Chapter-4.

BAYESIAN APPROACH TO SOME PROBLEMS IN LIFE TESTING AND RELIABILITY ESTIMATION.

4.1 Introduction:

When X:the life time of a system, has exponential distribution with an unknown *fixed* parameter related inferance abuut θ and also about the reliability functions is well known under classical theory of estimation.The parameter/s θ need not be fixed,generally the case in reliability estimation. Some times prior information about θ is known.In chapter-2 we studied how classical (sampling) theory fails when θ is not *fixed.* Also we have studied its incapability of incorporating prior information about parameter/s θ while doing reliability estimation and some other shortcomings. Bayesian inferance is one of the method which overtake these problems and superior to classical theory. We have also discussed suitability of Bayesian approach in reliability estimation in previous chapter.

In this chapter we consider Bayesian approach to some problems in reliability estimation credited to Basu and Ebrahimi (1989). Section-2 deals with reliability estimation for different models (k-out-of-n system, sterss-strength model) under squared error loss function assuming inverted gamma as a prior. Asymmetric loss function in section-3. Parametric empirical Bayes methods discussed in section-4 and in last section some concluding remarks.

4.2 Reliability Estimation Under Squared Error Loss Function For Different Models :

Model-I: Let T denote life time of a system with distribution function F(.), then the reliability function R_1 is the probability that the system will be in operating condition and functioning satisfactorily at time t. That is,

$$R_1 = P(T>t). \qquad \ldots(4.1).$$

Model=II: Let X and Y be two random variables with cumulative distribution function F(.) and G(.) respectively. Suppose Y be random strength of a component subject to random stress x. Naturally, component fails if at any moment the applied stress or load is greater than its strength or resistance (that is, it fails if X>Y). Hence reliability of the component in this case is given by

$$R_2 = P(X < Y). \qquad \ldots(4.2).$$

Such situations mostly occurs in structural and aircraft industries. As an example :A solid propellent rocket engine is successefully fired provided the chamber pressure (say X) generated by ignition stays below the brust pressure (say Y) of rocket chamber.

4.2.1 Different Systems:

A system is called simple if it consists of a single component, otherwise it is called a complex system. k-out-of-p system is one of the complex system with total p components. This functions if at least k of its components are functioning. As an example : Airoplane consists of three engines and is functioning if at least two of its are in good condition. This is 2-out-of-3 system.

Series system and parallel systems are perticular case of k-out-of-p system.

For k=p, it is p-out-of-p system and this will function if all components are functioning simultaniously-called "series system".

For k=1, it is 1-out-of-p system and this will function if at least one out of p is in working condition - called 'parallel system".

4.2.2 Estimation Of Reliability Function For *Model-I*:

Suppose life time of the system (say X) at which it fails follows exponential distribution. Conditional p.d.f. of r.v.X for given θ is given by

$$f_X(x/\theta) = \begin{cases} 1/\theta.\exp[-x/\theta]; & \text{if } x,\theta>0 \\ 0 \quad ; & \text{otherwise.} \end{cases} \quad \ldots(4.3).$$

where, Θ is a r.v. whose prior distribution is known to be inverted gamma with parameters (α,ν), denoted by $\Theta \to IG(\alpha,\nu)$ and is given by

$$g_\Theta(\theta) = \begin{cases} \alpha^\nu/\Gamma\nu.\exp[-\alpha/\theta].(1/\theta)^{\nu+1}; & \text{if } \theta,\alpha,\nu>0 \\ 0 & ; \text{otherwise.} \end{cases} \quad \ldots(4.4).$$

Let $\underset{\sim}{X} = (X_1,X_2,\ldots,X_n)$ be a random sample from (4.3), hence, likelihood of $\underset{\sim}{X}$ on conditioning to Θ is given by

$$L(\underset{\sim}{X}/\theta) = (1/\theta)^n . \exp\{-T/\theta\} \qquad \ldots(4.5).$$

where, $T = \sum_{i=1}^{n} X_i$.

