Chapter-2.

BURN-IN TEST AND OPTIMIZATION CRITERIA

2.1 Introduction:-

For the class of life time distribution, when the life timei is known to be exponential, methods of estimation and testing have been well studied.

Saunder and Myhere (1981) have developed theory and methods of estimation of parameters, when hazard rate decreases with age. Following are situations where the life time distributions having such property.

1.Newly born baby has high death rate and upto certain age death rate goes on decreasing

2.In software developments, initially it has very high failure rate, but as the debugging process goes on in order to search and remove faults, the failure rate decreases.

3.New instalation of complex system with many subsystems.

Davis and Feldstein (1979) studied the family of distributions for which the hazard rate, $h(t)$, which decreases from initial value of $(\theta + \lambda/\phi)$ to an ultimate value θ, can be modeled as

$$h(t) = \theta + \lambda/(t + \phi), \; t > 0 \; \& \; \phi > 0. \qquad ...(2.1).$$

Such models seems appropriate in engineering equipments.In addition an advantage of considering such parametric models can be demonstrated by finding optimal burn-in periods, which in a sense increases the quality of the unit.

For the model desceibed by (2.1), David and Feldstein (1979) provided the m.l.e.'s of parameters θ and λ when ϕ is assumed to be known, under progressively censored sampling and computational method for their evaluation.

Saunder and Myhere (1981) have considered reparametrised version of the above model and obtained m.l.e.'s of the parameters, and also have obtained optimum burn-in period based on various criteria. For the model considered by Saunder and Myhere(1981) the hazard function, h(.), is given by

$$h(t) = \left(\frac{\alpha\beta}{1 + t\beta}\right) + \alpha\beta\gamma, \qquad t,\alpha,\beta,\gamma > 0 \qquad \ldots(2.2).$$

where α,β,γ are positive unknowns.

The cumulative hazard function is given by

$$H(t) = \int_0^t h(x)\,dx$$
$$= \alpha\ \{log\ (1+t\beta) + \beta\gamma t\}; \quad t > 0 \qquad \ldots(2.3).$$

Saunder and Myhere (1981) have provided m.l.e.'s and computational procedures,for a class of decreasing hazard rate distribution with two unknown parameters for which

$$H(t) = \alpha\ Q(t\beta),$$

when certain assumptions are made about behaviour of $q = Q'$; namely for the functions ζ & ψ defined by

$$\psi(x) = x\ q(x);$$

$$\zeta(x) = 1 + x\frac{q(x)}{q(x)} \qquad \ldots(2.4).$$

satisfy i) ψ is increasing.

ii) q is log convex.

iii) ζ is bounded between 0 & 1; and is unimodal:
(that is, there exists $S = \{ x : \zeta(x) > c, 0 < c < 1\}$ is convex).

In section-2 we discuss the genesis of the model. Section-3 deals with obtaining m.l.e.'s of the parameters in model (2.2). these estimators are used to obtain optimum burn-in, under different conditions, to be discussed in section-4. We have established optimum burn-in period based on desired quality in section-5. Some concluding remarks are given in section-5.

2.2 The Genesis Of The Model:

Let us assume that each unit has constant hazard rate, say λ. Thus life times are exponential random variable with parameter λ. Suppose due to variation in manufacturing quality control, the failure rates of such unit vary slightly in some random manner.Preassume this variation has gamma distribution with parameters (α,β) and is given by

$$g(\lambda) = \frac{1}{\beta^{\alpha}\Gamma\alpha} \exp\{-\lambda/\beta\}.\lambda^{(\alpha-1)}; \quad \lambda > 0 \quad \& \quad \alpha,\beta \text{ positive constants}, \ldots(2.5).$$

