CHAPTER 4 .

ASYMMETRICAL FACTORIAL EXPERIMENT :-

In this chapter first we give introduction of asymmetrical factorial experiments along with its historical background. In section 4.2, we give the analysis for asymmetrical factorial experiments, especially for the pxq -factorial experiment and pxqxk -factorial experiments. In section 4.3 different methods of construction of asymmetrical factorials are given.

4.1:-INTRODUCTION :

In the previous chapter, we have considered only symmetrical factorial experiments i.e. the experiments in which the factors occur at same number of levels. However, in many situations it is unrealistic to expect the same number of levels for all factors. To remove this drawback of symmetrical factorial experiment, we consider an factorial experiment which allows different levels for different factors. Such an experiment in which different factors occur at different levels is called as a 'asymmetrical factorial experiment ' , or 'mixed factorial experiment ' [ochran and Cox (1950)]. It can be described in brief as follows-

Suppose an experiment [Das, Giri (1971)] involves n factors ; $A_1, A_2, ---, A_n$ with levels $s_1, s_2, ---, s_n$ respectively. In such experiments the total number of treatments are

$$v = s_1 \times s_2 \times --- \times s_n \quad .$$

And, such an experiment is called as

$$s_1 \times s_2 \times - - - \times s_n \text{ -factorial experiment .}$$

In the next paragraph we present the historical development of 'asymmetrical factorial ' experiments .

Yates (1935, 1937) was the first to tackle this problem. He proposed the confounded designs of the type $3^s \times 2^n$ together with the method of analysis. Further, Li (1944) suggested methods similar to that of Yates for constructing confounded designs for

$$4 \times 2 \times 2,\ 4 \times 3 \times 2,\ 4 \times 4 \times 2,\ 4 \times 3 \times 3,\ 5 \times 2 \times 2 .$$

Nair and Rao (1941, 1942, 1948) were the first to give the sufficient combinatorial conditions which lead to the construction of confounded designs. Thompson and Dick (1951) gave designs for factorial experiments involving 2 or 3 factors, derived from orthogonal latin squares. Kishen and Srivastava (1959) and Das (1960) have developed two different methods for constructing such designs. Kishen and Srivastava's approach is through the use of finite geometries while Das has given a technique of such designs by linking them with the fractional replicates of symmetrical experiments. Nishii (1981) , Bose and Iyer (1982) give 'irregular ' plans for asymmetrical factorials where estimates are balanced in some sense.

In the next section, we discuss the analysis of asymmetrical factorial experiment for $s_1 \times s_2 \times s_3$ - - - series .

4.2 . ANALYSIS OF ASYMMETRICAL FACTORIAL EXPERIMENTS

A) The p x q factorial experiments :-

Suppose there are two factors A and B at respective levels $s_1 = p$ and $s_2 = q$ . . The main effects A and B have (p-1) and (q-1) d.f. respectively. Each component of the main effect of A can be estimated seperately at each of the levels of B . thus each component of A contributes (q-1) d.f. to the AB interaction. This implies interaction AB has (p-1)(q-1)d.f. out of total d.f.

Suppose that the p x q factorial experiment is arranged in a randomised complete block design with r , replications. With slight modification the model used in the previous chapter can be rewritten as

$$Y_{ijg} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \delta_g + e_{ijg} \quad \text{-------}(4.2.1)$$

i = 1, 2, - - -, p ; j = 1, 2, - - -, g = 1, 2, - - , r .

Where,

y_{ijg} is the yield when factor A is at i th level ;B at j th level in g th replication.

δ_g : is the effect due to the g th replication.

