (0,1) = \Theta$ then for $x = (0,0,\dots,0)$ or $x = (1,1,\dots,1)$ the maximum of $l(\theta / \underline{x}) = \theta^{\Sigma x_i} (1-\theta)^{n-\Sigma x_i}$ occurs at $\hat{\theta} = 0$ and $\hat{\theta} = 1$ respectively and the $\hat{\theta}$ belogs to the clouser of Θ and it does not belong to Θ. If we consider $\Theta = [0, 1]$ then this problem does not occur.

<u>2.2.10</u> : Maximum likelihood estimator need not be consistent (Basu (1957))

Let $X_1, X_2, \dots, X_n$ be i.i.d $b[1, P(\theta)]$ where

$$P(\theta) = \begin{cases} \theta & \text{if } \theta \text{ is rational} \\ 1-\theta & \text{if } \theta \text{ is irrational} \end{cases} \qquad 0<\theta<1.$$

We shall prove (I) $\bar{X}$ is m.l.e. of θ

(II) $\bar{X} \xrightarrow{P} E_\theta(X_i) = P(\theta)$

Let here $l(\theta \mid \underline{x}_n) = P(\theta)^{\Sigma x_i} (1 - P(\theta))^{n-\Sigma x_i}$

$$= \begin{cases} \theta^{n\bar{x}} (1-\theta)^{n(1-\bar{x})} & \text{if } \theta \text{ is rational.} \\ (1-\theta)^{n\bar{x}} \theta^{n(1-\bar{x})} & \text{if } \theta \text{ is irrational.} \end{cases}$$

This implies $\hat{\theta} = \bar{x}$ is m.l.e. and since X_i's are

i.i.d. $\bar{X}_n \xrightarrow{P} E(X_1)$ from WLLN.

i.e. $\bar{X}_n \xrightarrow{P} P(\theta)$

and since $P(\theta) \neq \theta \; \forall \; \theta$.

$\bar{X}_n$ is not consistent so m.l.e. need not be consistent.

2.2.11 : Inconsistency of maximum likelihood estimator

(Ferguson (1982)):

Let $\hat{\theta}_n$ denote a maximum likelihood esti~~/~~mator of θ b sed on sample of size n if $\delta(\theta) \longrightarrow 0$ sufficiently fast as $\theta \longrightarrow 1$ with probability one as $n \longrightarrow \infty$ whatever be the true value of $\theta \in [0, 1]$

Proof : The following densities on [-1, 1] provide a continuous parameteriazation between the triangular distribution (when $\theta = 0$) and the uniform distribution when $\theta = 1$ with parameter space $\Theta = [0, 1]$.

$$f(x \mid \theta) = (1-\theta)\frac{1}{\delta(\theta)}\left[1 - \frac{|x-\theta|}{\delta(\theta)}\right] I_A(x) + \frac{\theta}{2} I_{[-1,1]}(x) \qquad \ldots(2.2.14)$$

where A represents the interval $[\theta - \delta(\theta), \theta + \delta(\theta)]$, $\delta(\theta)$ is a continuous decreasing function of θ with $\delta(0) = 1$ and $0<\delta(\theta) \leq 1-\theta$, for $0<\theta<1$. It is assumed that $X_1, X_2, \ldots$ are i.i.d. observations available from one of these distributions. Above set up satisfies the following conditions.

(a) The parameter space Θ is a compact interval on the real line.

(b) The obsevations are independent identically distributed accurating to a distribution $F(x \mid \theta)$ for some $\theta \in \Theta$.

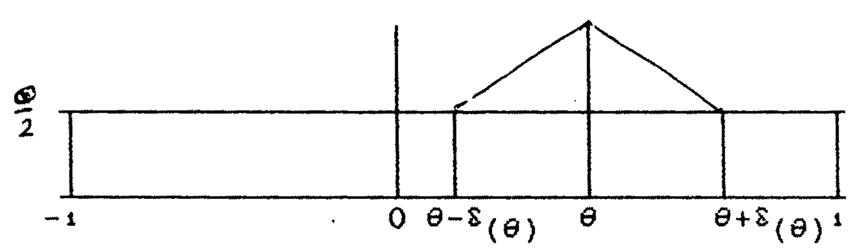
(c) Densities $f(x \mid \theta)$ with respect to some σ -finite measure (Lebesgue measure in above example) exists and are continuous in θ for all x.