Then R(t) : the reliability of any such unit selected at random is given by

$$R(t) = \int_0^\infty \exp\{-\lambda t\}.g(\lambda).d\lambda.$$

$$= \int_0^\infty \exp\{-\lambda t\}.\frac{1}{\beta^\alpha \Gamma\alpha}.\exp\{-\lambda/\beta\}.\lambda^{(\alpha-1)}.d\lambda.$$

$$= 1/(1 + t\beta)^\alpha \qquad \ldots(2.6).$$

That is,

$$R(t) = \exp\{ \alpha.(-log(1+t\beta) \} \qquad \ldots(2.7).$$

The failure rate of the unit then will be

$$h(t) = \frac{f(t)}{\bar{F}(t)}. \qquad \ldots(2.8).$$

Here $f(t) = \int_{R_\lambda} f(t/\lambda).g(\lambda).d\lambda$, (where R_λ:range of λ).

$$= \int_0^\infty \lambda.\exp\{-\lambda t\}.\frac{1}{\beta^\alpha \Gamma\alpha} \exp\{-\lambda/\beta\}.\lambda^{(\alpha-1)}.d\lambda.$$

$$= \alpha\beta/(1+t\beta)^{(\alpha+1)}, \qquad \ldots(2.9).$$

which is univariate Lomax or Pareto type-II distribution;

and $\bar{F}(t) = \int_0^\infty \exp\{-\lambda t\}.g(\lambda).d\lambda$

$$= (1+ t\beta)^{-\alpha} \qquad \ldots(2.10).$$

Using (2.9) & (2.10) in (2.8), we get

$$h(t) = \alpha\beta/(1+t\beta); \quad t > 0, \alpha, \beta > 0. \qquad \ldots (2.11).$$

Here the choice of the gamma as mixing distribution comes from two considerations, namely-

i>Flexible two parmmeter family capable of representing many situations adequatly; and ii>Gives closed form distribution.

Generally, such a distribution is called mixed exponential distribution' and sample from such a model are highly censored. This might be due to the cost involved in testing and / or a unit that does not fail early, is is considered "good" and probably, that unit does not fail during the subsequent test period.

2.3 M.L.E. For Progressively Censored Data:

Denote sample data with $\underset{\sim}{t} = (t_1, t_2, \ldots, t_k; t_{(k+1)}, \ldots, t_n)$; where $(t_1, t_2, \ldots, t_k)$ are ordered failure times and $(t_{(k+1)}, \ldots, t_n)$ are ordered censoring times. Note that experiment is to be continued until each item has failed or censored. The contribution of the i^{th} unit $g(t_i)$; to the likelihood is $\bar{F}(t_i)$ provided i^{th} unit is censored and $f(t_i)$ otherwise. That is,

$$g(t_i) = (f(t_i))^{\delta_i} . (\overline{F(t_i)})^{(1-\delta_i)} ,$$

where,

$$\delta_k = \begin{cases} 1, \text{ if death occurs to } k^{th} \text{ unit;} \\ 0, \text{ if loss occurs to } k^{th} \text{ unit.} \end{cases}$$

hence, likelihood function is given by

$$L(\alpha, \beta, \gamma / \underset{\sim}{t}) = \prod_{i=1}^{n} g(t_i) \qquad \ldots(2.12).$$

$$= \prod_{i=1}^{n} (f(t_i))^{\delta_i} . (\overline{F}(t_i))^{(1-\delta_i)} ,$$

Thus,

$$L(\alpha,\beta\gamma/\underset{\sim}{t}) = \prod_{i=1}^{k} f(t_i) . \prod_{i=(k+1)}^{n} \overline{F}(t_i)$$

$$= \prod_{i=1}^{k} \frac{f(t_i)}{\overline{F}(t_i)} . \prod_{i=(k+1)}^{n} \overline{F}(t_i) . \prod_{i=1}^{k} \overline{F}(t_i)$$

$$= . \prod_{i=1}^{k} h(t_i) . . \prod_{i=1}^{n} \overline{F}(t_i) \qquad ...(2.13).$$

