CHAPTER - II

A STUDY OF DOWNTON'S AND HAWKES MODEL

2.1 Introduction

The bivariate exponential distribution is well known in the study of reliability and availability of systems. The effect of any correlation between variables on the total reliability of a system would be of great interest, when the actual form of bivariate exponential distribution is not important.

While studying the Downton's bivariate exponential distribution (DBVED), it is important to note the construction of Marshall-Olkin bivariate exponential distribution (M-O BVED). It is obtained by supposing that failure is caused by three independent "Shocks" on a system of two component with arrival rates λ_1, λ_2 and λ_3 respectively .

The bivariate exponential distribution of Marshall-Olkin preservs the property of lack of memory that the residual life is independent of age. It suffers from a mathematical difficulty that, it is a mixture of singular distribution and continuous one. Marshall-Olkin model would be appropriate for situations where pairs of identical observations appear in the observed data. It also preservs weakened version of the lack of memory property of the univariate exponential distribution. However the advantages of this property of BVED is not yet established. In situations where the failure of one component weakens a second component, M-O model is not appropriate. Freund(1961) has proposed model for such situations. However Freund's model does not extend any property of the univariate exponential distribution. Where as Downton's BVED is an extension of univariate exponential distribution in which the concept of failure due to successive damage is generalised. First consider a single component which is subjected to shocks that occur according to a poisson process with failure rate λ. Suppose further that the probability that a shock is "fatal" is $1-\nu$, where $0< \nu <1$ independent of previous shocks, then the number of shocks N till the component fails is geometrically distributed. The time to failure T has exponential distribution with parameter $\lambda = (1 - \nu)\mu$. A proof of this result is given in Section 2.2.

Downton extended this idea to two component system in which each component is subjected to the shocks. There are two types of shocks say Type-1 and Type-2 arrive with random inter-arrival times $\{ X_n, n \geq 1 \}$ and $\{ Y_n, n \geq 1 \}$ having exponential distribution with means $1/\lambda_1$ and $1/\lambda_2$ respectively.

A Type-1 shock causes only failure to Component-1 and Type-2 shock causes only failure to Component-2. Let N_1 and N_2 respectively be the number of shocks required for the corresponding component to fail. Suppose that $(N_1, N_2)^T$ have a bivariate geometric distribution with joint probability generating function $\Pi(z_1, z_2)$. Also suppose that the two shock processes are independent of each other. Let $T_1 = \sum_{i=1}^{N_1} X_i$ and $T_2 = \sum_{j=1}^{N_2} Y_j$ be the failure times of Components 1 and 2 respectively and the joint distribution function of the life times of the two components say $F(t_1, t_2)$ is a bivariate exponential distribution. Downton considered a particular form of p.g.f. of bivariate geometric random vector $(N_1, N_2)^T$ and the resulting bivariate exponential distribution is called as Downton's model. Since Downton's model has only three parameters. It may not be fit for some data. It is necessary to construct another model for general distribution with more parameters.

Hawkes has considered a different bivariate geometric distribution. Suppose that P_{ij} be the probability that the first component is in state i and second component is in state j with $\sum_{i,j} P_{ij} = 1$, where $i,j \in \{0, 1\}$. Note that '0' and '1' stands for component is in failure state and functioning state respectively with $P_{11} + P_{10} = P_1$; $P_{11} + P_{01} = P_2$; $Q_1 = 1 - P_1$ and $Q_2 = 1 - P_2$.

A sequence of shocks occur according to Poisson process with

N_1 : be the number of shocks required to cause failure of exactly one of the component and

N_2 : be the number of shocks required to cause failure to the surviving component.

Hawkes bivariate exponential distribution consists five parameters which are used in the construction of bivariate exponential distribution. By putting $P_1 = P_2 = P = 1-Q$ and taking the limit as $P_{00} \rightarrow Q$ in Hawkes model then it corresponds to the same number of shocks having geometric distribution. The corresponding model is nothing but Downton's model.

In this chapter we discuss the concept of Downton's(1970) BVED and Hawkes(1972) BVED with study of some distributional properties. In Lemma(2.2.1) a property of the univariate exponential distribution and specifications of Downton's BVED is discussed. In Section 2.3 the joint p.d.f. of vector $(T_1, T_2)^T$ for Downton's model is obtained. Section 2.4 deals with some distibutional properties of DBVED compairing with M-O BVED. In Section 2.5 the Laplace transform of Hawkes BVED is derived. Section 2.6 deals with regression property of Hawkes model and finally in Section 2.7 it is verified that Downton's model is a particular case of Hawkes model under some conditions.

2.2 Specification of Downton's model

In this section we derive the bivariate exponential distribution which is due to Downton(1970). As a first step we give a property of the univariate exponential distribution which later is generalised to bivariate case. The following lemma is useful in this context.

Lemma(2.2.1) Let N be a geometric random variable with parameter ν having probability mass funtion

$$P_r[N = n] = \begin{cases} \nu^{n-1}(1-\nu) \; ; \; n \geq 1 , & 0 < \nu < 1 \\ 0 & ; \text{ otherwise.} \end{cases} \qquad (2.2.0)$$

Let $\{ X_k, k \geq 0 \}$ be a sequence of independent random variables such that $X_0 \equiv 0$, $\{X_k, k \geq 1 \}$ are i.i.d. having distribution function F. Define

$$S_n = \sum_{i=0}^{n} X_i \; ; \; n \geq 0 .$$

If N and $\{ X_i \}$ are independent random variables then the Laplace transform corresponding to the distribution function of S_N is given by

$$\tilde{F}_{S_N}(s) = \{ \tilde{F}(s)(1-\nu) \}/\{1 - \nu \tilde{F}(s)\},$$

where $\tilde{F}(s)$ is the Laplace transform of F.

Proof : Let $\{ X_k, k \geq 1 \}$ be sequence of i.i.d. random variables having distribution function F. Let $\tilde{F}$ be the Laplace transform of F, that is

$$\tilde{F}(s) = E\,[\exp(-s\, X_i)]$$

$$= \int_0^{\infty} \exp(-\, sx)\, dF(x). \qquad (2.2.1)$$

Let N be a geometric random variable with parameter ν having probability mass funtion defined in (2.2.0).

Consider $\tilde{F}_{S_N}$ be the Laplace transform of S_N. Then We have

$$\tilde{F}_{S_N}(\, s\,) = E\left[E_N\left\{ \exp(-s\, S_N) \right\} / N\right]$$

$$= E\left[E_N\left\{ \exp(-s \sum_{i=1}^{N} X_i) \right\} / N\right]$$

$$= E_N\left\{ \prod_{i=1}^{N} E\,[\exp(-s\, X_i) \mid N\,] \right\}$$

$$= E_N\left\{ \prod_{i=1}^{N} \tilde{F}(s) \right\}, \text{ since } X_i\text{'s are i.i.d.}$$

$$= E_N\left\{ \tilde{F}(s) \right\}^{N}. \qquad (2.2.2)$$

It may be noted that the right side of (2.2.2) is the probability generating function (p.g.f.) of N evaluated at $\tilde{F}(s)$. Since the p.g.f. $\Pi(z)$ of N is given by

$$\Pi(z) = z(1 - \nu)/(1 - \nu z),$$

We get

$$\tilde{F}_{S_N}(s) = [\, \tilde{F}(s)\, (1 - \nu)\,]/[\, 1 - \nu\, \tilde{F}(s)\,]. \qquad (2.2.3)$$

Hence the lemma. □

Theorem(2.2.1) If $\{ X_n, n \geq 1 \}$ is a sequence of i.i.d. random variables having an exponential distribution with parameter μ and if N is geometric random variable with parameter ν then the distribution of $S_N = \sum_{i=1}^{N} X_i$ is exponential with parameter $\mu(1-\nu)$.

Proof : Since X_i's are i.i.d. exponential random variables with parameter μ, the Laplace transform of the distribution function of X is given by

$$\begin{aligned}\tilde{F}(s) &= E\,[\exp(-s\ X_i)] \\ &= \int_0^\infty \exp(-\,sx)\ dF(x) \\ &= \int_0^\infty \exp(-\,sx)\ \mu \exp(-\,\mu x)\ dx \\ &= \mu \int_0^\infty \exp\{\ -(\mu+s)x\ \}\ dx \\ &= \mu/(\mu + s).\end{aligned}$$

Now by lemma(2.2.1) we write

$$\tilde{F}_{S_N}(s) = \mu(1 - \nu)/[\ \mu(1 - \nu) + s\ ] \qquad (2.2.4)$$

Since the right hand side of (2.2.4) corresponds to the Laplace transform of an exponential distribution with parameter $\mu(1 - \nu)$, by uniqueness theorem of Laplace transform; it follows that the distribution of S_N is exponential with parameter $\mu(1 - \nu)$. □

In what follows is the generalisation of lemma(2.2.1) to two variable case.