The other terns have same appropriate meaning as we have seen earlier.

$$\bar{y}_{...} = \sum_i \sum_j \sum_g \frac{Y_{ijg}}{rpq}$$

$$\bar{y}_{i..} = \frac{1}{p} \sum_j \sum_g y_{ijg}$$

$$\bar{y}_{.j.} = \frac{1}{q} \sum_{i} \sum_{g} y_{ijg}$$

$$\bar{y}_{..g} = \frac{1}{r} \sum_{i} \sum_{j} y_{ijg}$$

$$\bar{y}_{ij.} = \frac{1}{pq} \sum_{g} y_{ijg} \quad .$$

The usual least square estimates are given by

$$\hat{\mu} = \bar{y}_{...} \quad , \quad \hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...}$$

$$\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...} \quad , \quad (\hat{\alpha B})_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

and

$$\hat{\delta}_g = \bar{y}_{..g} - \bar{y}_{...} \quad .$$

The total sum of squares corrected for mean is given as

$$\text{T.S.S.} = \sum_{i} \sum_{j} \sum_{g} y_{ijg}^2 - rpq\, \bar{y}_{...}^2 \qquad \text{--------(4.2.2)}$$

which carries $rpq - 1$ d.f. One is lost because of the linear constraint

$$\sum_{i} \sum_{j} \sum_{g} (y_{ijg} - \bar{y}_{...}) = 0 \quad .$$

The sum of squares due to A is equal to

$$S\,S\,A = (rq \sum_{i} \bar{y}_{i..}^2 - rpq\, \bar{y}_{...}^2) \qquad \text{--------(4.2.3)}$$

with (p − 1) d.f.

The S.S due to B is equal to

$$S\,S\,B = (rp \sum_{j} \bar{y}_{.j.}^2 - rpq\, \bar{y}_{...}^2) \qquad \text{--------(4.2.4)}$$

with (q − 1) d.f.

The S.S. due to replicates is

$$SSR = \left(pq \sum_{g} \bar{y}^{2}_{..g} - rpq \, \bar{y}^{2}_{...} \right) \qquad \text{--------}(4.2.5)$$

And S.S. due to interaction AB is given by

$$SSAB = r \sum_{i} \sum_{j} \bar{y}^{2}_{ij.} - rq \sum_{i} \bar{y}^{2}_{i..} - rp \sum_{j} \bar{y}^{2}_{.j.} + rpq \, \bar{y}^{2}_{...} \qquad \text{--}(4.2.6)$$

with (pq − p − q + 1) d.f. And S.S. due to error, S.S.E. can be obtained by substracting the addition of SSA, SSB, SSAB and SSR from total S.S. That is, we have the relation.

$$T.S.S. = SSA + SSB + SSAB + SSR + SSE \; .$$

And splitting up of corrosponding d.f. is

$$pqr - 1 = p-1 + q-1 + (p-1)(q-1) + r-1 + (pq-1)(r-1) \; .$$

The hypothesis of under interest are

H_0 : Interaction effect is not significant

i.e. $(\alpha B)_{ij} = 0$, for every i, j .

against,

H_1 : Interaction effect is significant.

If above null hypothesis of non significance of interaction is accepted, then we test the following hypotheses.

H_0 : Factorial effect due to factor A is absent, and

H_0 : Factorial effect due to factor B is absent.

These all hypethoses can be tested in the usual way. The ANOVA is given as below −

Table No. 4.2.1 .

Analysis of Variance For $p \times q$ -factorial In R.B.D. with r -replications .

Source of Variation	d.f.	S.S.	M.S.	F
Replicate	r - 1	SSR	MSR	
A	p - 1	SSA	MSA	MSA/MSE
B	q - 1	SSB	MSB	MSB/MSE
A B	(p-1)(q-1)	SSAB	MSAB	MSAB/MSE .
Error		SSE	MSE	
Total	rpq - 1	T.S.S.		

Also sums of squares for main effects and interaction can be obtained by forming two-way tables for each pair of factors. Consider A by B two-way table. The total S.S. among cells has (pq-1) d.f. From the marginal totals in the table we complete the sum of squares for main effect A with (p-1) d.f. and that for main effect of B with (q-1) d.f. By substraction, the sum of squares for the interaction AB is obtained and it carries (p-1) (q-1) d.f.

In the similar way the analysis can be carried in other designs.

THE $p \times q \times k$ FACTORIAL EXPERIMENT .