1/k times log likelihood is given by

$$1/k . \log L(\alpha,\beta\gamma/\underset{\sim}{t}) = 1/k . \sum_{i=1}^{k} \log h(t_i) + 1/k . \sum_{i=1}^{n} \log \overline{F}(t_i)$$

$$= 1/k . \sum^{k} \log h(t_i) - 1/k . \sum_{i=1}^{n} H(t_i) \qquad ...(2.14).$$

where $h(t) = \dfrac{f(t)^{-1}}{\overline{F}(t)}$, $H(t) = \int_{o}^{t} h(x).dx$

$$= - \log \overline{F}(t_i).$$

Let

$$H(t) = \alpha . Q(t\beta:\gamma); \; h(t) = \alpha\beta . q(t\beta:\gamma) \qquad ...(2.15).$$

where, $Q(t\beta:\gamma) = \log(1+t\beta) + \gamma\beta t$;

$$q(t\beta:\gamma) = 1/(1+t\beta) + \gamma .$$

Differentiating (2.14) with respect to (w.r.t.)α and equating to zero we get

$$1/k . \sum_{i=1}^{k} \frac{1}{h(t_i)} . \frac{\delta}{\delta\alpha}(h(t_i)) - 1/k . \sum_{i=1}^{n} . \frac{\delta}{\delta\alpha}(H(t_i)) = 0$$

$$1/k . \sum_{i=1}^{k} \{\alpha\beta[\frac{1}{1+t_i\beta}+\gamma]\}^{-1} . \beta[\frac{1}{1+t_i\beta}+\gamma] - 1/k . \sum_{i=1}^{n} Q(t_i\beta:\gamma) = 0$$

(Substituting values of h(t) and H(t)).

$$1/\alpha = n/k . \overline{Q}(\beta:\gamma); \qquad ...(2.16).$$

where, $\bar{Q}(\beta:\gamma)=1/n.\sum_{i=1}^{n} Q(t_i\beta:\gamma)$.

Similerly, $\frac{\delta}{\delta\beta}\log L(\alpha,\beta\gamma/\underset{\sim}{t}) = 0$ gives

$$1/k.\sum_{i=1}^{k}\frac{1}{h(t_i)}\cdot\frac{\delta}{\delta\beta}(h(t_i)) - 1/k.\sum_{i=1}^{n}\cdot\frac{\delta}{\delta\beta}(H(t_i)) = 0$$

$$1/k.\sum_{i=1}^{k}\{\alpha\beta.q(t_i\beta:\gamma)\}^{-1}\{\alpha[q(t_i\beta:\gamma)+\beta.\frac{\delta}{\delta\beta}q(t_i\beta:\gamma)]\}-\alpha/k.\sum_{i=1}^{n}q(t_i\beta:\gamma) = 0$$

(by (2.15))

$$(1/\beta k).\sum_{i=}^{k}\{1+\beta\frac{q'(\beta ti:\gamma)}{q(\beta t_i:\gamma)}\} - \frac{\alpha n}{\beta k}.1/n.\sum_{i=1}^{n}\beta q(t_i\beta:\gamma) = 0$$

$$\bar{\zeta}(\beta:\gamma) = \frac{\alpha n}{\beta k}\,\bar{\psi}\,(\beta:\gamma), \qquad \text{...(2.17).}$$

where $\bar{\psi}\,(\beta:\gamma) = 1/n.\sum_{i=1}^{n}\beta q(t_i\beta:\gamma)$ & $\bar{\zeta}(\beta:\gamma) =(1/k).\sum_{i=}^{k}\{1+\beta\frac{q'(\beta ti:\gamma)}{q(\beta t_i:\gamma)}\}$.

Finally, $\frac{\delta}{\delta\gamma}\log L(\alpha,\beta\gamma/\underset{\sim}{t}) = 0$ gives

$$1/k.\sum_{i=1}^{k}\frac{1}{h(t_i)}\cdot\frac{\delta}{\delta\gamma}(h(t_i)) - 1/k.\sum_{i=1}^{n}\cdot\frac{\delta}{\delta\gamma}(H(t_i)) = 0$$

$$1/k.\sum_{i=1}^{k}\{\alpha\beta[\frac{1}{1+t_i\beta}+\gamma\ ]\}^{-1}.\alpha\beta -1/k.\sum_{i=1}^{n}\alpha\beta t_i = 0$$

$$1/k.\sum_{i=1}^{k}\{(1+t_i\beta)(1+\beta\gamma t_i+\gamma)^{-1}\} = \frac{n\alpha\beta}{k}\,\bar{t}\ , \qquad \text{...(2.18).}$$

where, $\bar{t} = 1/n.\sum_{i=1}^{n} t_i$.