(d) If $\theta \neq \theta'$ then $F(x \mid \theta)$ is not identical to $F(x \mid \theta')$ (Identifiability).

Then consider $\hat{\theta}_n$ the value of θ which maximizes the log likelihood function. That is

$$l_n(\theta) = \log \prod_{i=1}^{n} f(x_i \mid \theta) = \sum_{i=1}^{n} \log f(x_i \mid \theta) \qquad (2.2.15)$$

Since $\theta < 1$ then from (2.2.14) grap[h of density will be



$$f(x \mid \theta) \le \frac{1-\theta}{\delta(\theta)} + \frac{\theta}{2} < \frac{1}{\delta(\theta)} + \frac{1}{2} \qquad (2.2.16)$$

for each fixed positive number $\alpha < 1$

$$\max_{0 \le \theta \le \alpha} \frac{1}{n} l_n(\theta) = \frac{1}{\delta(\alpha)} + \frac{1}{2} < \infty.$$

Since $\delta(\theta)$ is decreasing we have whatever be value of θ,

$$\max_{0 \le \theta \le \alpha} \frac{1}{n} l_n(\theta) \longrightarrow \infty \quad \text{with probability one.}$$

Provided $\delta(\theta) \longrightarrow 0$ sufficiently fast as $\theta \longrightarrow 1$

Since $\hat{\theta}_n$ will be greater than α for preassigned $\alpha < 1$.

Let $M_n = \max \{X_1, X_2, \ldots, X_n\}$. Then $M_n \longrightarrow 1$ with probability one whatever be the true value of θ.

$$\begin{aligned} \max_{0 < \theta < 1} l_n(\theta) &= \Sigma \log f(x_i \mid \theta) \\ &\ge \Sigma \log f(x_i \mid M_n) \\ &\ge (n-1) \log \frac{M_n}{2} + \log \left[\frac{1-M_n}{\delta(M_n)} + \frac{M_n}{2}\right] \\ &\ge (n-1) \log \frac{M_n}{2} + \log \frac{1-M_n}{\delta(M_n)}. \end{aligned}$$

This implies that

$$\max_{0 < \theta < 1} \frac{1}{n} l_n(\theta) \ge \frac{n-1}{n} \log \frac{M_n}{2} + \frac{1}{n} \log \frac{1-M_n}{\delta(M_n)} \qquad (2.2.17)$$

So with probability one

$$\liminf_{n \to \infty} \max_{0 \le \theta \le 1} \frac{1}{n} l_n(\theta) \ge \log \frac{1}{2} + \liminf_{n \to \infty} \frac{1}{n} \log \left[\frac{1-M_n}{\delta(M_n)}\right].$$

Whatever be the value of θ, M_n converges to 1 at a certain rate, the slowest rate being for the triangular $(\theta = 0)$.

Since this distribution has smaller mass than any of the others in sufficiently small neighbourhoods of one. Thus we can shoose $\delta(\theta) \longrightarrow 0$ so fast as $\theta \longrightarrow 1$ that

$\frac{1}{n} \log \left[(1 - M_n) \mid \delta(M_n)\right]$ converges to infinity with probability one for the triangular and hence for all other possible true values of θ. This shows that $\hat{\theta}_n$ is m.l.e but not consistent for θ.

2.3 ORDER OF CONSISTENCY :

2.3.1 : Consistent estimator with order (C_n) (Akahira Takeuehi (1981)).

For any increasing sequence of positive numbers $\{C_n\}$ $(C_n \longrightarrow \infty)$ an estimator T_n is called consistent with order $\{C_n\}$ if for every $\varepsilon > 0$ and ϑ of Θ there exist a δ sufficiently large number and sufficiently small positive number L satisfying the condition

$$\overline{\lim_{n\to\infty}} \sup_{\theta:\|\theta-\vartheta\|<\delta} P_{\theta,n} \left\{C_n \| T_n - \theta \| \geq L\right\} < \varepsilon.$$

2.3.2 : Lemma : If T_n is a $\{C_n\}$ consistent estimator then T_n is consistent estimator. That is if

$$\overline{\lim_{n\to\infty}} \sup_{\theta:\|\theta-\vartheta\|<\delta} P_{\theta,n} \left\{C_n \| T_n - \theta \| \geq L\right\} \leq \varepsilon \tag{2.3.2}$$