Suppose there are three factors A, B and C at levels p . q and $s_3 = k$ respectively, and are to be tested in all combinations. The mathematical model may be given as -

$$Y_{ijg} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \gamma_l + (\alpha\gamma)_{il} + (\beta\gamma)_{jl} + (\alpha\beta\gamma)_{ijl} + p_g + e_{ijlg} \quad \text{--(4.2.7)}$$

$i = 1, 2, ---, p$

$j = 1, 2, ---, q$

$l = 1, 2, ---, k$

$g = 1, 2, ---, r$,

and different terms have same meaning. Suppose that the experiment is conducted according to RBD with `r ' replications. In the usual way, we calculate the sums of squares due to various components and further the `ANOVA ' is given as below--

Table No. 4.2.2 .

Analysis OF Variance For $p \times q \times k$ -Factorial in RBD With `r' -replications.

Sources OF Variation	d.f.	
Replicates	$r - 1$	
A	$p - 1$	
B	$q - 1$	
C	$k - 1$	
A B	$(p-1)(q-1)$	
A C	$(p-1)(k-1)$	
B C	$(q-1)(k-1)$	
A B C	$(p-1)(q-1)(k-1)$	
Error		
Total	$rpqk - 1$	

In the usual manner we test different, [Fedrer(19)] hypotheses of significance of main effects and interactions.

If the factor A is applied at p -different levels, it may be desirable to estimate the linear, quadratic and perhaps other responses. So the p - 1 d.f. may be partitioned into p-1 individual d.f. if contrasts are meaningful. Also the 'q-1' d.f. corrosponding to effect B are partitioned if contrasts are meaningful.

Same method is extended for the cases where there are more than three factors. In general, the sums of squares for main effects are calculated directly, and those for interactions are calculated by substraction.

THE $2^n \times 3^s$ SERIES FACTORIAL EXPERIMENTS .

These types of factorial experiments are very useful. It involves two factors say A and B at levels n and s respectively. For , n= s = 1, we get 2 x 3 -factorial experiment. For, n= 1, s = 2, we get 2 x 3 x 3 factorial experiment containing 18 treatment combinations and so on. The above all experiments can be conducted in RBD with r -replications. If we wish to use large number of treatment combination, then it is desirable to use one of the incomplete block designs. Confounding in such a type of experiments has been given by Yates (1937)

In the next section we will discuss about the construction of ' asymmetrical factorial experiments '.

4.3:- CONSTRUCTION OF ASYMMETRICAL FACTORIALS :

We have discussed indetail the confounding of symmetrical factorials in previous chapter. In symmetrical factorial experiment confounding of higher order interactions can be done without losing any information on main effects. But confounding in asymmetrical factorial experiment is some what complicated. However, there are different methods of confounding of 'asymmetrical factorial experiments'. Some of these are given below.

1) Construction of balanced confounded asymmetrical designs by linking them with the fractions of suitable symmetrical factorials.

2) Confounding in asymmetrical factorial with the use of Galois field and finite Geometries. [Raktoe, Hedayat and Fedrer (1981)].

3) Confounding of asymmetrical factorial with the help of pseudofactors.

In the literature on asymmetrical factorial experiments the concept "balance" is being used. It shows the relative loss of information on any affected interaction is the same.

Bose (1947) introduced the concept of balancing in symmetrical factorial experiments.

Definition :4.3.1:- In a partially confounded symmetrical factorial experiment, if each of the $(s-1)^{k-1}$ pencils of $(s-1)$ d.f carried by the $(k-1)$ th order interaction between factors $A_{i1}, A_{i2}, ---, A_{ik}$ is confounded in r_1 replications and remains unconfounded in r_2 replications, then we say that the interaction $A_{i1}, A_{i2}, ---, A_{ik}$ has been balanced.

We note that if the interaction $A_{i1}.A_{i2} ---. A_{ik}$ is balanced, there is a uniform loss of information equal to $r_1/(r_1+r_2)$ on every degree of freedom belonging to this interaction.

Bose (1947) further defined the following :

Definition : 4.3.2 :- If each of the pencils carrying s-1 d.f. of (k-1) th order interactions, is confounded in r_1 replications and unconfounded in r_2 replications, then (k-1) th order interaction is said to be completely balanced.