Now, to obtain m.l.e.'s of parameters one has to solve (2.16), (2.17) and (2.18).Following is an algorithm to obtain m.l.e.'s of the parameters.

i> Initially give values of α and β as α_{i-1} and β_{i-1} in equation (2.18).Solve it for γ say γ_i(>= 0).

ii> Solve (2.16) and (2.17) for α, β with fixed γ at γ_i. Say these values of α and β as α_i and β_i.

iii> Stop this procedure for appropriate accuracy.

Appendix is provides programming to obtain random sample from such a model and to estimate parameters of this model by making use of above steps.

2.4 Optimum Burn-in And Ultimate Hazard Rate:

Consider a unit which is having life length X,with hazard rate as given in (2.2).A burn -in of length $\tau \geq 0$ yields a random remaining life, say X_τ.Hazard rate corresponding to r.v.X_τ is given by

$$h(t+\tau) = f_{X_\tau}(t) / \bar{F}_{X_\tau}(t)$$

$$= \left[\frac{\alpha\beta}{1 + (t+\tau)\beta}\right] + \alpha\beta\gamma, \quad t,\alpha,\beta,\gamma > 0 \qquad \text{(by (2.2))}.$$

$$= \alpha\beta' \, q(t\beta' : \gamma') \qquad \ldots(2.19).$$

where $\beta' = \left(\frac{\beta}{1+\tau\beta}\right)$, $\gamma' = \gamma(1+\tau\beta)$.

Note that hazard rate corresponding to residual life time X_τ also belongs to the same family as that of X, with new parameters α, β' and γ'. After burning the unit in a laboratory, initial hazard rate is lowered to $\alpha\beta'(1+\gamma')$ from the value $\alpha\beta(1+\gamma)$. While the terminal (or ultimate)hazard rate remains the same as that of X, which is given by

$$\alpha\beta'\gamma' = \alpha\beta\gamma.$$

Upon obtaining the all the three estimators,one might look for the use of these estimators.These estimators can be used for determining optimum burn-in time, depending upon various criteria.These criteria are described bellow:

First criteria:

Let Rs.C be cost per unit time of burn-in, then the increased gain per unit (system) for a burn-in of period τ is

$$g(\tau) = B[\alpha\beta(1+\gamma) - \alpha\beta'(1+\gamma')] - C\tau$$

$$g(\tau) = \alpha\beta B[(1+\gamma) - (1+\tau\beta)^{-1}\{1+\gamma(1+\tau\beta)\}] - C\tau$$

$$= \alpha\beta B[1-(1+\tau\beta)^{-1}] - C\tau \qquad \ldots(2.20).$$

In order to maximize gain, solve $\frac{\delta}{\delta\tau}\, g(\tau) = 0$ for τ.

(2.20) gives $\qquad \alpha\beta^2 B.(1+\tau\beta)^2 = C$

$$\tau = \{\frac{\alpha\beta}{C}\}^{1/2} - \frac{1}{\beta}$$

Since burn-in cannot be negative, take optimum burn-in period as

$$\tau^* = \max\{(\{\frac{\alpha\beta}{C}\}^{1/2} - \frac{1}{\beta}),0\} \qquad \ldots(2.21).$$

Second Criteria:

Here we consider problem of minimizing the cost to bring the initial hazard rate within 100P% (where 0<P<1)of the ultimate hazard rate using green-run (without failure), that is we want to have

$$\alpha\beta'(1+\gamma') = \alpha\beta\gamma + P.(\alpha\beta\gamma),$$

(since $\alpha\beta\gamma$ is the ultimate hazard rate).

$$\left(\frac{\alpha\beta}{1+\tau\beta}\right).[1+\gamma(1+\tau\beta)] = \alpha\beta\gamma(1+P),$$

(substituting values of β', γ' in above equation).

Now, solving for τ we get

$$\tau = (1/\gamma P - 1)\beta^{-1}.$$

Therefore minimum burn-in time required according to this criteria is given by

$$\tau^{*} = \max\{(1/\gamma P - 1)\beta^{-1},\ 0\}, \qquad \ldots(2.22).$$

costs Rs.C.τ^{*}.