Theorem(2.2.2) Let $(N_1, N_2)^T$ be a vector follows a bivariate geometric distribution with p.g.f. $\Pi(z_1, z_2)$. So that

$$\Pi(z_1, 1) = (1 - P_1)z_1/(1 - P_1 z_1)$$

and $$\Pi(1, z_2) = (1 - P_2)z_2/(1 - P_2 z_2) .$$

The interval between the shocks are independent and exponentially distributed with scale parameters λ_1 and λ_2 for each component respectively. The joint distribution of their life times is $F(t_1, t_2)$ and corresponding Laplace transform is $\psi(s_1, s_2)$. If the probability generating function of a bivariate geometric distribution is of the form

$$\Pi(z_1, z_2) = z_1 z_2 /(1 + \alpha + \beta + \gamma - \alpha z_1 - \beta z_2 - \gamma z_1 z_2),$$

Where α, β and γ all are non negative constants. Then the Laplace transform $\psi(s_1, s_2)$ is given by

$$\psi(s_1, s_2) = \mu_1\mu_2/\{ (\mu_1 + s_1)(\mu_2 + s_2) - \rho s_1 s_2\},$$

where $\mu_1 = \lambda_1/(1 + \alpha + \gamma)$, $\mu_2 = \lambda_2/(1 + \beta + \gamma)$

and $\rho = \{ \alpha\beta + \beta\gamma + \alpha\gamma + \gamma + \gamma^2 \}/\{(1 + \alpha + \gamma)(1 + \beta + \gamma) \}$.

Proof : The probability generating function of a bivariate geometric distribution is of the form

$$\Pi(z_1, z_2) = z_1 z_2/(1 + \alpha + \beta + \gamma - \alpha z_1 - \beta z_2 - \gamma z_1 z_2).$$

Let $T = (T_1, T_2)$ be exponentially distributed random variable with scale parameters λ_1 and λ_2 respectively having the Laplace transform $\psi(s_1, s_2)$, which is defined as

$$\psi(s_1, s_2) = E [\exp(-s_1 T_1 - s_2 T_2)] \tag{2.2.5}$$

$$= \int_0^\infty \int_0^\infty \exp(-s_1 t_1 - s_2 t_2)\, dF(t_1, t_2).$$

Using bivariate geometric p.g.f. we write

$$\psi(s_1, s_2) = W_1W_2 \{ 1 + \alpha + \beta + \gamma - \alpha W_1 - \beta W_2 - \gamma W_1 W_2 \}^{-1},$$

where $W_1 = [\lambda_1/(\lambda_1 + s_1)]$ and $W_2 = [\lambda_2/(\lambda_2 + s_2)]$.

That is

$$\psi(s_1, s_2) = \lambda_1\lambda_2 [(\lambda_1 + s_1)(\lambda_2 + s_2)]^{-1} \Big\{(1 + \alpha + \beta + \gamma) - \alpha [\lambda_1/(\lambda_1 + s_1)] - \beta [\lambda_2/(\lambda_2 + s_2)] - \gamma [\lambda_1/(\lambda_1 + s_1)][\lambda_2/(\lambda_2 + s_2)]\Big\}^{-1}$$

$$= \lambda_1\lambda_2 \Big\{(1 + \alpha + \beta + \gamma)(\lambda_1 + s_1)(\lambda_2 + s_2) - \alpha \lambda_1(\lambda_2 + s_2) - \beta\lambda_2(\lambda_1 + s_1) - \gamma \lambda_1\lambda_2\Big\}^{-1}$$

$$= \lambda_1\lambda_2 \Big\{\lambda_1\lambda_2 + (1 + \beta + \gamma)\lambda_1 s_2 + (1 + \alpha + \gamma)s_1\lambda_2 + (1 + \alpha + \beta + \gamma)s_1 s_2\Big\}^{-1}$$

We multiply and divide by $(1 + \beta + \gamma)(1 + \alpha + \gamma)$ so that

$$\psi(s_1, s_2) = \mu_1\mu_2 [\mu_1\mu_2 + \mu_1 s_2 + \mu_2 s_1 + (1 - \rho)s_1 s_2]^{-1},$$

where $\mu_1 = \lambda_1/(1 + \alpha + \gamma)$, $\mu_2 = \lambda_2/(1 + \beta + \gamma)$

and $\rho = \{ \alpha\beta + \beta\gamma + \alpha\gamma + \gamma + \gamma^2 \}\{(1 + \alpha + \gamma)(1 + \beta + \gamma)\}^{-1}$.

Finally,

$$\psi(s_1, s_2) = \mu_1\mu_2/\{(\mu_1 + s_1)(\mu_2 + s_2) - \rho\, s_1 s_2\}. \qquad (2.2.6)$$

The Equation-(2.2.6) is called the Laplace transform of bivariate exponential distribution due to Downton.

In order to have a proper generalisation of the property of univariate exponential distribution given in Theorem(2.2.1). We need bivariate geometric distribution whose marginals are geometric. Then we have to identify a bivariate p.g.f. Π such that, $\Pi(z_1, 1)$ and $\Pi(1, z_2)$ correspond to marginal p.g.f.'s and are of the form considered in the above Theorem(2.2.2). That is

$$\Pi(z_1, 1) = (1 - P_1)z_1/(1 - P_1z_1), \quad (2.2.7)$$

and $\Pi(1, z_2) = (1 - P_2)z_2/(1 - P_2z_2)$. (2.2.8)

One such bivariate p.g.f.

$$\Pi(z_1, z_2) = z_1z_2/\{ 1 + \alpha + \beta + \gamma - \alpha z_1 - \beta z_2 - \gamma z_1 z_2\}, \quad (2.2.9)$$

Where α, β and γ all are non negative constants.

In this case $\Pi(z_1, 1)$ and $\Pi(1, z_2)$ are of the form as given in (2.2.7) and (2.2.8) respectively with $P_1 = (\alpha + \gamma)/(1 + \alpha + \gamma)$ and $P_2 = (\alpha + \beta)/(1 + \alpha + \beta)$. If $(N_1, N_2)^T$ is a vector having the p.g.f. given in Equation (2.2.9) then N_1 and N_2 have geometric distribution with correlation coefficient ρ.

Let $\{ X_n \}$ be a sequence of i.i.d. random variables having an exponential distribution with parameter μ_1 and $\{ Y_n \}$ be a sequence of i.i.d. random variables having an exponential distribution with parameter μ_2.

Define $T_{1n} = \sum_{i=0}^{n} X_i$, $T_{2n} = \sum_{i=0}^{n} Y_i$. Then the joint distribution of the vector $(T_1 = T_{1N}, T_2 = T_{2N})^T$ has a distribution with exponential marginals and is derived in the following section.

2.3 The Property of the DBVED

We derive the distribution corresponding to the Laplace transform obtained in (2.2.6). Hence we state the following Lemma.

Lemma(2.3.1). If $\tilde{f}$ be the Laplace transform corresponding to f. That is

$$\tilde{f}(s) = E[\exp(-st)] = \int_0^\infty \exp(-st)\, dF(t),$$

then

$$\tilde{f}(s - a) = \int_0^\infty \exp(-st)\exp(at)\, dF(t).$$

So that the inverse of the Laplace transform of $\tilde{g}$, Where $\tilde{g}(s) = \tilde{f}(s - a)$ is given by $f(t)\exp(at)$.

Theorem (2.3.1). The joint density function corresponding to the Laplace transform $\psi(s_1, s_2)$ obtained in (2.2.6) is given by

$$f(t_1, t_2) = \mu_1\mu_2/(1 - \rho) \exp\left\{ - (\mu_1 t_1 + \mu_2 t_2)/(1 - \rho)\right\} I_0[2(1 - \rho)^{-1}(\rho\mu_1\mu_2 t_1 t_2)^{1/2}],$$

with $\mu_1, \mu_2 > 0$ and $0 < \rho < 1$. Where I_0 is the modified Bessel function of first kind of order zero.