The above definition of 'balancing in factorial experiments' due to Bose (1947) fails when the block size is not a prime or not a prime power. To remove draw back of above definition, Shah (1958) provided a definition of 'complete balance '.

Definition 4.3.3 :-(Balanced Factorial Experiment)

A factorial experiment will be called a balanced factorial experiment (BFE) if the following conditions are satisfied.

1) Each of the treatment combination is replicated the same number of times, say r,

2) Each of the block is of the same size, say, k.

3) Estimates of contrasts belonging to different interactions are uncorrelated with each other.

4) Complete balance is achieved over each of the interactions.

We discuss below a method of construction of balanced confounded asymmetrical factorial designs by linking them with the fractions of suitable symmetrical factorials. This method is first given by Kishen and Srivastava (1959) and then by Das(1960) And it appears that the method of Das is more general than the method due to Kishen and Srivastava.

4.3.1 CONSTRUCTION OF BALANCED CONFOUNDED ASYMMETRICAL FACTORIALS

Consider an experiment with n factors $A_1, A_2, --, A_n$ such that factor A_i, is at s_i levels (i = 1, 2, - - ,n) and all s_i's are not equal. We will have

$$v = s_1 \times s_2 \times --- \times s_n$$

total number of treatment combinations. Let R denote the size of block. In this method of combination it is required that

$\frac{v}{R} = N$, say is either a prime or a prime power. Let $N = s^k$ where, s is a prime and k any integer.

Each of the n factors which has s levels, is called a 'real' factor. The real factors are denoted by A, B, C, - - etc. or by A_1, A_2, A_3 - - - etc. And each of other factors who is not at s levels is called a 'factor of asymmetry'. These are denoted by X, Y, Z or by X_1, X_2, X_3 etc. The technique of construction [Das(1960)] consists of converting the asymmetrical factorial to a suitable fraction of corresponding symmetrical factorial, by denoting the levels of each factor of asymmetry by the combinations of a requiste number of factors each at s levels. These latter factors are called as 'pseudofactors' corrosponding to that factor of asymmetry. The levels of each of real and pseuofactors are denoted by the elements of GF(s). The number of pseudofactors 'n_i' corrosponding to a factor of asymmetry is determined from

$$s^{n_i - 1} < s_i < s^{n_i} ,$$

Where, s_i denotes the levels of factor A_i . If $s_i < s$, then any s_i of s elements of GF(s) are used to denote the levels of A_i . Then any set of s_i combinations of the s^{n_i} factorials is used to denote the levels of A_i and remaining combinations $s^{n_i} - s_i$ are omitted and the above design becomes a fraction of, s^m where

$$m = to + \sum_i n_i$$

to = number of real factors

and $\sum_i n_i$ = number of pseudofactors

corrosponding to all the factors of asymmetry .

We know that the number of blocks per replication in the asymmetrical factorial is s^k . Therefore the corrosponding symmetrical factorial is to be split into s^k blocks by confounding a suitable set of interactions. The treatment combinations,which containing those combinations of the pseudo factors are not used for desigination of levels of factors of asymmetry, are not to be taken as the block contents. And interaction involving only pseudo factors are not confounded as this lead to confounding of main effects of the corrosponding factors of asymmetry. Similarly, if an interaction involving only real factors is confounded, it will be confounded completely as in symmetrical factorials.

Let I_s , an interaction of corrosponding symmetrical factorial s^m is confounded which leads to the confounding of some interaction say, I_A of the asymmetrical factorial and we say that interaction I_s corrosponds to the interaction I_s. When I_s does not contain any pseudofactors , I_s and I_A are identical.

When I_s contains one or more pseudofactors corrosponding to one factor of asymmetry, say X , then the corresponding I_A is obtained from I_s by replacing the set of pseudofactors in it by X . If I_s contains pseudofactors corrosponding to two factors of asymmetry, say X and Y, then I_A is obtained from I_s by replacing the set of pseudofactors corrosponding to y by y .