Now, conditional distribution of Θ for given $\underset{\sim}{X}$ called posterior distribution of Θ using Bayes' theorem is

$$g_{\Theta}(\theta/\underset{\sim}{X}) = \frac{f(\underset{\sim}{X}/\theta).g(\theta)}{\int_{R_\theta} f(\underset{\sim}{X}/\theta).g(\theta).d\theta}$$

$$= \frac{(1/\theta)^n .\exp\{-\Sigma X_i/\theta\}\alpha^{\nu}/\Gamma\nu.\exp\{-\alpha/\theta\}.(1/\theta)^{\nu+1}}{\int_0^{\infty}(1/\theta)^n .\exp\{-\Sigma X_i/\theta\}.\alpha^{\nu}/\Gamma\nu.\exp\{-\alpha/\theta\}.(1/\theta)^{\nu+1}.d\theta}$$

$$= \frac{(\alpha+T)^{n+\nu} \quad (1/\theta)^{(n+\nu+1)} .\exp\{-(T+\alpha)/\theta\}}{\Gamma(n+\nu) \int^{\infty}(1/\Gamma(\nu+n).(\alpha+T)^{(n+\nu)}.. (1/\theta)^{(n+\nu+1)} .\exp\{-(T+\alpha)/\theta\}.d\theta}$$

$$= \frac{(\alpha+T)^{n+\nu}}{\Gamma(n+\nu)}.\exp\{-(T+\alpha)\}.(1/\theta)^{(n+\nu+1)} \qquad \ldots(4.6).$$

therefore, $(\Theta/\underset{\sim}{X}) \rightarrow IG(\ \alpha+T;\ n+\nu\ )$.

Under squared error loss function, Bayes estimate of mean life θ and its variance respectively given by

$$\hat{\theta}_B = E(\Theta/\underset{\sim}{X})$$

$$= \int_0^{\infty} \theta . g_{\Theta}(\theta/\underset{\sim}{X}) . d\theta$$

$$= \int_0^{\infty} \theta \frac{(\alpha+T)^{n+\nu}}{\Gamma(n+\nu)} . \exp\{-(T+\alpha)\} . (1/\theta)^{(n+\nu+1)} . d\theta$$

$$= (\alpha+T)/(n+\nu-1) \qquad \ldots(4.7).$$

and,

$$Var(\hat{\theta}_B) = E(\theta^2/\underset{\sim}{X}) - [E(\theta/\underset{\sim}{X})]^2$$

$$= (\alpha+T)^2/\{(n+\nu-1)^2(\alpha+\nu-2)\}; \qquad (n+\nu)>2, \qquad \ldots(4.8).$$

(since, $E(\theta^2/\underset{\sim}{X}) = (\alpha+T)^2/(n+\nu-1)(n+\nu-2)$).

Estimation Of Reliability Function $R_1(t)$ *Under Squared Error Loss Function:*

$$R_1(t) = P(X > t/\theta)$$

$$= \exp(-t/\theta) \qquad \ldots(4.9).$$

Bayes estimate of $R_1(t)$, under squared error loss function is expected value of $R_1(t)$ under posterior distribution of Θ, given by

$$[\hat{R}_1(t)]_B = \int_0^{\infty} \exp(-t/\theta) g_{\Theta}(\theta/\underset{\sim}{X}) . d\theta$$

$$= \int_0^{\infty} \exp(-t/\theta) . \frac{(\alpha+T)^{n+\nu}}{\Gamma(n+\nu)} . \exp\{-(T+\alpha)\} . (1/\theta)^{(n+\nu+1)} . d\theta$$

$$= [1+t/(\alpha+T)]^{-(n+\nu)} \qquad \ldots(4.10).$$

and,

$$Var([\hat{R}_1(t)]_B) = E[R_1^2(t)/\underset{\sim}{X}] - \{E[R_1(t)/\underset{\sim}{X}]\}^2$$

$$= [1+2t/(\alpha+T)]^{-(n+\nu)} - [1+t/(\alpha+T)]^{-2(n+\nu)} \qquad \ldots(4.11).$$

(since, $E[R_1^2(t)/\underset{\sim}{X}] = \int^{\infty} \exp(-2t/\theta) . \dfrac{(\alpha+T)^{n+\nu}}{\Gamma(n+\nu)} . \exp\{-(T+\alpha)\} . (1/\theta)^{(n+\nu+1)} . d\theta$

$= [1+2t/(\alpha+T)]^{-(n+\nu)}$).