Proof : Consider the Laplace transform given in (2.2.6)

$$\begin{aligned}\psi(s_1, s_2) &= \mu_1\mu_2/\{(\mu_1 + s_1)(\mu_2 + s_2) - \rho s_1 s_2\}\\ &= \mu_1\mu_2/\{\mu_2(\mu_1 + s_1) + \mu_1 s_2 + s_1 s_2 - \rho s_1 s_2\}\\ &= \mu_1\mu_2/\{(\mu_1 + s_1)\mu_2 + [\mu_1 + (1 - \rho)s_1]s_2\}.\end{aligned}$$

That is

$$\psi(s_1, s_2) = \mu_1\mu_2/E1 \left\{(\mu_1 + s_1)\mu_2/E1 + s_2\right\}, \qquad (2.3.1)$$

where $E1 = \mu_1 + (1 - \rho)s_1$. Inverting $\psi(s_1, s_2)$ with respect to s_2 by treating s_1 as a constant, we get the inverse Laplace transform with respect to s_2 as follows

$$\begin{aligned}L^{-1}(s_2) &= \mu_1\mu_2/E1 \exp\left\{- [(\mu_1 + s_1)\mu_2/E1]\, t_2\right\}\\ &= \mu_1\mu_2/E1 \exp\left\{- B1\, t_2\right\}, \qquad (2.3.2)\end{aligned}$$

Where $B1 = (\mu_1 + s_1)\mu_2/E1$. Now B1 can be simplified as

$$B1 = \left\{\mu_2/(1-\rho)\right\}\left\{1 - [\rho\mu_1/((1-\rho)(\mu_1/(1-\rho) + s_1))]\right\}$$

$$= \mu_2/(1-\rho) - [\rho\mu_1\mu_2/((1-\rho)^2 D1)],$$

Where $D1 = \mu_1/(1-\rho) + s_1$. So that $L^{-1}(s_2)$ is to be writen as

$$L^{-1}(s_2) = \mu_1\mu_2/E1 \exp\left\{(-\mu_2/(1-\rho) + [\rho\mu_1\mu_2/((1-\rho)^2 D1)])t_2\right\}$$

$$= \left\{\mu_1\mu_2/\{(1-\rho)\, D1\}\right\} \exp\left\{-\mu_2 t_2/(1-\rho)\right\}$$

$$\exp\left\{[\{\rho\mu_1\mu_2 t_2/(1-\rho)^2\}]/D1\right\}.$$

$$= \mu_1\mu_2/(1-\rho) \exp\left\{-\mu_2 t_2/(1-\rho)\right\} \left\{D1^{-1} \exp(\alpha_1 t_2/D1)\right\}$$

where $\alpha_1 = \rho\mu_1\mu_2/(1-\rho)^2$. By using Lemma-(2.3.1), we write

$$\exp\left\{\mu_1 t_1/(1-\rho)\right\} f(t_1, t_2) = \mu_1\mu_2/(1-\rho) \exp\left\{-\mu_2 t_2/(1-\rho)\right\}$$

$$L^{-1}\left\{D1^{-1} \exp(\alpha_1 t_2/D1)\right\}.$$

That is

$$f(t_1, t_2) = \mu_1\mu_2/(1-\rho) \exp\{-(\mu_1 t_1 + \mu_2 t_2)/(1-\rho)\}$$

$$L^{-1}\left\{D1^{-1} \exp(\alpha_1 t_2/D1)\right\}.$$

Using Erdelyi et al.(1954, p.245, Equation 35) we have

$$f(t_1, t_2) = \mu_1\mu_2/(1-\rho) \exp\left\{-(\mu_1 t_1 + \mu_2 t_2)/(1-\rho)\right\}$$

$$I_0[2(1-\rho)^{-1}(\rho\mu_1\mu_2 t_1 t_2)^{1/2}]. \quad (2.3.4)$$

The Equation (2.3.4) gives the density of the bivariate vector corresponding to Downton's model. In this way the marginals are exponential and hence we call the distribution a bivariate exponential distribution .

In the following section we will study some properties of Downton's model and compare with Marshall-Olkin model.

2.4 Properties of DBVED

In this section some important properties of DBVED are given. That is, we obtain correlation coefficient, k^{th} order conditional moment about zero of T_1 given $T_2 = t_2$, conditional expectation of T_1 given $T_2 = t_2$ for DBVED and the same for M-O BVED .

Theorem(2.4.1). If $(T_1, T_2)^T$ is a random vector having the DBVED. Then we have

$$E(T_1) = \mu_1^{-1}, \quad E(T_2) = \mu_2^{-1}, \ Var(T_1) = \mu_1^{-2},$$

$$Var(T_1) = \mu_2^{-2} \quad \text{and} \quad Corr(T_1, T_2) = \rho.$$

Proof : Consider the Laplace transform given in Equation (2.2.6)

$$\psi(s_1, s_2) = \mu_1\mu_2/\{(\mu_1 + s_1)(\mu_2 + s_2) - \rho\, s_1 s_2\}$$

We know that

$$E(T_i) = (-1)\ \partial\psi(s_1, s_2)/\partial s_i \ ; \qquad E(T_i^2) = \partial^2\psi(s_1, s_2)/\partial s_i^2 \ ;$$

and $$E(T_1 T_2) = \partial^2\psi(s_1, s_2)/\partial s_1\partial s_2$$

are evaluated at $s_i = 0$; $i = 1, 2$.

Since $$\partial\psi(s_1, s_2)/\partial s_1 = -\ \mu_1\mu_2(\mu_2 + (1 - \rho)s_2)/V^2, \qquad (2.4.1)$$

$$\partial\psi(s_1, s_2)/\partial s_2 = -\ \mu_1\mu_2(\mu_1 + (1 - \rho)s_1)/V^2, \qquad (2.4.2)$$

$$\partial^2\psi(s_1, s_2)/\partial s_1^2 = 2\ \mu_1\mu_2(\mu_2 + (1 - \rho)s_2)^2/V^3, \qquad (2.4.3)$$

$$\partial^2\psi(s_1, s_2)/\partial s_2^2 = 2\ \mu_1\mu_2(\mu_1 + (1 - \rho)s_1)^2/V^3, \qquad (2.4.4)$$

and

$$\partial^2\psi(s_1, s_2)/\partial s_1\partial s_2 = 2\mu_1\mu_2\{\mu_1\mu_2 + \mu_1(1 - \rho)s_2 + \mu_2(1 - \rho)s_1$$
$$(1 - \rho)^2\, s_1 s_2\}/V^4 \qquad (2.4.5)$$

Where $V = (\mu_1 + s_1)(\mu_2 + s_2) - \rho s_1 s_2$. On simplification we write $E(T_i) = \mu_i^{-1}$, $Var(T_i) = \mu_i^{-2}$, $i = 1, 2$

and $E(T_1 T_2) = (1 + \rho)\mu_1^{-1}\mu_2^{-1}$.

Thus we get

$$Cov(T_1, T_2) = \rho \mu_1^{-1}\mu_2^{-1} \quad \text{and} \quad Corr(T_1, T_2) = \rho. \qquad (2.4.6)$$

□

Theorem(2.4.2). Let $\psi(s_1, s_2)$ be Laplace transform given in Equation (2.2.6). Then

$$E^*[T_1^k|t_2] = \int_0^\infty E(T_1^k|t_2) \exp(-st_2)dt_2,$$

Where $E^*[T_1^k|t_2]$ is the k^{th} order conditional moment about zero of T_1 given $T_2 = t_2$. That is

$$(-1)^k \partial^k \psi(s_1, s_2)/\partial s_1^k\Big|_{s_1 = 0} = E[T_1^k \exp(-s_2 T_2)]$$

$$= \mu_2 E^*[T_1^k|(\mu_2 + s_2)]$$

$$\text{and} \quad E^*[T_1^k|t_2] = (k!)[t_2(1-\rho) + \rho\mu_2]^k/(\mu_1^k t_2^{k+1}).$$

Proof : In order to obtain $E^*[T_1^k|t_2]$, we have to consider the Laplace transform given in (2.2.6). By differentiating Equation (2.2.5) with respect to s_1 on both sides we get

$$\left\{-\mu_1\mu_2(\mu_2 + (1-\rho)s_2)\right\}\left\{\mu_1(\mu_2 + s_2) + s_1\mu_2 + (1-\rho) s_1 s_2\right\}^{-2}$$

$$= -E\left\{T_1 \exp(-s_1 T_1 - s_2 T_2)\right\}.$$

Putting $s_1 = 0$ we get

$$\left\{-\mu_1\mu_2(\mu_2 + (1-\rho)s_2)\right\}\left\{\mu_1(\mu_2 + s_2)\right\}^{-2}$$

$$= -E\left\{T_1 \exp(-s_2 T_2)\right\}. \qquad (2.4.7)$$

Consider right hand side of (2.4.7)

$$\text{R.H.S.} = E\left\{ T_1 \exp(-s_2T_2)\right\}$$

$$= E\Big[E\,[T_1 \exp(-s_2T_2)\,|\,T_2 = t_2] \Big]$$

$$= \int_0^\infty \int_0^\infty t_1 \exp(-s_2t_2)\; f(t_1, t_2)\, dt_1\, dt_2$$

$$= \int_0^\infty \left\{\exp(-s_2t_2)\; f(t_2) \int_0^\infty t_1\, f(t_1 |\, t_2)\, dt_1\right\} dt_2$$

$$= \int_0^\infty [\, f(t_2)\; E(T_1|t_2)\,]\, \exp(-s_2t_2)\; dt_2.$$

Hence Equation (2.4.7) becomes

$$\left\{-\mu_1\mu_2(\mu_2 + (1-\rho)s_2)\right\}\left\{\mu_1(\mu_2+s_2)\right\}^{-2}$$

$$= \int_0^\infty [\, f(t_2)\; E(T_1|t_2)\,]\, \exp(-s_2t_2)\, dt_2. \qquad (2.4.8)$$