The same procedure is continued to obtain I_A from I_s when there are pseudofactors in I_s corrosponding to more than two factors of asymmetry. The real factors in I_s remain as they are in I_A.

Example:-4.2.1 We consider the problem of construction of 3x2x2 factorial in the blocks of size six.Since 12/6=2, hence the corrosponding symmetrical factorial is 2^4 with factors X_1, X_2, A and B . X_1 and X_2 are the factors of asymmetrical where as A and B are real factors.

We obtain the first replication of the corrosponding symmetrical factorial by confounding $X_1.X_2.AB$. We take the defining contrasts as

$$I = X_1 = X_2 = X_1 X_2$$

and 2^4 combinations are divided into four groups. To get the fraction, the combination 11 of the factors X_1 and X_2 is omitted.

So the defining contrasts for the fraction are

$$I = X_1 = X_2 = X_1 X_2$$

The aliases of $X_1 X_2 AB$ are as below

$$X_1 X_2 AB = X_2 AB = X_1 AB = AB \ .$$

Therefore, the interaction confounded in the asymmetrical factorial are

XAB, XAB, XAB, AB .

So a balanced design is obtained by confounding the three interactions viz. $X_1 AB$, $X_2 AB$ and $X_1 X_2 AB$ each of which corresponds to XAB, in three replication. And AB is confounded due to fractionation.

The plan is given as below -

Table No. 4.3.1

Plan Of The Confounded Asymmetrical Factorial 3x2x2 In Six Plot Block .

Confounded Interaction	Replication 1 $X_1 AB$		Replication 2 $X_2 AB$		Replication 3 $X_1 X_2 AB$	
	Block I	Block II	Block I	Block II	Block I	Block II
	0000	0001	0000	0001	0000	0001
	0011	0010	0011	0010	0011	0010
	0111	0110	1011	1010	0101	0100
	0100	0101	1000	1001	0110	0111
	1010	1011	0101	0100	1010	1011
	1001	1000	0110	0111	1001	1000

By recoding the levels of X denoting 00 by 0, 01 by 1 and 10 by 2, the design is converted to original design. This is given as below -

Table No. 4.3.2

Replication 1		Replication 2		Replication 3	
Block I	Block II	Block I	Block II	Block I	Block II
000	001	000	001	000	001
011	010	011	010	011	010
111	110	211	210	101	100
100	101	200	201	110	111
210	211	101	100	210	211
201	200	110	111	201	200

4.3.2 CONSTRUCTION OF ASYMMETRICAL FACTORIALS WITH THE HELP OF FINITE GEOMETRIES .

White and Hultguist (1965) extended the use of finite fields in the construction of asymmetrical plans. They define the addition and multiplication of elements from distinct finite fields after mapping them on a finite cummulative subring containing subrings ismorplic to each of the fields under consideration. Then they applied the standard procedure of constructing confounded symmetrical factorial experiments. It is as below

Consider an asymmetric factorial experiment with n factors $A_1, A_2, ---, A_n$, i th factor A_i, being at s_i, levels. And suppose we are interested in constructing a confounded asymmetrical experiment $s_1 \times s_2 \times --- \times s_n$ in s_1 blocks, in each replication and let s_1 is a prime.

And further, let

$$s_1 > s_2 > - - - > s_n .$$

For x, primitive of $GF(s_1)$, the elements of $GF(s_1)$ can be enumerated as

$$\alpha_0 = 0, \alpha_1 = x, \alpha_2 = x^2, - - -, \alpha_{s_1} = x^{s_1 - 1} = 1 .$$

Let us identify these elements with the s_1 levels of factor A_1. And the s_i levels of factor A_i can be selected as any s_i elements of the elements of $GF(s_1)$. In connection with this Kishen and Srivastava (1959a,b) described a very nice way of constructing a polynomial over $GF(s_1)$ that takes s_i specified values. Due to this polynomial we can restrict the levels of A_i to any s_i elements of $GF(s_1)$ in an arbitrary manner. After suitable choosing the levels of the n factors, let the $s_1 . s_2 . - - - . s_n$ treatment combinations be denoted by $(x_1, x_2, - - -, x_n)$, where x_i is takes the s_i suitably selected elements of $GF(s_1)$.