Estimation Of Reliability Function For k-out-of-p System:

Consider the complex system with p distinct units. Let X_i denote life time (failure time) of the i^{th} component. Suppose, conditional distribution of life time of i^{th} unit is given as

$f_{X_i}(x_i/\theta_i) = (1/\theta_i).\exp[-x_i/\theta_i]; x_i, \theta_i > 0$ for i = 1,2,...,p.

...(4.12).

Let, X: life time of k-out-of-p system. Reliability function of the system at time t is given by

$R_s(t) = P(\ X>t\ )$

=P(k or more components work at time greater than t)

$$= \sum_{j=k}^{p} \sum_{\alpha_i} \prod_{i=1}^{j} \bar{F}_{\alpha_i}(t) \prod_{i=j+1}^{p} F_{\alpha_i}(t) \qquad \ldots(4.13).$$

where, $\bar{F}_{\alpha_i}(t) = 1-F_{\alpha_i}(t)$

$=\exp\{-t/\theta_{\alpha_i}\}$: reliability function of α_i^{th} unit;

the sum $\sum_{\alpha_i}$ is over all P_j distinct combinations of the integers {1,2,...,P} taken j at a time such that exactly j of the X_i's are greater than t and hence remaining (P-j) X_i's are less than or equal to t); j≥k.

Now, one can increase the complexity of above system by assuming prior distributions of parameters θ_i's, for i =

$1,2,\ldots,p$. In the Bayesian framework we assume priorily that $\theta_1,\theta_2,\ldots,\theta_p$ are independent with $\theta_i \rightarrow IG(\alpha_i,\nu_i)$, for $i = 1,2,\ldots,p$. Let $(X_{ij};\ j = 1,2,\ldots,n_i, i = 1,2,\ldots,p)$ be p independent random samples of sizes $n_1,n_2,\ldots,n_p$ respectively. The i^{th} sample is from following population.

$$f_X(x/\theta_i) = \begin{cases} 1/\theta_i.\exp[-x/\theta_i]; & \text{if } x,\theta_i>0 \text{ for } i = 1,2,\ldots,p \\ 0 \quad ; & \text{otherwise.} \end{cases} \qquad \ldots(4.14).$$

Define statistic $T_i = \sum_{j=1}^{n_i} X_{ij}; \text{for } i=1,2,\ldots,p.$ Note that $(T_1,T_2,\ldots,T_p)$ is sufficient statistics for $(\theta_1,\theta_2,\ldots,\theta_p)$. Also note, posterior distribution of $\theta_i \rightarrow IG(\alpha_i+T_i\ ,\ n_i+\nu_i)$; for $i = 1,2,\ldots,p$ (using (3.6)). Thus under squared error loss functionfunction, Bayes estimate of reliability function for i^{th} unit is given by

$$(\hat{\bar{F}}_i(t))_B = [1+t/(\alpha_i+T_i)]^{-(n_i+\nu_i)} \qquad \ldots(4.15).$$

(using (3.10)). Hence, Bayesian estimate of $R_s(t)$ is given by

$$(\hat{R}_s(t))_B = \sum_{j=k}^{p} \sum_{\alpha_i} \prod_{i=1}^{j} (\hat{\bar{F}}_{\alpha_i}(t))_B \prod_{i=j+1}^{p} (\hat{F}_{\alpha_i}(t))_B$$
$$= \sum_{j=k}^{p} \sum_{\alpha_i} \prod_{i=1}^{j} [1+t/(\alpha_{\alpha_i}+T_{\alpha_i})]^{-(n_{\alpha_i}+\nu_{\alpha_i})}$$
$$* \prod_{i=j+1}^{p} \{1-[1+t/(\alpha_{\alpha_i}+T_{\alpha_i})]^{-(n_{\alpha_i}+\nu_{\alpha_i})}\} \qquad \ldots(4.16).$$

