CHAPTER - III

HYPOEXPONENTIAL DISTRIBUTION AND ITS PROPERTIES

3.1 Introduction

There are number of processes that can be divided into sequential phases. If the time for process spends in each phase is independent and identically distributed then the over all time is hypoexponentially distributed. Such type of distributions arises in case of " service time for Input/Output operations in a computer system " and also in many context of reliability theory. In the earlier chapter we have studied the bivariate exponential distributions introduced by Marshall-Olkin, Downton and Hawkes. In the present chapter we discuss some important properties of hypoexponential distribution and relationship with bivariate exponential distribution discussed in chapter II.

Section-3.2 deals with definition of hypoexponential distribution and some important properties such as a Survival function, Laplace transform, moment about origin and reproductive property of hypoexponential distribution. We also illustrate different situations in which this model arises. The relationship between hypoexponential distribution with Downton's BVED, Marshall-Olkin BVED and Hawkes BVED is established in Section-3.3. Finally, In Section-3.4 of this chapter contains procedure of model sampling from hypoexponential distribution. The observations on hypoexponential distribution are obtained for the parameters $\eta = 1$ and $\xi = 1$ respectively .

3.2 Hypoexponential distribution and its properties.

In this section a definition of hypoexponential distribution and its properties are given. Further the model is introduced through illustrations

Definition(3.2.1) let T be a continuous random variable with distribution function F given by

$$F(t;\, \eta,\, \xi) = \begin{cases} 0 & ;\ t < 0 \\ [\, \xi(\xi - \eta)^{-1}\{1 - \exp(-\eta t\,)\}] \\ \quad - [\ \eta(\xi - \eta)^{-1}\{1 - \exp(-\xi t\,)\}] & ;\ t \geq 0 \end{cases}$$

with parameters η and ξ. The function F is called as hypoexponential distribution function. The corresponding density function $f(t;\, \eta,\, \xi)$ is given by

$$f(t;\, \eta,\, \xi) = \begin{cases} 0 & ;\ t < 0 \\ \eta\, \xi\ (\xi - \eta)^{-1}\ [\exp(-\eta t) - \exp(-\xi t)] & ;\ t \geq 0 \end{cases}$$

(3.2.1)

In the following important properties of hypoexponential distribution are studied.

Theorem(3.2.1) If T is a random variable with p.d.f. given by the Equation (3.2.1) then the Survival function $\bar{F}(t)$ is

$$\exp(-\eta t) + \eta(\xi - \eta)^{-1}\ [\exp(-\eta t) - \exp(-\xi t)]\,.$$

Proof : From (3.2.1), we obtain

$$\bar{F}(t) = \Pr[\ T > t\ ]$$

$$= 1 - \Pr[\ T \leq t\ ]$$

$$= 1 - \int_0^t \xi\eta\ (\xi - \eta)^{-1}[\exp(-\eta t) - \exp(-\xi t)]\ dt.$$

$$= 1 + \xi(\xi - \eta\ )^{-1}\ \exp(-\eta t) + \eta(\xi - \eta)^{-1} \cdot e^{-\xi t} - 1$$

$$= \exp(-\eta t\ ) + \eta(\xi - \eta)^{-1}\ [\exp(-\eta t) - \exp(-\xi t)]\,. \qquad (3.2.2)$$

□

Theorem(3.2.2) If T is hypoexponential random variable with p.d.f. as given in (3.2.1), then the Laplace transform of T is

$$\eta\, \xi\ (s + \eta)^{-1}\ (s + \xi)^{-1}.$$

Proof : Let $\tilde{f}(s)$ be the Laplace transform of $f(\,.,\ \eta,\ \xi)$ and it is defined as

$$\tilde{f}(s) = \int_0^\infty \exp(-\,st)\ f(t;\ \eta,\ \xi)\ dt$$

$$= \eta\xi\ (\xi - \eta)^{-1}\int_0^\infty \left\{ \exp(-\,[\eta + s]t\,) - \exp(-\,[\xi + s]t\,)\right\} dt$$

