CHAPTER-III

ACCEPTANCE SAMPLING BY VARIABLES FOR EXPONENTIAL DISTRIBUTION

3.1 Introduction:

In this chapter, we shall consider that the measurements on the items in the lot has an exponential distribution. We consider, one parameter exponential distribution when lower and upper specification limit is given.
Thus we study the sampling plans by variable such as

Case-I : Exponential distribution with one parameter :

When i) lower specification limit L is given,

ii) upper specification limit U is given,

3.2 One parameter case :

Let X be the random variable that the time it takes for a certain kind of Television tube to wear out.
The density function of exponential distribution is given by

$$f(x,\sigma) = \frac{1}{\sigma} e^{-x/\sigma} \qquad x > 0 \qquad (3.2.1)$$

Here σ is the parameter of the distribution and it is usually unknown.

(i) Lower specification limit L is given :

Let θ be the probability of an item being defective then,

$$\theta = P_\sigma [X_1 \leq L]$$

$$= P_\sigma \left[\frac{2X_1}{\sigma} \leq \frac{2L}{\sigma} \right]$$

$$= Q_2 \left(\frac{2L}{\sigma} \right)$$

that is,

$$\frac{2L}{\sigma} = Q_2^{-1} (\theta) \qquad (3.2.2)$$

Now the criteria of accepting or rejecting the lot be as follows.

Accept the lot if $\theta \leq \theta_o$ and reject it otherwise. If θ is known the problem is quite trivial. If θ is unknown then we have to estimate $\hat{\theta}$ by choosing an appropriate method.

Let $X_1, X_2, \ldots, X_n$ be the measurements on n items, so that $X_1, X_2, \ldots, X_n$ are i.i.d. exponential with mean σ. In order to obtain the minimum variance unbiased estimate of θ define,

$$T_1 = \begin{cases} 1 & \text{if } X_1 \leq L \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T_1 is unbiased for θ, then by using Rao-Blackwell-Lehmann-Scheffe Theorem, MVUE is given by

$$E(T_1 \mid \Sigma X_i) = P[X_1 \leq L \mid \Sigma X_i = t]$$

$$= P[\frac{X_1}{\Sigma X_i} \leq \frac{L}{t} \mid \sum_{i=1}^{n} X_i = t]$$

$$= P[\frac{2X_1}{2X_1+2\sum_{2}^{n} X_i} \leq \frac{L}{t} \mid \sum_{i=1}^{n} X_i = t] \quad .. (3.2.3)$$

Now the distribution of $2X_1/2X_1 + 2\sum_{2}^{n} X_i$ is the beta distribution with parameter 1 and (n-1) and this distribution is independent of σ. So by Basu's Theorem it follows that the right hand side of (3.2.3) is equivalent to

$$= P[\beta_{(1,n-1)} \leq L/t]$$

Hence, the MVUE is given by

$$\hat{\theta} = 1 - (1 - L/_{n\bar{x}})^{n-1} \quad (3.2.4)$$

where $t = n\bar{x}$, so accept the lot if $\hat{\theta} \leq \theta_0$ otherwise reject the lot. Using (3.2.4) the OC function can be written as,

$$L(\sigma) = P_\sigma \text{ [Accepting the lot]}$$

$$= P_\sigma [1-(1-L/n\bar{x})^{n-1} \leq \theta_0]$$

$$= P_\sigma [(1-L/n\bar{x})^{n-1} \geq (1-\theta_0)]$$

$$= P_\sigma [1 - \frac{L}{n\bar{x}} \geq (1-\theta_0)^{1/n-1}]$$

$$= P_\sigma [L/n\bar{x} \leq 1 - (1 - \theta_o)^{1/n-1}]$$

$$= P_\sigma [n\bar{x} \geq \frac{L}{1 - (1 - \theta_o)^{1/n-1}}]$$

$$= P_\sigma [\frac{2n\bar{x}}{\sigma} \geq \frac{2L}{\sigma[1-(1- \theta_o)^{1/n-1}]}]$$

$$= P_\sigma [\chi^2_{2n} \geq \frac{2KL}{\sigma}]$$

$$= 1 - Q_{2n}(\frac{2KL}{\sigma}) \qquad (3.2.5)$$

where $k = \frac{1}{[1 - (1- \theta_o)^{1/n-1}]}$ and Q_{2n} is distribution function of chi-square random variable with 2n d.f. We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β). Using (3.2.5) we get the following two equations.