Remark-I:

Since series and parallel systems are perticular case of k-out-of-p system; reliability function for series system is given by

$$R_4(t) = \prod_{i=1}^{p} \bar{F}_i(t) \qquad \ldots(4.17).$$

and its Bayesian estimator is given by

$$(\hat{R}_4(t))_B = \prod_{i=1}^{p} [1+t/(\alpha_i+T_i)]^{-(n_i+\nu_i)} \qquad \ldots(4.18).$$

(using (4.13) & (4.16) respectively).

Similerly, reliability function for parallel system is given by

$$R_5(t) = 1 - \prod_{i=1}^{p} (1-\bar{F}_i(t)) \qquad \ldots(4.19).$$

and its Bayesian estimator is given by

$$(\hat{R}_5(t))_B = 1 - \prod_{i=1}^{p} \left\{ 1-[1+t/(\alpha_i+T_i)]^{-(n_i+\nu_i)} \right\} \qquad \ldots(4.20).$$

Remark II:

If X_i's are independant and identically distributed (i.i.d.) with p.d.f. as given by (4.3) reliability function of k-out-of-p system is given by

$$R_\sigma(t) = \sum_{j=k}^{p} \binom{p}{j} (\bar{F}(t))^{j} (1-\bar{F}(t))^{p-j} \, ;$$

which can be rewritten as

$$R_\sigma(t) = \sum_{j=k}^{p} \sum_{s=0}^{p-j} (-1)^{-1} \binom{p}{j} \cdot \binom{p-j}{s} (\bar{F}(t))^{j} (\bar{F}(t))^{s} \, ;$$

$$R_\sigma(t) = \sum_{j=k}^{p} \sum_{s=0}^{p-j} (-1)^{-1} \binom{p}{j} \cdot \binom{p-j}{s} . \exp[-tj/\theta] . \exp[-ts/\theta]$$

$$R_\sigma(t) = \sum_{j=k}^{p} \sum_{s=0}^{p-j} (-1)^{-1} \binom{p}{j} \cdot \binom{p-j}{s} \exp[-t(j+s)/\theta] \qquad \ldots(4.21)$$

Now, if $(X_1, X_2, \ldots, X_n)$ be a random sample of size n from (4.3) then Bayesian estimate of $R_\sigma(t)$ is given by

$$(\hat{R}_{\sigma}(t))_B = \sum_{j=k}^{p} \sum_{s=0}^{p-j} (-1)^{-1} \binom{p}{j} \cdot \binom{p-j}{s} \cdot [1 + t(j+s)/(\alpha+T)]^{-(n+\nu)}$$

...(4.22)

where, $T = \sum_{i=1}^{n} X_i$. (using (4.6) and (4.10)).

4.2.3 Estimation Of Reliability Function For *Model-II*:

Assume that X: random stress, and Y: random strength following exponential distribution with parameters θ_1 and θ_2 respectively; where $\theta_i \rightarrow IG(\alpha_i, \nu_i)$, for i = 1,2.Data available is of the form $(X_1, X_2, \ldots, X_{n1}; Y_1, Y_2, \ldots, Y_{n2})$.

Problem is to estimate reliability function given by

$$\begin{aligned} R_2 &= P(X < Y) \\ &= \theta_2/(\theta_1+\theta_2) \\ &= \lambda/(1+\lambda) \end{aligned} \qquad \ldots(4.23).$$

$$\text{for, } P(X < Y/\theta_1, \theta_2) = \int_0^{\infty} P(x<Y/\theta_2).dF(x/\theta_1)$$

$$\begin{aligned} &= \int_0^{\infty} \exp[-x/\theta_2].(1/\theta_1).\exp[-x/\theta_1].dx. \\ &= \int_0^{\infty} (1/\theta_1).\exp[-x(1/\theta_1+1/\theta_2].dx. \\ &= \theta_2/(\theta_1+\theta_2) \\ &= \lambda/(1+\lambda) \end{aligned} \qquad \text{(where, } \lambda = \theta_2/\theta_1\text{)}.$$