Therefore

$$\int_0^\infty [\, f(t_2)\; E(T_1|t_2)\,]\, \exp(-s_2t_2)\, dt_2$$

$$= \mu_2 \int_0^\infty \exp(-[\mu_2 + s_2]t_2)\; E(T_1|t_2)\, dt_2.$$

Thus

$$E\left\{T_1 \exp(-s_2T_2)\right\} = \mu_2\; E^*(T_1|(\mu_2 + s_2)). \qquad (2.4.9)$$

Hence Equation (2.4.7) becomes

$$E^*(T_1|(\mu_2 + s_2)) = \{(\mu_2 + s_2)(1-\rho) + \rho\,\mu_2\}\; [\mu_1(\mu_2 + s_2)]^{-1} \qquad (2.4.10)$$

Now differentiating Equation (2.4.7) with respect to s_1 and putting $s_1 = 0$ we get

$$\left\{2\mu_1\mu_2\; [\mu_2 + (1-\rho)s_2]^2\right\}$$

$$\left\{\mu_1(\mu_2 + s_2) + s_1\mu_2 + (1-\rho)\; s_1s_2\right\}^{-3}\Big|_{s_1 = 0}$$

$$= -E\left\{ T_1^2\; \exp(-s_1T_1 - s_2T_2)\right\}. \qquad (2.4.11)$$

That is

$$\mu_2 (2!) [(\mu_2 + s_2)(1 - \rho) + \rho \mu_2]^2 [\mu_1^2(\mu_2 + s_2)^3]^{-1}$$

$$= (-1)^2 \mu_2 E^*(T_1^2 | (\mu_2 + s_2))$$

Therefore

$$E^*(T_1^2|(\mu_2 + s_2)) = (2!)[(\mu_2 + s_2)(1 - \rho) + \rho \mu_2]^2 [\mu_1^2(\mu_2 + s_2)^3]^{-1} \tag{2.4.12}$$

In general if we differentiate the Laplace transform given in Equation (2.2.6) with respect to s_1; k times we get

$$(-1)^k \partial^k \psi(s_1, s_2)/\partial s_1^k \Big|_{s_1 = 0} = (-1)^{2k} E[T_1^k \exp(- s_2 T_2)]$$

By comparing Equations (2.4.10) and (2.4.12) we write

$$E^*(T_1^k|(\mu_2 + s_2)) = (k!) [(\mu_2 + s_2)(1 - \rho) + \rho \mu_2]^k [\mu_1^k (\mu_2 + s_2)^{k+1}]^{-1}. \tag{2.4.13}$$

By substituting $\mu_2 + s_2 = t_2$ in Equation (2.4.13) it reduces to

$$E^*(T_1^k|t_2) = (k!) [t_2(1 - \rho) + \rho \mu_2]^k [\mu_1^k t_2^{k+1}]^{-1}. \tag{2.4.14}$$

Hence the result follows. □

Now we obtain conditional mean and conditional variance of T_1 given $T_2 = t_2$ in the following Theorem.

Theorem(2.4.3). If $\psi(s_1, s_2)$ is the Laplace transform given in (2.2.6) then the conditional expectation and conditional variance of T_1 given $T_2 = t_2$ is given by

$$E(T_1|t_2) = (1 - \rho) \mu_1^{-1} + \rho \mu_2 t_2 \mu_1^{-1}$$

and

$$Var(T_1|t_2) = (1 - \rho) \mu_1^{-1} \left\{(1 - \rho) \mu_1^{-1} + 2 \rho \mu_2 t_2 \mu_1^{-1}\right\}$$

respectively.

Proof : Consider Equation (2.4.8) which can be written as

$$\left\{\mu_2\mu_1[\rho\mu_2 + (1-\rho)(\mu_2 + s_2)]\right\}[\mu_1\mu_2 + \mu_1 s_2]^{-2} \qquad //$$

$$= \int_0^\infty h(t)\exp(-s_2 t_2)\,dt_2,$$

Where $h(t) = f(t_2)\,E(T_1|t_2)$. Hence

$$(1-\rho)\mu_1^{-1}[\mu_2(\mu_2+s_2)^{-1}] + \left\{\rho\,\mu_1^{-1}[\mu_2(\mu_2+s_2)^{-1}]^2\right\}$$

$$= \int_0^\infty h(t)\exp(-s_2 t_2)\,dt_2$$

That is

$$(1-\rho)\,\mu_1^{-1}\,\mu^* + \left\{\rho\,\mu_1^{-1}\,\mu^{*2}\right\} = \int_0^\infty h(t)\,Q_1^*\,dt_2,$$

Where $\mu^* = [\mu_2(\mu_2+s_2)^{-1}]$ and $Q_1^* = \exp(-s_2 t_2)$.

By inverting with respect to s_2 we write

$$h(t) = (1-\rho)\,\mu_1^{-1}[\mu_2\exp(-(\mu_2 t_2))] + \rho\,\mu_1^{-1}\{\mu_2^2\exp(-(\mu_2 t_2))\}$$

$$= (1-\rho)\,\mu_1^{-1}\,\mu_2\,Z1 + \rho\,\mu_1^{-1}\,\mu_2^2\,Z1.$$

Where $Z1 = \exp(-(\mu_2 t_2))$.

That is

$$\mu_2\,Z1\,E(T_1|t_2) = \left\{(1-\rho)\,\mu_1^{-1} + \rho\,\mu_1^{-1}\,\mu_2 t_2\right\}\mu_2\,Z1$$

Therefore

$$E(T_1|t_2) = \left\{(1-\rho)\,\mu_1^{-1} + \rho\,\mu_1^{-1}\mu_2 t_2\right\} \qquad (2.4.15)$$

Now differenting Equation (2.4.1) with respect to s_1 on both sides yields

$$\left\{2\mu_1\mu_2[\mu_2 + (1-\rho)s_2]^2\right\}$$

$$\left\{\mu_1(\mu_2+s_2) + s_1\mu_2 + (1-\rho)\,s_1 s_2\right\}^{-3}\Big|_{s_1=0}$$

$$= -E\left\{T_1^2\,\exp(-s_1T_1 - s_2T_2)\right\}. \qquad (2.4.16)$$

Compairing with right hand side of (2.4.8) we have

$$\int_0^{\infty} [\ f(t_2)\ E(T_1^2|t_2)\ ]\ \exp(-\ s_2 t_2)\ dt_2$$

$$= 2\ \mu_1^{-2}\ \mu^{*}\ \left\{(1\ -\ \rho)\ +\ \rho\ \mu^{*}\ \right\}^2.$$

That is

$$\left\{2(1\ -\ \rho)^2\ \mu_1^{-2}\ \mu^{*}\right\}\ +\ \left\{2\rho^2\ \mu_1^{-2}\ \mu^{*^3}\right\}$$

$$+\ \left\{4\rho(1\ -\ \rho)\ \mu_1^{-2}\ \mu^{*^2}\ \right\}\quad =\int_0^{\infty} h(t)\ Q_1^{*}\ dt_2.$$

By inverting with respect to s_2 we get

$$h(t)\ =\ \{2(1\ -\ \rho)^2\ \mu_1^{-2}\ \mu_2\ Z1\}\ +\ \{\rho^2\ \mu_1^{-2}\mu_2^3\ Z1\ \ t_2^2\}$$

$$+\ \{4\rho\ (1\ -\ \rho)\ \mu_1^{-2}\ \mu_2^2\ Z1\ \ t_2\}.$$

That is

$$\mu_2\ Z1\ E(T_1^2|t_2)$$

$$=\mu_2 Z1\ \ \mu_1^{-2}\left\{2(1\ -\ \rho)^2\ +\ \rho^2\ \mu_2^2\ t_2^2\ +\ 4\rho\ (1\ -\ \rho)\ \mu_2\ t_2\right\}.$$

That is

$$E(T_1^2|t_2)\ =\ \mu_1^{-2}\ \left\{2(1\ -\ \rho)^2\ +\ \rho^2\ \mu_2^2\ t_2^2\ +\ 4\rho\ (1\ -\ \rho)\ \mu_2 t_2\right\}\tag{2.4.17}$$

Therefore after simplification we get an expression for the conditional variance of T_1 given $T_2\ =\ t_2$ as

$$Var(T_1|t_2)\ =\ (1\ -\ \rho)\ \mu_1^{-1}\ \left\{(1\ -\ \rho)\ \mu_1^{-1}\ +\ 2\ \rho\ \mu_2\ t_2\ \mu_1^{-1}\right\}\tag{2.4.18}$$

□

In order to study some properties of M-O BVED. We obtain first Laplace transform for M-O model and we derive the conditional expectation for the same in following two Theorems.