$$= \eta\xi\ (\xi - \eta)^{-1} \left\{ [\eta + s]^{-1} - [\xi + s]^{-1} \right\}$$

$$= \eta\ \xi\ (s + \eta)^{-1}\ (s + \xi)^{-1}. \tag{3.2.3}$$

□

Theorem(3.2.3) If T is hypoexponential random variable with p.d.f. as given in (3.2.1) then the r^{th} Order moment about origin is given by

$$(r!)\ (\eta\ \xi)^{-r}\ (\xi - \eta)^{-1}\ [\,\xi^{r+1} - \eta^{r+1}].$$

Proof : The r^{th} Order moment about origin is defined as

$$\mu_r' = E[\ T^r\ ]$$

$$= \int_0^\infty t^r\ f(t;\ \eta,\ \xi)\ dt$$

$$= \eta\xi\ (\xi - \eta)^{-1}\int_0^\infty t^r \left\{ \exp(-\,[\eta\ t]\,) - \exp(-\,[\xi\ t]\,) \right\} dt$$

$$= \eta\xi\ (\xi - \eta)^{-1} \left\{ \Gamma(r+1)/\eta^{r+1} - \Gamma(r+1)/\xi^{r+1} \right\}$$

$$= (r!)\ (\eta\xi)^{-r}\ (\xi - \eta)^{-1}\ [\,\xi^{r+1} - \eta^{r+1}]. \tag{3.2.4}$$

□

Theorem(3.2.4) If T_1 and T_2 are independent but not identically distributed exponential random variables with parameters η and ξ respectively then the distribution of sum of two random variables is a hypoexponential distribution with parameters η and ξ.

Proof : Given that T_1 and T_2 are independent but not identically distributed exponential random variables with parameters η and ξ

respectively, with p.d.f.'s

$$f_{T_1}(t_1) = \begin{cases} \eta \exp(-[\eta t_1]); & t_1 \geq 0, \eta > 0 \\ 0 & ; \text{otherwise,} \end{cases}$$

$$f_{T_2}(t_2) = \begin{cases} \xi \exp(-[\xi t_2]); & t_2 \geq 0, \xi > 0 \\ 0 & ; \text{otherwise.} \end{cases}$$

Let $T = T_1 + T_2$ then by convolution theorem we can write

$$f_T(t) = \int_0^t f_{T_1}(t_1)\, f_{T_2}(t - t_1)\, dt_1$$

$$= \xi\eta\,(\eta - \xi)^{-1}[1 - \exp(-(\eta - \xi)t)]\cdot\exp\{-\xi t\}$$

$$= \xi\eta\,(\xi - \eta)^{-1}[\exp(-\eta t) - \exp(-\xi t)],\ t \geq 0;\ \xi > \eta. \tag{3.2.5}$$

We observe that the Equation (3.2.5) is the probability density function of hypoexponential distribution. □

In the following some applications of hypoexponential distribution are given.

EXAMPLE(3.2.1) : Let there be three components, two of which are required to be in working order for the system to function properly such a system is known as Triple modular redundancy(TMR) system. Thus after one failure, the system reduces to a series system of two components, an improvement over this simple scheme, known as TMR/simplex, detects a single component failure, discards the failed component, and it reverts to one of the nonfailing simplex components.

Let X, Y, and Z denote the times to failure of the three components. Let W denote the residual time to failure of the selected surviving component. Let X, Y and Z be mutually independent and exponentially distributed with parameter λ. If L denote the time to failure of TMR/simplex then

$$L = \min[X, Y, Z] + W.$$

Now since the exponential distribution is memoryless it follows that the life time W of the surviving component exponentially distributed with parameter λ. Also from the property of order statistic it follows that $\min[X,Y,Z]$ is exponentially distributed with parameter 3λ. By using Theorem-(3.2.4) we say that L has a two stage hypoexponential distribution with parameters 3λ and 3λ. Therefore using Definition-(3.2.1) we have

$$F_L(t) = 1 - [3\lambda]/[2\lambda] \exp(-\lambda t) + \lambda/[2\lambda] \exp(-3\lambda t); \; t \geq 0$$

$$= 1 - 3/2 \exp(-\lambda t) + 1/2 \exp(-3\lambda t).$$

EXAMPLE(3.2.2) Consider the TMR system and let X, Y and Z denote the life times of the three components. assume that these random variables are mutually independent and exponentially distributed with parameter λ.