$$\frac{2kL}{\sigma_1} = Q_{2n}^{-1} (\alpha) = \chi^2_{2n,\alpha} \qquad (3.2.6)$$

and

$$\frac{2kL}{\sigma_2} = Q_{2n}^{-1} (1 - \beta) = \chi^2_{2n,1-\beta} \qquad (3.2.7)$$

Using equation (3.2.2), then equations (3.2.6) and (3.2.7) can be written as

$$k \cdot Q_2^{-1}(\theta_1) = Q_{2n}^{-1}(\alpha)$$

that is,

$$k\,\chi^2_{2,\theta_1} = \chi^2_{2n,\alpha} \qquad (3.2.8)$$

and

$$k\,\chi^2_{2,\theta_2} = \chi^2_{2n,1-\beta} \qquad (3.2.9)$$

dividing equation (3.2.9) by (3.2.8) we get

$$\frac{\chi^2_{2,\theta_2}}{\chi^2_{2,\theta_1}} = \frac{\chi^2_{2n,1-\beta}}{\chi^2_{2n,\alpha}} \qquad (3.2.10)$$

using (3.2.10) by trial we can find the value of n, from equation (3.2.8) we have

$$k = \frac{\chi^2_{2n,\alpha}}{\chi^2_{2,\theta_1}} \qquad (3.2.11)$$

and from equation (3.2.9) we get

$$k = \frac{\chi^2_{2n,1-\beta}}{\chi^2_{2,\theta_2}} \qquad (3.2.12)$$

It is found that from (3.2.11) the resulting OC function of the plan passes through the producer's risk point

$(\theta_1, 1-\alpha)$ and if it found that from (3.2.12) it passes through the consumer's risk point (θ_2, β).

(ii) Upper specification limit U is given :

In this case θ is given by

$$\theta = P_\sigma (X_1 \geq U)$$

$$= P_\sigma(\frac{2X_1}{\sigma} \geq \frac{2U}{\sigma})$$

$$= 1 - Q_2 (\frac{2U}{\sigma})$$

that is

$$\frac{2U}{\sigma} = Q_2^{-1} (1- \theta) \qquad (3.2.13)$$

Now our criteria of accepting or rejecting the lot be as follows.

Accept the lot if $\theta \geq \theta_o$ and reject it otherwise.

In order to obtain minimum variance unbiased estimate of θ define,

$$T_2 = \begin{cases} 1 & \text{if } X_1 \geq U \\ 0 & \text{otherwise.} \end{cases}$$

Clearly T_2 is unbiased for θ. Then by using Rao-Blackwell-Lehmann-Scheffe Theorem, the MVUE is given by

$$E(T_2 | \Sigma X_i) = P_\sigma [X_1 \geq U \mid \Sigma X_i = t]$$

$$= P_\sigma [\frac{X_1}{\Sigma X_i} \geq \frac{U}{t} \mid \sum_{i=1}^{n} X_i = t]$$

$$= P_\sigma [\frac{2X_1}{2X_1 + 2 \sum_{2}^{n} X_i} \geq \frac{U}{t} \mid \sum_{i=1}^{n} X_i = t] \quad ..(3.2.14)$$

Now, the distribution of $2X_1 / 2X_1 + 2 \sum_{2}^{n} X_i$ is the beta distribution with parameter 1 and n-1 and this distribution is independent of σ. So, by Basu's Theorem it follows that the right hand side of (3.2.14) is equivalent to

$$= P [\beta_{(1,n-1)} \geq \frac{U}{t}]$$

$$= (1 - \frac{U}{t})^{n-1}$$

that is

$$\hat{\theta} = (1 - \frac{U}{n\bar{x}})^{n-1} \quad (3.2.15)$$

where $t = n\bar{x}$, so accept the lot if $\hat{\theta} \not\leq \theta_0$ otherwise reject the lot. Using (3.2.15), the OC function can be written as

$$L(\sigma) = [\text{Accepting the lot}]$$

$$= P_\sigma \left[\left(1 - \frac{U}{n\bar{x}}\right)^{n-1} \geq \theta_o \right]$$

$$= P_\sigma \left[1 - \frac{U}{n\bar{x}} \geq (\theta_o)^{1/n-1} \right]$$

$$= P_\sigma \left[n\bar{x} \geq \frac{U}{1 - (\theta_o)^{1/n-1}} \right]$$

$$= P \left[\chi^2_{2n} \geq \frac{2kU}{\sigma} \right]$$

$$= 1 - Q_{2n} \left(\frac{2kU}{\sigma} \right) \tag{3.2.16}$$

where $k = \dfrac{1}{1 - (\theta_o)^{1/n-1}}$ and Q_{2n} is the distribution function of chi-square random variable with 2n d.f. We find n and k so that the resulting plan has OC function passing through the producer's risk point $(\theta_1, 1-\alpha)$ and consumer's risk point (θ_2, β) using (3.2.16) we get the following two equations.