Theorem(2.4.4). If $(T_1,\ T_2)^T$ is a random vector having the M-O BVED with parameters λ_1, λ_2 and λ_3 respectively. Then the

Laplace transform is given by

$$\psi_M(s_1, s_2) = \left\{ \mu_1\mu_2/\{(\mu_1 + s_1)(\mu_2 + s_2)\}\right\} \left\{1 + \rho(\mu_1 + \mu_2) s_1 s_2 \mu_1^{-1}\mu_2^{-1} (\mu_1 + \mu_2 + (1 - \rho)(s_1 + s_2))\right\}$$

Where $\mu_1 = \lambda_1 + \lambda_3$, $\mu_2 = \lambda_2 + \lambda_3$ and $\rho = \lambda_3/\lambda$.

Proof : Let $(T_1, T_2)^T$ is a random vector having the M-O BVED with parameters λ_1, λ_2 and λ_3 respectively having survival function $\bar{F}(t_1, t_2)$. By Equation (2.2.5) we write

$$\psi_M(s_1, s_2) = E[\exp(- s_1 T_1 - s_2 T_2)]$$

$$= \int_0^\infty\int_0^\infty \exp(-s_1 t_1 - s_2 t_2)\, d\bar{F}_M(t_1, t_2), \qquad (2.4.19)$$

Where

$$\bar{F}_M(t_1, t_2) = \begin{cases} \exp\{-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_3 t_1\} & ; \text{if } t_1 > t_2 \\ \exp\{-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_3 t_2\} & ; \text{if } t_1 < t_2. \end{cases}$$

Note that Young(1917) has given the result on integration by parts in two or more dimensions. Here we state that result as follows

If $G(0, t_2) \equiv 0 \equiv G(t_1, 0)$ and G is of bounded variation on finite intervals then

$$\int_0^\infty \int_0^\infty G(t_1, t_2)\, d\bar{F}(t_1, t_2) = \int_0^\infty \int_0^\infty \bar{F}(t_1, t_2)\, dG(t_1, t_2) .$$

This change is of particular use, when $G(t_1, t_2)$ is absolutely continuous and $\bar{F}(t_1, t_2)$ is easy to compute. To satisfy the conditions for G, we replace $\exp(- s_1 t_1 - s_2 t_2)$ of the Laplace transform by

$$G(t_1, t_2) = [1 - \exp(-s_1 t_1)]\ [1 - \exp(-s_2 t_2)].$$

By using result due to Young(1917) in Equation (2.4.19) the Laplace transform of M-O BVED becomes

$$\psi_M(s_1, s_2) = \phi(s_1, s_2) - \phi(\infty, s_2) - \phi(s_1, \infty) + 1, \qquad (2.4.20)$$

Where $\phi(s_1, s_2)$, $\phi(\infty, s_2)$ and $\phi(s_1, \infty)$ are given by

$$\phi(s_1, s_2) = \int_0^\infty \int_0^\infty [1 - \exp(-s_1 t_1)]\ [1 - \exp(-s_2 t_2)]\ dF(t_1, t_2)$$

$$= \int_0^\infty \int_0^\infty \bar{F}(t_1, t_2)\ s_1\ s_2\ \exp(-s_1 t_1 - s_2 t_2)\ dt_1\ dt_2$$

$$= s_1\ s_2 \int_0^\infty \exp(-s_1 t_1)\ dt_1 \left\{ \int_0^\infty \bar{F}(t_1, t_2)\ \exp(-s_2 t_2)\ dt_2 \right\}$$

$$= s_1\ s_2 \left\{ \int_0^\infty \left\{ \int_{t_1}^\infty \exp(-a_1 t_2)\ dt_2 \right\} \exp(-a_2 t_1)\ dt_1 + \int_0^\infty \left\{ \int_0^{t_2} \exp(-a_3 t_2)\ dt_2 \right\} \exp(-a_4 t_1)\ dt_1 \right\},$$

Where $a_1 = \lambda_2 + \lambda_3 + s_2$, $a_2 = \lambda_1 + s_1$, $a_3 = \lambda_2 + s_2$, and $a_4 = \lambda_1 + \lambda_3 + s_1$. That is

$$\phi(s_1, s_2) = s_1\ s_2 \left\{ a_1^{-1} \int_0^\infty \exp(-b_1 t_1)\ dt_1 + a_3^{-1} \int_0^\infty \exp(-b_2 t_1)\ dt_1 - a_3^{-1} \int_0^\infty \exp(-b_1 t_1)\ dt_1 \right\},$$

Where $b_1 = \lambda + s_1 + s_2$, $b_2 = \lambda_1 + \lambda_3 + s_1$ and $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

Hence

$$\phi(s_1, s_2) = s_1\ s_2 \left\{ a_1^{-1}\ b_1^{-1} + a_3^{-1}\ [\ a_3^{-1} - b_1^{-1}\ ] \right\}$$

$$= s_1\ s_2\ b_1^{-1} \left\{ a_1^{-1} + a_3^{-1} \right\}.$$

Thus

$$\phi(s_1, s_2) = s_1 s_2 a_1^{-1} b_1^{-1} a_3^{-1} [b_1 + \lambda_3] . \tag{2.4.21}$$

Now

$$\phi(\infty, s_2) = \lim_{s_1 \to \infty} [\phi(s_1, s_2)] = s_2 / a_1,$$

$$\phi(s_1, \infty) = \lim_{s_2 \to \infty} [\phi(s_1, s_2)] = s_1 / a_3,$$

By substituting $\phi(s_1, s_2)$, $\phi(\infty, s_2)$ and $\phi(s_1, \infty)$ in Equation (2.4.20) we get

$$\psi_M(s_1, s_2) = a_1^{-1} b_1^{-1} a_3^{-1} [b_1(s_1 s_2 - a_3 s_2 - a_1 s_1 + a_1 a_3) + \lambda_3 s_1 s_2]$$

On simplification it reduces to

$$\psi_M(s_1, s_2) = a_1^{-1} b_1^{-1} a_3^{-1} [(\lambda_1 + \lambda_3) (\lambda_2 + \lambda_3) b_1 + \lambda_3 s_1 s_2]$$

$$= [\mu_1 \mu_2 b_1 + s_1 s_2 \lambda_3] [(\mu_1 + s_1)(\mu_2 + s_2) b_1]^{-1} \tag{2.4.22}$$

Where $\mu_1 = \lambda_1 + \lambda_3$ and $\mu_2 = \lambda_2 + \lambda_3$.

The equation (2.4.22) can be written as

$$\psi_M(s_1, s_2) = \mu_1 \mu_2 [(\mu_1 + s_1)(\mu_2 + s_2)]^{-1} \left\{ 1 + \rho s_1 s_2 \lambda \mu_1^{-1} \mu_2^{-1} b_1^{-1} \right\} \tag{2.4.23}$$

Consider

$$b_1/\lambda = (\lambda + \lambda_3)(\lambda + \lambda_3)^{-1}(s_1 + s_2) \lambda^{-1} + 1$$

$$= 1 + (1 + \rho)(s_1 + s_2)(\mu_1 + \mu_2)^{-1}$$

$$= [\mu_1 + \mu_2 + (1 + \rho)(s_1 + s_2)] (\mu_1 + \mu_2)^{-1}.$$

By substituting in Equation (2.4.23) we get

$$\psi_M(s_1, s_2) = \left\{ \mu_1 \mu_2 / [(\mu_1 + s_1)(\mu_2 + s_2)] \right\} \left\{ 1 + \rho (\mu_1 + \mu_2) s_1 s_2 \mu_1^{-1} \mu_2^{-1} (\mu_1 + \mu_2 + (1 - \rho)(s_1 + s_2))^{-1} \right\} \tag{2.4.24}$$

Hence the result follows. □

In the following, using the Laplace transform of M-O BVED as given in Equation (2.4.), we can easily obtain conditional expectation of T_1 given $T_2 = t_2$.

Theorem(2.4.5). If $(T_1, T_2)^T$ is a random vector having the M-O BVED with Laplace transform given in Equation (2.4.24). Then the conditional expectation of T_1 given $T_2 = t_2$ is given by

$$E(T_1|t_2) = (1+\rho)\,[\mu_1 - \rho\,\mu_2]^{-1} - \left\{\rho\,(\mu_1+\mu_2)^2\,(\mu_1\mu_2(1+\rho)\,[\mu_1-\rho\,\mu_2])^{-1} \exp(-\,[\,(\mu_1-\rho\,\mu_2)/(1+\rho)]\,t_2)\right\}.$$

Proof : In order to obtain conditional expectation, we differentiate the Laplace transform given in Equation (2.4.24) with respect to s_1 and letting $s_1 = 0$. Thus

$$\partial\psi_M(s_1, s_2)/\partial s_1 = \left\{\mu_1\mu_2/(\mu_2+s_2)\right\}\left\{\partial(\mu_1+s_1)^{-1}/\partial s_1\right\} + \left\{\rho(\mu_1+\mu_2)s_2/(\mu_2+s_2)\right\}\partial/\partial s_1\left\{s_1[\,(\mu_1+s_1)(K_1+(1+\rho))]^{-1}\right\}$$

evaluated at $s_1 = 0$; Where $K_1 = \mu_1 + \mu_2 + (1+\rho)\,s_2$.