Now to confound a k -factor interaction involving F_1, we form s_1 blocks acccrding to the s_1 flats of the pencil.

$$x_1 + (a_{i2} x_{12} + - - - - + a_{ik-1} x_{ik-1}) = \alpha \qquad ----(4.3.2.1)$$

$$\alpha \in FG(s_1) .$$

$$a_{ir} \in GF(s_1), \quad r = 2, 3, - - -, k - 1 .$$

In the plan, the interaction of the factors $F_{i1}, F_{i2}, --, F_{ik-1}$ carried by the pencil (4.3.2.1) is intensionally confounded. But the main effect of F_1 and all the interaction of F_1 with $F_{i2}, F_{i3}, ---, F_{ik-1}$ will automatically unintensionally get confounded. When there are at least two factors at s_1 levels each, no main effect will be partially confounded.

Example:4.3.2.1 Consider a $3^2 \times 2$ experiment. The element of GF(3) are 0, 1, 2. Total number of treatment combination 18 let these are denoted by (x_1, x_2, x_3), $x_1, x_2 = 0, 1, 2$ & $x_3 = 0, 1$

We can obtain balance in four replications, by confounding the four pencils, one in each replication.

$$x_1 + x_2 + x_3 = 0, 1, 2$$

$$x_1 + x_2 + 2x_3 = 0, 1, 2$$

$$x_1 + 2x_2 + x_3 = 0, 1, 2$$

$$x_1 + 2x_2 + 2x_3 = 0, 1, 2 .$$

4.3.3 CONFOUNDING WITH THE HELP OF PSEUDOFACTORS

Consider an asymmetrical factorial experiment $t^m \times s^n$, Where the levels of factors are different powers of the same prime. i.e. $t = p^{\alpha}$ and $s = p^{\beta}$, p -being a prime number and , B are positive integers. The t -levels of a factor can be identified with all treatment combinations of pseudofactors, and the s levels of other factor can be identified with all treatment combinations of B pseudofactors. Thus our original experiment $t^m \times s^n$ can be converted into $p^{m\alpha+n\beta}$ and treated as symmetrical in $m\alpha+n\beta$ pseudofactors each at p-levels. Then using the well know techniques of confounding for symmetrical experiment, confounding can be done. To save main effects of asymmetrical factorial experiment, only interactions containing pseudofactors are not confounded.

As an example, let us consider 4×2^2 -factorial experiment with 3 factors A, B and C at levels 4, 2 and 2 respectively. Let levels of A are denoted as 0, 1, 2, 3 and of B and C 0, 1 respectively. Let us identify the four levels 0, 1, 2, 3 of factor A by the treatment combinations 00, 01, 10, 11 of a factorial experiment 2^2 with two pseudofactors D & E each at 2 levels. So the original problem 4×2^2 is converted into 2^4, symmetrical factorial. Now suppose BCDE is confounded with blocks. The key block is constituted by the solutions of the equation

$$x_2 + x_3 + x_4 + x_5 = 0 \qquad \text{----------(4.3.3.1)}$$

And it contains

(0000, 1000, 0011, 0101, 0110, 1010, 1101, 1110)

Another block is obtained from the key block. Hence the complete plan is given as below

Table No 4.3.3.1

Block	Contents of Blocks
1	(0000, 1000, 0011, 0110, 1011, 1101, 1110, 0101)
2	(0001, 1001, 0010, 0111, 1010, 1100, 1111, 0100)

Resubstituting 00 as 0, 01 as 1, 10 as 2 and 11 as 3, the above plan can be rewritten for the original experiments.

Table No. 4.3.3.2

Block	Contents of Block
1	(000, 200, 011, 110, 211, 301, 310, 101)
2	(001, 201, 010, 111, 210, 300, 311, 100)

We have reached the end of this dissertation and since the objectives were limited we could not cover all the concepts arising in factorial experiment theory. More concepts and construction methods can be found in the published literature.