Theorem 4.2.1 : The posterior density of λ is given by

$$g(\lambda/T_1,T_2) = [\beta(n_1+\nu_1;\ n_2+\nu_2)]^{-1} \frac{u^{(n_2+\nu_2)}.\lambda^{(n_1+\nu_1-1)}}{(\lambda+u)^{(n_1+\nu_1+n_2+\nu_2)}} \ ; \quad \ldots(4.24).$$

Where, $T_1 = \sum_{i=1}^{n_1} X_i$; $T_2 = \sum_{i=1}^{n_2} X_i$ and $u = (T_1+\alpha_1)/(T_2+\alpha_2)$.

Proof: Note that

$$g(\theta_1,\theta_2/T_1,T_2) = \frac{g(\theta_1\theta_2,T_1,T2)}{f(T_1,T_2)}$$

$$= \frac{g(T_1,\theta_1).g(T_2,\theta_2)}{f(T_1).f(T_2)}$$

(since (T_1,θ_1) and (T_2,θ_2) are indipendent).

$$= \frac{f(T_1/\theta_1).g(\theta_1).f(T_2/\theta_2).g(\theta_2)}{[\int_{R_{\theta_1}} f(T_1/\theta_1).g(\theta_1).d\theta_1].[\int_{R_{\theta_2}} f(T_2/\theta_2).g(\theta_2).d\theta_2]}$$

...(4.25).

Consider,

$$\int_{R_{\theta_1}} f(T_1/\theta_1).g(\theta_1).d\theta_1$$

$$= \int_0^\infty (1/\theta_1)^{n_1}(\Gamma n_1)^{-1} \exp\{-T_1/\theta_1\}(T_1)^{n_1-1} (\alpha_1)^{\nu_1}(\Gamma\nu_1)^{-1} \exp\{-\alpha_1/\theta_1\}(1/\theta_1)^{\nu_1+1}.d\theta_1$$

$$= (T_1)^{n_1-1}.(\alpha_1)^{\nu_1}.(\Gamma\nu_1)^{-1}.(\Gamma n_1)^{-1}\int_0^\infty \exp\{-(\alpha_1+T_1)/\theta_1\}(1/\theta_1)^{(n_1+\nu_1+1)}.d\theta_1$$

$$= (T_1)^{n_1-1}.(\alpha_1)^{\nu_1}.(\Gamma\nu_1)^{-1}.(\Gamma n_1)^{-1}.(\Gamma(n+\nu)).(T_1+\alpha_1)^{-(n_1+\nu_1)}$$

(4.26).

Similerly,

$$\int_{R_{\theta_1}} f(T_1/\theta_1).g(\theta_1).d\theta_1$$

$$= (T_2)^{n_2-1}.(\alpha_2)^{\nu_2}.(\Gamma\nu_2)^{-1}.(\Gamma n_2)^{-1}.(\Gamma(n_2+\nu_2)).(T_2+\alpha_2)^{-(n_2+\nu_2)}$$

...(4.27).

(4.27).

Using (4.26) and (4.27) in equation (4.25) we get,

$$g(\theta_1,\theta_2/T_1,T_2) = (\Gamma(n_1+\nu_1))^{-1}.(T_1+\alpha_1)^{(n_1+\nu_1)}(\Gamma(n_2+\nu_2))^{-1}.(T_2+\alpha_2)^{(n_2+\nu_2)}$$
$$.(\theta_2/\theta_1)^{(n_1+\nu_1+1)}.(1/\theta_2)^{(n_1+\nu_1+n_2+\nu_2+2)}.\exp\{-[\theta_2/\theta_1(T_1+\alpha_1)+(T_2+\alpha_2)]/\theta_2\}.$$

(4.28).

Let, $\lambda = (\theta_2/\theta_1)$ and $\theta = \theta_2$

$J:(\theta_1,\theta_2)\not\to(\lambda,\theta) = \theta/\lambda^2$.