Therefore

$$\partial\psi_M(s_1, s_2)/\partial s_1 = \left\{\mu_1\mu_2/(\mu_2+s_2)\right\}\left\{-\,(\mu_1+s_1)^{-2}\right\} + \left\{\rho(\mu_1+\mu_2)s_2/(\mu_2+s_2)\right\}\ H_1,$$

Where $$H_1 = \left\{[\,\mu_1K_1 + (\mu_1(1+\rho)+K_1)\,s_1 + s_1^2\,(1+\rho)] - s_1\,[\,\mu_1(1+\rho)+K_1+2(1+\rho)\,s_1]\right\}\left\{\mu_1K_1+(\mu_1(1+\rho)+K_1)s_1+s_1^2\,(1+\rho)\right\}^{-2}. \qquad (2.4.25)$$

Putting $s_1 = 0$ then Equation (2.4.25) we write

$$\partial\psi_M(s_1, s_2)/\partial s_1\Big|_{s_1=0} = -\mu_1^{-1}\Big\{\mu_2/(\mu_2 + s_2)\Big\}\Big\{\rho\,\mu_1^{-1}\,\mu'\,s_2$$

$$\{(\mu' + s_2)(\mu_2 + s_2)\}^{-1}\Big\}, \qquad (2.4.26)$$

Where $\mu' = (\mu_1 + \mu_2)/(1 + \rho)$. Now consider

$$s_2/\{(\mu' + s_2)(\mu_2 + s_2)\} = \{A/(\mu' + s_2)\} + \{B/(\mu_2 + s_2)\}, \qquad (2.4.27)$$

Where A and B are arbitrary constants. Thus

$$s_2 = A\,(\mu_2 + s_2) + B\,(\mu' + s_2) \qquad (2.4.28)$$

Substituting $s_2 = -\mu_2$ and $s_2 = -\mu'$ in (2.4.28) respectively we get

$$A = -\mu'/(\mu_2 - \mu') = -\mu_2(1 + \rho)/(\mu_1 - \rho\mu_2)$$

and

$$B = -\mu_2/(\mu' - \mu_2) = (\mu_1 + \mu_2)/(\mu_1 - \rho\mu_2).$$

By substituting the values of A and B in Equation (2.4.27) and using this expression in Equation (2.4.26) yields

$$\partial\psi_M(s_1, s_2)/\partial s_1\Big|_{s_1=0} = -\mu_2(1 + \rho)/\{(\mu_1 - \rho\mu_2)(\mu_2 + s_2)\}$$
$$+ \{\rho(\mu_1 + \mu_2)^2/[\mu_1(1 + \rho)(\mu_1 - \rho\mu_2)]\}$$
$$\{(\mu_1 - \rho\mu_2)/(1 + \rho) + (\mu_2 + s_2)\}^{-1}.$$

Using the form of Equation (2.4.3) the above expression can be written as

$$-\int_0^\infty [\,f(t_2)\,E(T_1|t_2)\,]\exp(-s_2t_2)\,dt_2$$

$$= -\mu_2(1 + \rho)/\{(\mu_1 - \rho\mu_2)(\mu_2 + s_2)\}$$
$$+ \{\rho(\mu_1 + \mu_2)^2/[\mu_1(1 + \rho)(\mu_1 - \rho\mu_2)]\}$$
$$\{(\mu_1 - \rho\mu_2)/(1 + \rho) + (\mu_2 + s_2)\}^{-1}.$$

By inverting with respect to s_2 we write

$$f(t_2)\,E(T_1|t_2) = \mu_2(1+\rho)\exp(-\mu_2 t_2)/(\mu_1-\rho\mu_2)$$
$$- \{\rho(\mu_1+\mu_2)^2/[\mu_1(1+\rho)(\mu_1-\rho\mu_2)]\}$$
$$\exp\left\{-[(\mu_1-\rho\mu_2)/(1+\rho)+\mu_2]\,t_2\right\}.$$

Therefore

$$E(T_1|t_2) = (1+\rho)/(\mu_1-\rho\mu_2)$$
$$- \{\rho(\mu_1+\mu_2)^2/[\mu_1(1+\rho)(\mu_1-\rho\mu_2)]\}$$
$$\exp\left\{-[(\mu_1-\rho\mu_2)/(1+\rho)]\,t_2\right\}. \qquad (2.4.29)$$

The Equation (2.4.29) gives the required expression for conditional expectation of T_1 given $T_2 = t_2$ for M-O BVED model.

In the following we will discuss about the Hawkes model.

2.5. DERIVATION OF HAWKES BVED.

We have discussed the probability generating function of bivariate geometric distribution in Section 2.2. Using the form of probability generating function as given in Equation (2.2.9) containing five parameters, Downton used that form of p.g.f.which reduced into distribution which has only three parameters.

Hawkes used following idea for consruction of BVED. Let us define two events A_1 and A_2 with joint probabilities given by

	A_1	$\bar{A}_1$	
A_2	P_{11}	P_{01}	P_2
$\bar{A}_2$	P_{10}	P_{00}	Q_2
	P_1	Q_1	

(2.5.1)

where $P_1 + Q_1 = P_2 + Q_2 = 1$.

There are number of independent trials denoted by T_i ($i = 1,2.$) which are completed up to the first occurence of the event A_1 and A_2.

In this Section we derive the bivariate exponential distribution which is due to Hawkes(1972). First step is to obtain Laplace transform $\phi_H(s_1, s_2)$. As we discussed in Equation (2.5.1) the p.g.f. is given by

$$\Pi(z_1, z_2) = z_1 z_2 \Big\{P_{11} + P_{10}P_2 z_2 (1 - Q_2 Z_2)^{-1}$$

$$+ P_{01}P_1 z_1 (1 - Q_1 z_1)^{-1} + P_{00} \Pi(z_1, z_2)\Big\}. \qquad (2.5.2)$$

That is

$$\Pi(z_1, z_2)[1 - P_{00}] = z_1 z_2 \Big\{(1 - Q_2 Z_2)^{-1} (1 - Q_1 z_1)^{-1}\Big\}$$

$$\times \Big\{[P_{11}(1 - Q_2 Z_2)(1 - Q_1 z_1)] + [P_{10} P_2 z_2$$

$$(1 - Q_1 z_1)] + [(1 - Q_2 Z_2) P_{01} P_1 z_1]\Big\}.$$

That is

$$\Pi(z_1, z_2) = z_1 z_2 [1 - P_{00}]^{-1} (1 - Q_2 Z_2)^{-1} (1 - Q_1 z_1)^{-1}$$

$$\Big\{P_{11} - z_1(P_{11}Q_1 - P_{01} P_1) - z_2(P_{11}Q_2 - P_{10}P_2)$$

$$- z_1 z_2 (P_{10}Q_1 P_2 + P_1 P_{01} Q_2 - P_{11} Q_1 Q_2)\Big\}. \qquad (2.5.3)$$

□

In the following we obtain the Laplace transform for Hawkes BVED.

Theorem(2.5.1) If $(T_1, T_2)^T$ is a random vector with p.g.f. given by the Equation (2.5.3). Then the Laplace transform for Hawkes BVED is given by

$$\psi_H(s_1, s_2) = \left\{\mu_1\mu_2/\{(\mu_1 + s_1)(\mu_2 + s_2)\}\right\}$$

$$\left\{1 + [P_{oo} - (1 - P_1)(1 - P_2)]\ s_1 s_2\ ((\mu_1 + P_1 s_1)\right.$$

$$\left.(\mu_2 + P_2 s_2) - \mu_1\mu_2 P_{oo})^{-1}\right\}$$

Where $\mu_i = \lambda_i P_i$, $i = 1, 2$.