Third Criteria:

Suppose the benefit due to increased reliability is Rs.B per unit time of increased expected life and Rs.C be cost per unit time of burn-in . Here the corresponding optimization function is given by

$$g(\tau) = B.E(X_\tau - X_o).$$

Hence maximum gain is determined from $\frac{\delta}{\delta\tau} g(\tau) = 0$, which is equivalant to

$$\frac{\delta}{\delta\tau} EX_\tau = C/B$$

$$h(\tau).\exp\{H(\tau)\}*{}_{\tau}\int^{\infty} \exp\{-H(t)\}.dt = (B+C)/B \qquad \ldots(2.23).$$

$$(\text{since}, EX_\tau = {}_0\int^{\infty} \bar{F}_{X_\tau}(t).dt$$

$$= {}_0\int^{\infty} \bar{F}(t+\tau)/\bar{F}(\tau).dt$$

$$= \int_0^{\infty} \exp\{-\int_0^{(t+\tau)} h(x).dx\} / \int_0^{\infty} \exp\{-\int_0^{\tau} h(x)\, dx\}.dt$$

$$= \int_0^{\infty} \exp\{-H(t+\tau)+H(\tau)\}.dt;\&$$

$$\frac{\delta}{\delta\tau} EX_{\tau} = \int_0^{\infty} [h(\tau)-h(t+\tau)].\exp\{-H(t+\tau)+H(\tau)\}.dt$$

$$= h(\tau).\exp\{H(\tau)\} * \int_{\tau}^{\infty} \exp\{-H(t)\}.dt - 1.)$$

The equation (2.23) can be solved for τ by numerical method.

In the following section we now establish the optimum burn-in period for desired quality.

2.5 Optimum Burn-in Period For Desired Quality :

I. Optimum Burn-in Period For Desired Quality (R_o) *At Period* t_o :

Suppose survival function of life time of a unit is as given by equation (2.10). After burning the unit for a period of τ, survival function of residual life time, X_{τ}, is given by

$$\bar{F}_{\tau}(t) = \left\{ 1 + \frac{\beta t}{(1+\tau\beta)} \right\}^{-\alpha}. \qquad \ldots(2.24).$$

We know that, this survival function is more than that of new unit.

Now, if producer wanted to convince the consumer with the warranty that, his unit is of quality (reliability) at least R_o at period t_o, then problem is how much duration of time producer has to burn the unit in a laboratory in order to attain this criteria with minimum cost. Mathematically, find minimal τ such that

$$\bar{F}_{\tau}(t_o) \geq R_o \qquad \ldots(2.25).$$

That is,

$$\left\{ 1 + \frac{\beta t}{(1+\tau\beta)} \right\}^{-\alpha} \geq R_o .$$

Which on simplification optimal period (say τ^*) is given by

$$\tau^* = \max \left\{ 0, \left[\frac{R_o^{1/\alpha}}{1 - R_o^{1/\alpha}} - \frac{1}{\beta} \right] \right\} \qquad \ldots(2.26).$$

Again, if Rs. C is cost per unit burn-in, plus compitition cost function (say C(t), which is increasing in t) due to other producers unit of the same type. Therefore, total cost incured to achive above criteria is given by

$$(\text{Total cost at } \tau^*) = C.\tau^* + \int_0^{\tau^*} C(t).dt$$

II> Optimum Burn-in Period Criteria For Desired Quality Level Over Life Length Of A Unit :

Above criteria gives warranty R_o at period t_o only. Suppose, instead of giving desired quality at period t_o only, producer wants to convince consumer by giving desired quality level, say $S_o(t)$. Now problem is how much duration of time burn-in test be conducted in order to achive given desired quality level, $S_o(t)$ with minimum cost. Mathematically, find minimum τ such that

$$\bar{F}_\tau(t) \geq S_o(t), \quad \text{for all } t. \qquad \ldots(2.26).$$

For example:

1. Let desired quality is given by

$$S_{10}(t) = \begin{cases} \exp\{-\lambda t\}, & \text{if } t,\lambda > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad \ldots(2.27).$$