Then we have to prove

$$\overline{\lim_{n\to\infty}} P_{\theta,n} \left\{\|T_n-\theta\|\geq\varepsilon'\right\} = 0 \text{ for } \varepsilon' \text{ any positive number.} \tag{2.3.3}$$

(2.3.2) implies that

$$\overline{\lim_{n\to\infty}}\ P_{\vartheta,n}\left\{ \| T_n - \vartheta \| > \frac{L}{C_n} \right\} < \varepsilon \tag{2.3.4}$$

As L is finite and as $n \longrightarrow \infty$, $C_n \longrightarrow \infty$ let n_o be such that

$$\frac{L}{C_n} < \varepsilon', \quad \forall \ n \geq n_o.$$

From (2.3.4) we get

$$\overline{\lim_{n\to\infty}}\ P_{\vartheta,n}\left\{ \| T_n - \vartheta \| > \varepsilon' \right\} < \varepsilon$$

Now by letting $\varepsilon > 0$ we get

$$\overline{\lim_{n\to\infty}}\ P_{\vartheta,n}\left\{ \| T_n - \vartheta \| > \varepsilon' \right\} = 0$$

Hence

$$\lim_{n\to\infty} P_{\theta,n}\left\{ \| T_n - \theta \| > \varepsilon' \right\} = 0.$$

<u>2.3.3</u> : <u>Lemma</u> : If T_n is $\{C_n\}$ consistent and

$\lim_{n\to\infty} \left[\frac{d_n}{C_n} \right] = K$, $(0<K<\infty)$ then T_n is dn consistent.

<u>Proof</u> : Consider $\lim_{n\to\infty} \sup_{\theta:\|\theta-\vartheta\|<\delta} P_{\theta,n}\left\{\| T_n - \theta \| \geq \frac{L}{d_n}\right\}$

$$= \lim_{n\to\infty} \sup_{\theta:\|\theta-\vartheta\|<\delta} P_{\theta,n}\left\{C_n \| T_n - \theta \| \geq \frac{C_n}{d_n}.L\right\}$$

$$\leq \lim_{n\to\infty} \sup_{\theta:\|\theta-\vartheta\|<\delta} P_{\theta,n}\left\{C_n \| T_n - \theta \| > \frac{K}{2}.L\right\}$$

$$\text{since } \frac{C_n}{d_n} \geq 0 \tag{2.3.5}$$

Since $\{T_n\}$ is $\{C_n\}$ consistent there exist L' such that

$$\overline{\lim_{n\to\infty}} \sup_{\theta:\|\theta-\vartheta\|<\delta} P_{\theta,n} \left\{C_n \| T_n - \theta \| > L'\right\} < \varepsilon \qquad (2.3.6)$$

Now choose L such that

$\frac{K}{2} L > L'$ with this choice the R.H.S. of (2.3.5) is

less than the L.H.S. of (2.3.6).

Hence the result.

It is obvious that if $d_n \leq C_n$ and T_n is $\{C_n\}$ consistent then T_n is $\{dn\}$ consistent.

Example : 2.3.3 : Let $X_1, X_2, \ldots, X_n$ be i.i.d. $U(0, \theta)$ and $T_n = X_{(n)}$. We know that

$$P_\theta \left\{ | T_n - \theta | > \varepsilon' \right\} = \begin{cases} (1 - \frac{\varepsilon'}{\theta})^n & \text{if } \varepsilon' < \theta \\ 0 & \text{if } \varepsilon' \geq \theta. \end{cases}$$

Hence $\overline{\lim_{n\to\infty}} \sup_{\theta:\|T_n-\vartheta\|<\delta} P_{\theta,n} \left\{C_n \| T_n - \theta \| \geq L\right\}$

$$\theta < \left[1 - \frac{L}{C_n(\vartheta+\delta)}\right]^n$$

$$= \left[1 - \frac{L}{(\vartheta+\delta).n} \frac{n}{C_n}\right]^n$$

$$= e^{-\frac{L}{\vartheta+\delta}.a} \qquad \text{where } \frac{n}{C_n} \longrightarrow a.$$

$$< \varepsilon$$

Hence $\{T_n\}$ is n consistent whenever

$$L > - \frac{(\vartheta + \delta) \log \varepsilon}{a}.$$