Proof : By definition of $\psi_H(s_1, s_2)$ we write

$$\psi_H(s_1, s_2) = E\left\{\exp\{- s_1 T_1 - s_2 T_2\}\right\}$$

$$= \Pi\left\{\lambda_1/(\lambda_1 + s_1),\ \lambda_2/(\lambda_2 + s_2)\right\}$$

by making use of $\mu_i = \lambda_i P_i$, $i = 1,2$ we get

$$\psi_H(s_1, s_2) = \text{NUMERATOR / DENOMINATOR} \qquad (2.5.4)$$

Where

$$\text{NUMERATOR} = \left\{\lambda_1/(\lambda_1 + s_1)\right\}\left\{\lambda_2/(\lambda_2 + s_2)\right\}$$

$$\left\{P_{11} - \lambda_1/(\lambda_1 + s_1)(P_{11}Q_1 - P_{01}P_1) - \lambda_2/(\lambda_2 + s_2)\right.$$

$$(P_{11}Q_2 - P_{10}P_2) - \lambda_1/(\lambda_1 + s_1)\ \lambda_2/(\lambda_2 + s_2)$$

$$\left.(P_{10}Q_1P_2 + P_1P_{01}Q_2 - P_{11}Q_1Q_2)\right\}$$

$$= L1\left\{\lambda_1\lambda_2[P_{11} - P_{11}Q_1 + P_{01}P_{11} - P_{11}Q_2 + P_{10}P_2\right.$$

$$- P_{10}Q_1P_2 - P_1P_{01}Q_2 + P_{11}Q_1Q_2]$$

$$+ \lambda_1 s_2(P_{11} - P_{11}Q_1 + P_{01}P_1)$$

$$\left.+ \lambda_2 s_1(P_{11} - P_{11}Q_2 + P_{10}P_2 + s_1 s_2 P_{11})\right\},$$

where $L1 = \left\{\lambda_1\lambda_2(\lambda_1 + s_1)^{-2}(\lambda_2 + s_2)^{-2}\right\}$

Hence

$$NUMERATOR = L1 \left\{ \lambda_1\lambda_2[P_{11}P_2(1 - Q_1) + P_1P_2(P_{01} + P_{10})] + \lambda_1 s_2 P_1(P_{11}+P_{01}) + \lambda_2 s_1 P_2(P_{11}+P_{10}) + s_1 s_2 P_{11}\right\}$$

$$= L1 \left\{\mu_1\mu_2(1 - P_{00}) + \mu_1 s_2 P_2 + \mu_2 s_1 P_1 + s_1 s_2 P_{11}\right\}$$

$$= L1 \left\{ \mu_1\mu_2 + \mu_1 s_2 P_2 + \mu_2 s_1 P_1 + s_1 s_2 P_1 P_2 - P_1 P_2 s_1 s_2 - \mu_1\mu_2 P_{00} + s_1 s_2 P_{11}\right\}$$

$$= L1 \left\{[(\mu_1 + s_1 P_1)(\mu_2 + P_2 s_2) - \mu_1\mu_2 P_{00}] + s_1 s_2 P_{11} - P_1 P_2 s_1 s_2\right\}$$

Multiplying and dividing by P_1P_2 we get

$$NUMERATOR = \left\{ \mu_1\mu_2 [(\mu_1 + s_1)(\mu_2 + s_2)]^{-1} [(\lambda_1 + s_1)(\lambda_2 + s_2)]^{-1}\right\} \left\{[(\mu_1 + s_1 P_1)(\mu_2 + P_2 s_2) - \mu_1\mu_2 P_{00}] + s_1 s_2 P_{11} - P_1 P_2 s_1 s_2\right\} \quad (2.5.5)$$

And

$$DENOMINATOR = \left\{[1 - Q_1\lambda_1/(\lambda_1 + s_1)] [1 - Q_2\lambda_2/(\lambda_2 + s_2)] [(1 - P_{00}) \lambda_1/(\lambda_1 + s_1) \lambda_2/(\lambda_2 + s_2)]\right\}$$

$$= (\lambda_1 P_1 + s_1)(\lambda_2 P_2 + s_2) [(\lambda_1 + s_1)(\lambda_2 + s_2)]^{-2} \times \left\{\lambda_1\lambda_2 + s_2\lambda_1 + \lambda_2 s_1 + s_1 s_2 - P_{00}\lambda_1\lambda_2\right\}$$

$$= \left\{P_1 P_2 \; [(\mu_1 + s_1)(\mu_2 + s_2)] \, [(\mu_1 + P_1 s_1)(\mu_2 + P_2 s_2)]^{-2}\right\}$$

$$\times \left\{ (\mu_1 + s_1 P_1)(\mu_2 + P_2 s_2) - \mu_1 \mu_2 P_{oo} \right\},$$

Where $\mu_i = \lambda_i P_i$, $i = 1,2$, that is $\mu_i / P_i = \lambda_i$, $i = 1, 2$

Thus

$$\text{DENOMINATOR} = (\mu_1 + s_1)(\mu_2 + s_2)\left\{(\mu_1 + s_1 P_1)(\mu_2 + P_2 s_2)(\lambda_1 + s_1)\right.$$

$$\left.(\lambda_2 + s_2)\right\}^{-1} \left\{(\mu_1 + s_1 P_1)(\mu_2 + P_2 s_2) - \mu_1 \mu_2 P_{oo}\right\} \tag{2.5.6}$$

By substituting (2.5.5) and (2.5.6) in (2.5.4) we write

$$\psi_H(s_1, s_2) = \left\{ \mu_1 \mu_2 / \{(\mu_1 + s_1)(\mu_2 + s_2)\}\right\} \left\{1 + s_1 s_2 \, [P_{11} - P_1 P_2]\right.$$

$$\left.([\mu_1 + P_1 s_1] \, [\mu_2 + P_2 s_2] - \mu_1 \mu_2 P_{oo})^{-1}\right\}$$

$$\psi_H(s_1, s_2) = \left\{\mu_1 \mu_2 / ((\mu_1 + s_1)(\mu_2 + s_2)\right\} \left\{1 + [P_{oo} - (1 - P_1)\right.$$

$$\left.(1 - P_2)] \; s_1 s_2 ((\mu_1 + P_1 s_1)(\mu_2 + P_2 s_2) - \mu_1 \mu_2 P_{oo})^{-1}\right\} \tag{2.5.7}$$

Hence the result follows.

□

In the next section we will study regression property for Hawkes BVED.

2.6 Regression property of Hawkes model

In previous two Sections we have studied regression property for DBVED and M-O BVED. In this Section we study the regression property of Hawkes BVED. So we prove the following Theorem.

Theorem(2.6.1) If $(T_1, T_2)^T$ be a random vector having Hawkes BVED with Laplace transform given in (2.5.7) and $E^*[T_1|s]$ be the Laplace transform with respect to t_2 of the conditional expectation $E[T_1|t_2]$. Then

$$E^*[T_1|s] = [\mu_1(Q_2 - P_{oo})]^{-1} \left\{P_1Q_2s^{-1} - (P_{oo} - Q_1 \quad Q_2)\right.$$

$$\left.(1 - P_{oo}) \; [P_2 \; (\mu_2 \; [Q_2 - P_{oo}]/P_2 + s)]^{-1}\right\}$$

Where $E^*[T_1| \; s] = - \mu_2^{-1} \; \partial \; \psi_H(s_1, \; s - \mu_2)/\partial s_1 \Big|_{s_1 = 0}$

Proof : We substitute $s_2 = s - \mu_2$ in the Laplace transform given in (2.5.7).

By differentiating (2.5.7) with respect to s_1 on both sides and it is evaluated at $s_1 = 0$,

$$\partial \; \psi_H(s_1, \; s - \mu_2)/\partial s_1 \Big|_{s_1 = 0} = - \mu_2\mu_1^{-1}s^{-1} + [(P_{oo} - Q_1Q_2)(s - \mu_2)\mu_2]$$

$$[\mu_1 s \; (\mu_2 + P_2(s - \mu_2) - \mu_2 P_{oo})]^{-1}$$

$$- \mu_2^{-1} \; \psi_H(s_1, \; s - \mu_2)/\partial s_1 \Big|_{s_1 = 0}$$

$$= \mu_1^{-1}s^{-1} + \mu_2 \; G1 \; \mu_1^{-1} \; P_2^{-1}\{s \; [G2 + s]\}^{-1} - G1\{\mu_1 P_2 \; (G2 + s)\}^{-1},$$

Where $G1 = P_{oo} - Q_1Q_2$ and $G2 = \mu_2(Q_2 - P_{oo}) \; P_2^{-1}$.

That is

$$- \mu_2^{-1} \; \partial \; \psi_H(s_1, \; s - \mu_2)/\partial \; s_1 \Big|_{s_1 = 0}$$

$$= \mu_1^{-1}s^{-1} + \mu_2 \; G1 \; \mu_1^{-1} \; P_2^{-1} \left\{s^{-1} \; G2^{-1} - \{G_2 \; (G2 + s)\}^{-1}\right\}$$

$$- G1 \; [\; \mu_1 P_2 \; (G2 + s \;)]^{-1}$$

$$= P_1Q_2 \; [\mu_1(Q_2 - P_{oo}) \; s]^{-1}$$

$$- G1 \; (1 - P_{oo}) \; [(Q_2 - P_{oo}) \; \mu_1(G2 + s \;)]^{-1}$$

On simplification we get

$$E^{*}[T_1|s] = [\mu_1(Q_2 - P_{oo})]^{-1}\Big\{P_1Q_2s^{-1} - (P_{oo} - \; Q_1Q_2)$$

$$(1 - P_{oo}) \; [\mu_2 \; [Q_2 - P_{oo}]/P_2 + s \;\;]^{-1}\Big\}$$

(2.6.1)

Hence the result follows. □

In order to obtain conditional expectation of T_1 given $T_2 = t_2$ for Hawkes BVED. We prove the following Theorem.