Therefore,

$$g(\lambda,\theta/T_1,T_2) = (\Gamma(n_1+\nu_1))^{-1}.(T_1+\alpha_1)^{(n_1+\nu_1)}(\Gamma(n_2+\nu_2))^{-1}.(T_2+\alpha_2)^{(n_2+\nu_2)}$$
$$.(\lambda)^{(n_1+\nu_1-1)}.(1/\theta)^{(n_1+\nu_1+n_2+\nu_2+1)}.\exp\{-[\lambda(T_1+\alpha_1)+(T_2+\alpha_2)]/\theta\}$$

(4.29).

$$g(\lambda,\theta/T_1,T_2) = (\Gamma(n_1+\nu_1))^{-1}.(T_1+\alpha_1)^{(n_1+\nu_1)}(\Gamma(n_2+\nu_2))^{-1}.(T_2+\alpha_2)^{(n_2+\nu_2)}$$
$$.(\lambda)^{(n_1+\nu_1-1)}.(1/\theta)^{(n_1+\nu_1+n_2+\nu_2+1)}.\exp\{-[\lambda(T_1+\alpha_1)+(T_2+\alpha_2)]/\theta\}$$

(4.29).

$$g(\lambda/T_1,T_2) = \int_{R_\theta} g(\lambda,\theta/T_1,T_2)\, d\theta.$$

$$= (\Gamma(n_1+\nu_1))^{-1}.(T_1+\alpha_1)^{(n_1+\nu_1)}(\Gamma(n_2+\nu_2))^{-1}.(T_2+\alpha_2)^{(n_2+\nu_2)}$$
$$.(\lambda)^{(n_1+\nu_1-1)}\int_0^\infty (1/\theta)^{(n_1+\nu_1+n_2+\nu_2+1)}.\exp\{-[\lambda(T_1+\alpha_1)+(T_2+\alpha_2)]/\theta\}d\theta$$

$$= (\Gamma(n_1+\nu_1))^{-1}.(T_1+\alpha_1)^{(n_1+\nu_1)}(\Gamma(n_2+\nu_2))^{-1}.(T_2+\alpha_2)^{(n_2+\nu_2)}$$
$$.(\lambda)^{(n_1+\nu_1-1)}.\Gamma(n_1+\nu_1+n_2+\nu_2+).[\lambda(T_1+\alpha_1)+(T_2+\alpha_2)]^{-(n_1+\nu_1+n_2+\nu_2)}$$

Let $u = (T_1+\alpha_1)/(T_2+\alpha_2)$ therefore above equation can be written as

$$g(\lambda/T_1,T_2) = [\beta(n_1+\nu_1;\ n_2+\nu_2)]^{-1}\frac{u^{(n_2+\nu_2)}.\lambda^{(n_1+\nu_1-1)}}{(\lambda+u)^{(n_1+\nu_1+n_2+\nu_2)}};$$

hence result.

Bayes Estimate Of R_2 Under Squared Error Loss Function :

Using above theorem Bayes estimate of R_2 is gives by

$$(\hat{R}_2)_B = E_{\lambda/(T_1,T_2)}[\lambda/(1+\lambda)]$$

$$= [\beta(n_1+\nu_1;\ n_2+\nu_2)]^{-1}(u)^{(n_2+\nu_2)}.\int_0^{\infty}[\lambda/(1+\lambda)](\lambda)^{(n_1+\nu_1-1)}(\lambda+u)^{(n_1+\nu_1+n_2+\nu_2)}.d\lambda.$$

4.3 Parametric Empirical Bayes Method :

In an empirical Bayes analysis the prior parameters are unknown and need to be estimated from the data. Sometimes prior itself is unknown which is to be estimated from data, such methods are also called as an empirical Bayes methods.

Here we consider parametric empirical Bayes procedure for the estimation of parameters of failure processes. These systems are assumed to be repairable with negligible repair time. Suppose successive failures of a physical systems, (for example : computers, aeroplanes, vehicals, etc.), follows a homogenious Poisson process with rate λ. Therefore, times between failures, after repairing, follows exponential with mean time to failure is equal to (1λ). We assume λ to be random with a prior distribution chosen to reflect our prior knowledge about λ.