$$\frac{2kU}{\sigma_1} = Q^{-1}_{2n}(1-\alpha) = \chi^2_{2n,1-\alpha} \tag{3.2.17}$$

and

$$\frac{2kU}{\sigma_2} = Q^{-1}_{2n}(1-\beta) = \chi^2_{2n,1-\beta} \tag{3.2.18}$$

using equation (3.2.13) then (3.2.17) and (3.2.18) becomes

$$k\ Q_2^{-1}(1-\theta_1) = Q_{2n}^{-1}(1-\alpha)$$

that is

$$k\ \chi^2_{2,(1-\theta_1)} = \chi^2_{2n,1-\alpha} \tag{3.2.19}$$

and

$$k\ \chi^2_{2,(1-\theta_2)} = \chi^2_{2n,(1-\beta)} \tag{3.2.20}$$

Dividing equation (3.2.19) by (3.2.18) we get

$$\frac{\chi^2_{2,(1-\theta_2)}}{\chi^2_{2,(1-\theta_1)}} = \frac{\chi^2_{2n,(1-\beta)}}{\chi^2_{2n,1-\alpha}} \tag{3.2.21}$$

Using (3.2.21) we can find the value of n by trial. In order to find the value of k, using (3.2.18) we get

$$k = \frac{\chi^2_{2n,1-\alpha}}{\chi^2_{2,(1-\theta_1)}} \tag{3.2.22}$$

and from equation (3.2.19) we have

$$k = \frac{\chi^2_{2n,1-\beta}}{\chi^2_{2,(1-\theta_2)}} \tag{3.2.23}$$

The equation (3.2.22) which passes through producer's risk point $(\theta_1, 1-\alpha)$ and equation (3.2.23) passes through consumer's risk pCint (θ_2, β).

Example 3.1 :

Suppose that we are given the following quantities $\alpha = .1$, $\beta = .1$, $\theta_1 = .01$ and $\theta_2 = .0383$. Using chi-square table for equation (3.2.10) we can find the value of n by trial and that is n = 4. In order to find the value of k, from equation (3.2.11) we get k = 173.6318 and from equation, (3.2.12) we have k = 171.0883. Now taking the average we get k = 172.3600.

In tables given below I to IV for different values of α, β and θ_2 the values of n and k is computed and is compared with attribute plan parameters.

TABLE-I

$\alpha = .1, \quad \beta = .1, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.0383	174	3	4	172.8600
.0329	243	4	5	240.5042
.0253	465	7	8	461.5440
.0206	863	12	13	857.5891
.0176	1536	20	21	1527.1294

TABLE-II

$\alpha = .01, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters		Variable plan parameters	
	n	c	n	k
.0942	82	3	4	80.1241
.0716	127	8	5	124.9823
.0412	350	8	9	346.1374
.0356	476	10	11	471.3217
.0319	609	12	13	603.3221
.0305	677	13	14	671.3392
.0293	746	14	15	739.8054
.0258	1034	18	19	1025.0647
.0252	1106	19	20	1096.9303
.0246	1181	20	21	1171.8827

TABLE-III

$\alpha = .05, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters n	Attribute plan parameters c	Variable plan parameters n	Variable plan parameters k
.1335	35	1	2	34.2276
.077	47	2	3	79.8258
.0465	197	4	5	194.1601
.0275	616	10	11	611.4396
.0263	692	11	12	686.0817
.0253	768	12	13	762.2984
.0237	923	14	15	916.9455
.0215	1241	18	19	1234.0279
.0207	1403	20	21	1395.3626

TABLE-IV

$\alpha = .1, \quad \beta = .05, \quad \theta_1 = .01$

θ_2	Attribute plan parameters n	Attribute plan parameters c	Variable plan parameters n	Variable plan parameters k
.0376	243	4	5	239.7421
.0334	314	5	6	311.6480
.0242	700	10	11	696.1497
.0225	864	12	13	857.4569