Theorem(2.6.2) If the Laplace transform given in (2.5.7). Then the conditional expectation of T_1 given $T_2 = t_2$ is given by

$$E[T_1|t_2] = [\mu_1(Q_2 - P_{oo})]^{-1} \Big\{P_1Q_2 - (P_{oo} - \; Q_1Q_2)$$

$$(1 - P_{oo}) \; P_2^{-1}\Big\} \exp\Big\{- (\mu_2 \; [Q_2 - P_{oo}]/P_2 \;) \; t_2\Big\}$$

$$= \mu_1^{-1}\Big\{1 + C1 \; P_{10}^{-1} \; [\, 1 - (1 - P_{oo}) \; P_2^{-1} \; \exp\{- \lambda_2 P_{10} t_2\}] \Big\},$$

Where $C1 = P_{oo}P_{11} - P_{01}P_{10}$ is the cross ratio in the probability ratio given in (2.5.1).

Proof : Differentiating the Laplace transform given in Equation (2.5.7) with respect to s_1 we get

$$\partial\psi_H(s_1, s_2)/\partial s_1 \Big|_{s_1 = 0}$$

$$= - \mu_2\mu_1^{-1}(\mu_2 + s_2)^{-1} + \mu_1\mu_2 \; G1 \; s_2(\mu_2 + s_2)^{-1}$$

$$\Big\{(\mu_1 + P_1s_1)(\mu_2 + P_2s_2) - \mu_1\mu_2P_{oo} - s_1 \; [P_1(\mu_2 + P_2s_2)]\Big\}\Big\{(\mu_1 + \;\; s_1)^{-2}$$

$$[\,(\mu_1 + P_1s_1)(\mu_2 + P_2s_2) - \mu_1\mu_2P_{oo}]^{-1}\Big\}\Big|_{s_1 = 0},$$

Where $G1 = P_{oo} - Q_1Q_2$.

That is

$$\partial\psi_H(s_1, s_2)/\partial s_1\Big|_{s_1=0}$$

$$= -\mu_2\mu_1^{-1}(\mu_2 + s_2)^{-1} + \mu_2 G1\ \mu_1^{-1}\ P_2^{-1}\Big\{s_2(\mu_2 + s_2)^{-1}(G2 + s_2)^{-1}\Big\}, \tag{2.6.2}$$

Where $G2 = (1 - P_{oo})\ \mu_2 P_2^{-1}$. Consider

$$s_2(\mu_2 + s_2)^{-1}(G2 + s_2)^{-1} = A\ (\mu_2 + s_2)^{-1} + B\ (G2 + s_2)^{-1} \tag{2.6.3}$$

That is $s_2 = A\ (G2 + s_2)^{-1} + B\ (\mu_2 + s_2)^{-1}$ (2.6.4)

By substituting $s_2 = -G2$ and $s_2 = -\mu_2$ in Equation (2.6.4) respectively we get

$$A = -P_1/(Q_1 - P_{oo}) \quad \text{and} \quad B = (1 - p_{oo})/(Q_1 - P_{oo}).$$

Now substitute values of A and B in Equation (2.6.3) then the Equation (2.6.2) can be writen as

$$\partial\psi_H(s_1, s_2)/\partial s_1\Big|_{s_1=0}$$

$$= -\mu_2\ \mu_1^{-1}(\mu_2 + s_2)^{-1} + \Big\{1 + G1/(Q_2 - P_{oo})\Big\}$$

$$- \Big\{G1/(Q_2 - P_{oo})\Big\}\ G2\ (G2 + s_2)^{-1}$$

$$= -\mu_2\mu_1^{-1}(Q_2 - P_{oo})^{-1}\Big\{Q_2\ P_1(\mu_2 + s_2)^{-1} - (P_{oo} - Q_1Q_2)$$

$$\times\ (1 - P_{oo})\ P_2^{-1}\ [\mu_2(Q_2 - P_{oo})/P_2 + (\mu_2 + s_2)]^{-1}\Big\}. \tag{2.6.5}$$

Compairing with right hand side of (2.4.8) we write

$$\int_0^\infty [\ f(t_2)\ E(T_1|t_2)\ ]\ \exp(-s_2t_2)\ dt_2$$

$$= \mu_2\ \mu_1^{-1}\ (Q_2 - P_{oo})^{-1}\Big\{Q_2\ P_1(\mu_2 + s_2)^{-1} - (P_{oo} - Q_1Q_2)$$

$$(1 - P_{oo})\ P_2^{-1}\ [\mu_2(Q_2 - P_{oo})/P_2 + (\mu_2 + s_2)]^{-1}\Big\}. \tag{2.6.6}$$

By inverting the Equation (2.6.6) with respect to s_2 we get

$$E\,[T_1 |\ t_2] = [\mu_1(Q_2 - P_{00})]^{-1}\Big\{P_1Q_2 - (P_{00} - Q_1Q_2)(1 - P_{00}) \quad P_2^{-1} \exp\{-(\mu_2[Q_2 - P_{00}]/P_2 + s)\ t_2\}\Big\}. \quad (2.6.7)$$

Hence the proof. □

Now we simplify again the Equation (2.6.7) as follow

$$\begin{aligned} E[T_1|t_2] &= \mu_1^{-1}\Big\{[(P_{10} + P_{11})(P_{10} + P_{00}P_{10}^{-1}] \\ &\quad - (Q_2 - P_{00})Q_1Q_2\ (1 - P_{00})\ P_2^{-1}\ P_{10}^{-1} \exp\{-(\lambda_2 P_{10} t_2)\}\Big\} \\ &= \mu_1^{-1}\Big\{1 + [P_{11}P_{00}P_{10}^{-1} - P_{01}]\ [P_{11}P_{00}P_{10}^{-1} - P_{01}] \\ &\qquad (1 - P_{00})\ P_2^{-1} \exp\{-(\lambda_2 P_{10} t_2)\}\Big\} \\ &= \mu_1^{-1}\Big\{1 + C1\ P_{10}^{-1}\ [1 - (1 - P_{00})P_2^{-1}\ \exp\{-\lambda_2 P_{10} t_2\ \}\Big\}. \end{aligned}$$

(2.6.8)

Where $C1 = P_{00}P_{11} - P_{01}P_{10}$. The Equation (2.6.8) gives required simplified form of conditional expectation of T_1 given $T_2 = t_2$.

In the following section we discuss the Hawkes model is a generalisation of Downton's BVED.

2.7 Downton's model is a particular case of Hawkes model

In order to verify that Downton's BVED is a particular case of Hawkes BVED we proceed as follows.

Theorem(2.7.1) If $P_1 = P_2 = P = 1 - Q$ and as $P_{00} \to Q$ in Equation (2.5.7) then resulting Laplace transform is given by

$$\psi(s_1, s_2) = \mu_1\mu_2/\{(\mu_1 + s_1)(\mu_2 + s_2) - Q\ s_1 s_2\}$$

Proof : Let us consider Laplace transform given in (2.5.7). Letting $P_1 = P_2 = P = 1 - Q$ and as $P_{00} \to Q$ we write,

$$\psi_H(s_1, s_2) = \left\{\mu_1\mu_2/\{(\mu_1+ s_1)(\mu_2+ s_2)\}\right\} \left\{1 + [Q - (1 - P)^2]\right.$$

$$\left. s_1 s_2 \ ((\mu_1 + P_1 s_1) \ (\mu_2 + P_2 s_2) - \mu_1\mu_2 Q)^{-1}\right\}$$

$$= G \left\{\mu_1\mu_2(1 - Q) + \mu_1 P s_2 + \mu_2 P \ s_1 + Q \ s_1 s_2\right.$$

$$\left. - s_1 s_2 + 2 P \ s_1 s_2\right\} P^{-1}\left\{\mu_1(\mu_2 + s_2)\mu_2 s_1 + P \ s_1 s_2\right\}^{-1},$$

Where $G = \mu_1\mu_2/\{(\mu_1+ s_1)(\mu_2+ s_2)\}$. Thus

$$\psi_H(s_1, s_2) = G \ \{(\mu_1 + s_1)(\mu_2 + s_2)\}$$

$$[(\mu_1 + s_1)(\mu_2 + s_2) - Q \ s_1 s_2]^{-1}$$

$$= \mu_1\mu_2/\{(\mu_1 + s_1)(\mu_2 + s_2) - Q \ s_1 s_2\}. \qquad (2.7.1)$$

The Equation (2.7.1) is Laplace transform of Downton's model.

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