Now consider N similar systems following N independant homogenious Poisson processes with parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$. These are subject to failure at random points in time having repair facility with negligible repair time. Even though $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ are unobservable, assume that it is a random sample from a gamma prior with parameters (α, β); where α, β are unknown. Let

$$\underset{\sim}{t} = (t_{11}, t_{12}, \ldots, t_{1n_1}; t_{21}, t_{22}, \ldots, t_{2n_2}; \ldots\ldots; t_{N1}, t_{N2}, \ldots, t_{Nn_N})$$

be observed failure times of sizes $n_1, n_2, \ldots, n_N$ from systems 1,2,...N respectively. Since failure times for each system are independant exponentially distributed and systems operate independantaly, joint density of $\underset{\sim}{t}$ given $\underset{\sim}{\lambda}$ is given by

$$p(\underset{\sim}{t}/\underset{\sim}{\lambda}) = \prod_{i=1}^{N} \left\{ \prod_{i=1}^{n_i} \lambda_i . \exp[- \lambda_i t_{ij}] \right\}$$

$$= \prod_{i=1}^{N} \lambda_i^{(n_i)} . \exp[- \lambda_i \sum_{j=1}^{n_i} t_{ij}]$$

$$= \prod_{i=1}^{N} \lambda_i^{(n_i)} . \exp[- \lambda_i S_i] \qquad \ldots(4.30).$$

$$(\text{where } S_i = \sum_{j=1}^{n_i} t_{ij}).$$

Joint density of $\underset{\sim}{\lambda}$ is given by

$$g(\underset{\sim}{\lambda}/\alpha, \beta) = (\beta^{\alpha}/\Gamma\alpha)^N \prod_{i=1}^{N} (\lambda_i)^{\alpha-1} . \exp[-\beta\lambda_i] \qquad \ldots(4.31).$$

Hence,

$$p(\underset{\sim}{t}/\alpha,\beta) = \int_{R_{\underset{\sim}{\lambda}}} p(\underset{\sim}{t},\underset{\sim}{\lambda}/\alpha,\beta)d\underset{\sim}{\lambda}.$$

$$= \int_{R_{\underset{\sim}{\lambda}}} p(\underset{\sim}{t}/\underset{\sim}{\lambda})p(\underset{\sim}{\lambda}/\alpha,\beta)d\underset{\sim}{\lambda}.$$

$$= \int_0^\infty \prod_{i=1}^{N} \lambda_i^{(n_i)} .\exp[-\lambda_i S_i] .(\beta^\alpha/\Gamma\alpha)^N. (\lambda_i)^{\alpha-1}.\exp[-\beta\lambda_i].d\lambda_i$$

$$=(\beta^\alpha/\Gamma\alpha)^N \prod_{i=1}^{N} \int_0^\infty (\lambda_i)^{n_i+\alpha-1}.\exp[-(\beta+S_i)\lambda_i]d\lambda_i$$

...(4.32).

Point estimates of λ can be obtained based on the posterior distribution of λ_i given the estimated values of α and β (Basu and Rigdon(1986)). The marginal maximum likelihood estimators of the parameters of the prior distribution can be obtained by using numerical techniques. In general, the parametric empirical Bayes point estimates are less disperse than the classical m.l.e.'s. When there are few systems and many observed failure times per systems, the point estimates are only slightly less disperse. Parametric empirical Bayes posterior probability interval estimates are generally narrower than the classical confidance intervals.

4.4 Asymmetric Loss Function :

In certain situations over estimation/under estimation of reliability function is savier. For example : From consumers point of view, over estimation of reliability (or under estimation of failure rate) is dangerous to him if actually that component or system is low reliable. On the other hand, from producers point of view, under estimation of reliability function (or over estimation of failure rate) is savier to him if actually his product is of high reliable quality.