Therefore, we have to find minimal τ such that

$$\bar{F}_{\tau}(t) \geq S_{10}(t), \qquad \text{for all } t > 0.$$

$$\left\{ 1 + \frac{\beta t}{(1+\tau\beta)} \right\}^{-\alpha} \geq \exp\{-\lambda t\} \geq \exp\{-\lambda t\} \qquad \text{for all } t > 0.$$

On simplification for τ,we get

$$\tau \geq \max_{t>0} \left\{ \frac{t}{\exp\{(\lambda/\alpha).t\} -1} - \frac{1}{\beta} \right\} \qquad \ldots(2.28).$$

Let $g(t) = \left\{ \frac{t}{\exp\{(\lambda/\alpha).t\} -1} - \frac{1}{\beta} \right\}$, which is maximum at $t = 0$,

and $\lim_{t\to 0} g(t) = \lim_{t\to 0} \left\{ \frac{t}{\{1+(\lambda/\alpha)t+(\lambda/\alpha)^2 t^2 + ..\}-1} - \frac{1}{\beta} \right\}$

$$= (\alpha/\beta) - (1/\beta). \qquad \ldots(2.29).$$

Thus, optimum burn-in period is given by

$$\tau^* = \begin{cases} (\alpha/\lambda) - (1/\beta), & \text{if } /\lambda) > (1/\beta) \\ 0 & , \text{ otherwise.} \end{cases} \qquad \ldots(2.30).$$

[using (2.28) and (2.29)].

2. Let desired quality level is given by

$$S_{20} = \begin{cases} (1 + t\beta')^{-\alpha}, & \text{if } t > 0 \text{ and } \beta' < \beta, \\ 0 & \text{otherwise.} \end{cases} \qquad \ldots(2.31).$$

Now, we have to find minimal τ so that

$$\bar{F}_{\tau}(t) \geq S_{20}(t) \qquad \text{for all } t > 0.$$

That is,

$$\left\{ 1 + \frac{\beta t}{(1+\tau\beta)} \right\}^{-\alpha} \geq (1 + t\beta)^{-\alpha} \qquad \text{for all } t > 0.$$

On simplification, optimum burn -in period, τ^*, is given by

$$\tau^* = \begin{cases} (\beta - \beta')/(\beta.\beta'), & \text{if } \beta' < \beta, \\ 0 & \text{, otherwise.} \end{cases} \qquad \ldots(2.32).$$

3. Let desired quality level is given by

$$S_{30} = \begin{cases} (1 + t\beta)^{-\alpha'}, & \text{if } t > 0 \text{ and } \alpha' < \alpha, \\ 0 & \text{otherwise.} \end{cases} \qquad \ldots(2.33).$$

Similarly, we have to obtain minimum τ such that

$$\bar{F}_{\tau}(t) \geq S_{30}(t) \qquad \text{for all } t > 0.$$

Which gives,

$$\tau \geq \max_{t>0} \left\{ \frac{1}{[(1+t\beta)^{(\alpha'/\alpha)}-1]} - \frac{1}{\beta} \right\} \qquad \ldots(2.34).$$

Let, $g(t) = \left\{ \frac{1}{[(1+t\beta)^{(\alpha'/\alpha)}-1]} - \frac{1}{\beta} \right\}$

Differentating g(t) with respect to t and equating to zero, we get

$$(1+t\beta)^{(\alpha'/\alpha)-1} + [(\alpha'\beta/\alpha)-\beta]t - 1 = 0 \quad \text{...(2.35)}.$$

For given values of α, α' and β, solve this equation for t. Putting this value of t in equation (4.34) we will get optimal burn-in period, τ^* say, which gives desired quality level.

2.6 Concluding Remarks:

From available data following stated model, M.L.E.'s for parameters α, β, γ can be obtained by method described above. These estimators are used in equations (2.21), (2.22) and (2.23) to obtain optimum burn-in period under respective criteria. Note that under all these criteria, optimum burn-in period for exponential life time distribution is equal to zero.

Instead of single unit one can go for a system having two or more than two units working together. If each units life length following model (2.2), with same or different parameters, one can go for the same problem with respect to the system.

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