From this point of view, Basu and Ebrahimi (1989) considered loss function which takes in to account both these facts under some ristrictions on constants in their loss function and is given by

$$L(\Delta) = b.\exp[a.\Delta] - c.\Delta - b, \quad a,c \neq 0, \quad b > 0.$$

where $\Delta = (\hat{\theta}/\theta - 1)$. ...(4.33).

i> Here L(0) = 0 and for minimia to occur at zero, L'(o) = 0 gives ab = c.

ii> For a = 1,

$$L(\Delta) - L(-\Delta) = b\,[\exp\{\Delta\} - \exp\{-\Delta\}] - 2c\Delta$$

$$= b\,[\exp\{\Delta\} - \exp\{-\Delta\} - 2\Delta\,]$$

(since, ab = c).

$$= b\,[\,(1+\Delta+\Delta^2/2!+\Delta^3/3!+.....) - (1-\Delta+\Delta^2/2!-\Delta^3/3!+.....) - 2\Delta\,]$$

$$= b\,[\,2\Delta^3/3!+2\Delta^5/5!+.....\,] > 0.$$

Therefore, in this situation over estimation is more savier than under estimation.

iii> For $a < 0$, $L(\Delta)$ rises exponentially when $\Delta < 0$ (underestimation) and almost linearely when $\Delta > 0$ (overestimatio).

iv> Finally, for small $|a|$,

$$L(\Delta) = b[1+a\Delta+(a\Delta)^2/2!+(a\Delta)^3/3!+....]-c\Delta-b$$
$$\cong b[1+a\Delta+(a\Delta)^2/2!]-c\Delta-b$$
$$= ba^2/\Delta^2 2!,$$

which is symmetric function.

4.4.1 Estimation Of θ Under Asymmetric Loss Function $L(\Delta)$:

Let $\underset{\sim}{X} = X_1, X_2, ..., X_n$ be n observations from p.d.f. (3.3), problem is to estimate θ, say $\hat{\theta}(\underset{\sim}{X})$, under asymmetric loss function so that posterior risk is minimum. Mathematically, estimate θ by $\hat{\theta}_B$ so that

$$\hat{\theta}_B = \min_{\hat{\theta}} \left\{ E_{\theta/\hat{\theta}(\underset{\sim}{X})}[L(\Delta)] \right\} \qquad ...(4.34).$$

Consider,

$$E_{\theta/\hat{\theta}(\underset{\sim}{X})}[L(\Delta)] = \int_0^\infty \left\{ b.\exp[a(\hat{\theta}/\theta-1)-c(\hat{\theta}/\theta-1)-b]. \frac{(\alpha+T)^{n+\nu}}{\Gamma(n+\nu)}.\exp[-(T+\alpha)].(1/\theta)^{(n+\nu+1)} \right\}.d\theta.$$

$$= \left\{ b.\exp[-a].(\alpha+T)^{(n+\nu)}.(\alpha+T-a\hat{\theta})^{-(n+\nu)} - c.\hat{\theta}(n+\nu)/(\alpha+T) + c - b \right\} \qquad ...(4.35).$$

Therefore, to obtain minima of $E_{\theta/\hat{\theta}(\underset{\sim}{X})}[L(\Delta)]$, differentiate (4.33) with respect to $\hat{\theta}$ and equate to zero, we get

$$ab.\exp[-a].(\alpha+T)^{(n+\nu)}.(n+\nu).(\alpha+T-a\hat{\theta})^{-(n+\nu+1)} - c.(n+\nu)/(\alpha+T) = 0.$$

On simplification, we get

$$\hat{\theta} = \left\{ [1-\exp[-a/(n+\nu+1)]].(\alpha+T)/a \right\}$$

Therefore, Bayes estimator of θ under asymmetric loss function is given by

$$\hat{\theta}_B = \left\{ [1-\exp[-a/(n+\nu+1)]].(\alpha+T)/a \right\} \quad \ldots(4.36).$$

4.5 Concluding Remarks :

Here we studied some problems by considering exponential distribution as a model, which is applicable in number of physical situations. Here we have only considered gamma (or inverted gamma) priors. One can carry out similar analysis for other priors. Also results can be extended to other physical models and to censored samples.

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