let L denote the life time of the TMR system then

$$L = \min[X, Y, Z] + \min[U, V]$$

Hence U and V denotes the residual life times of the two surviving components after the first failure. By the memoryless property of the exponential distribution we conclude that U and V are exponentially distributed with parameter λ.

Therefore $\min[X, Y, Z]$ is exponentially distributed with parameter 3λ. $\min[U, V]$ has exponential distribution with parameter 2λ. Thus by using Theorem-(3.2.4) we say that L has a two stage hypoexponential distribution with parameters 3λ and 2λ. Therefore using Definition-(3.2.1) we write the distribution function of L can be written as

$$F_L(t) = [6\lambda]/[2\lambda] (1 - \exp(-2\lambda t)) \pm 6\lambda/[3\lambda] (1 - \exp(-3\lambda t)); \quad t \geq 0$$

$= 1 - 3\exp(-2\lambda t) + 2\exp(-3\lambda t)$

and hence corresponding density function of L is obtained by taking derivative of $F_L(t)$ with respect to t we get

$$f_L(t) = 6\lambda\exp(-2\lambda t) - 6\lambda\exp(-3\lambda t)) \; ; \; t \geq 0. \qquad \square$$

EXAMPLE(3.2.3) Consider a module, consisting of a functional unit together with an on-line fault detector.

Let T be the time to failure of the unit and C be the time to failure of the detector. After the unit fails, a finite time D is required to detect the failure. Failure of the detector, however, is detected instantaneously. Let X denote the time to failure indication and Y denote the time to failure occurence (of either the detector or the unit). Clearly, $X = \min[T + D, C]$ and $Y = \min[T, C]$. If the detector fails before the unit, then a false alarm is said to have occured. If the unit fails before the detector, then the unit keeps producing erroneous output during the detection phase and thus propogates the effect of the failure. The purpose of the detector is to reduce the detection time D. Assume that T, Dand C are mutually independent and exponentially distributed with parameters λ, δ and α respectively. Then clearly Y is exponentially distributed with parameter $\lambda + \alpha$ and also by using Theorem-(3.2.4) we say that T + D follows hypoexponential distribution with distribution function is given by

$$F_{T+D}(t) = 1 - \delta/(\delta - \lambda)\exp(-\lambda t) + \lambda/(\delta - \lambda)\exp(-\delta t).$$

$\square$

3.3 Hypoexponential distribution and some standard BVED's

In this section, the relationship between

i) hypoexponential distribution and Downton's BVED,

ii) hypoexponential distribution and Marshall and Olkin's BVED, and

iii) hypoexponential distribution and Hawkes BVED are studied.

3.3.1. Relation with Downton's BVED

In the following we obtain relationship between hypoexponential distribution with Downton's bivariate exponential distribution.

Theorem(3.3.1) Suppose that $(T_1, T_2)^T$ has a Downton's BVED then the distribution of $T_1 + T_2$ is hypoexponential distribution with parameters $\theta_1/(1-\rho)$ and $\theta_2/(1-\rho)$.

Proof : Let $\tilde{f}_1(s)$ be the Laplace transform corresponding to the distribution of $T_1 + T_2$. That is

$$\tilde{f}_1(s) = E\left\{\exp[-s(T_1 + T_2)]\right\}.$$

The $\tilde{f}_1(s)$ is to be obtained by substituting $s_1 = s_2 = s$ in the Laplace transform of Downton's BVED as given in Equation (2.2.6). That is

$$\tilde{f}_1(s) = \mu_1\mu_2/((\mu_1 + s)(\mu_2 + s) - \rho s^2)$$

$$= \left\{\mu_1\mu_2/(1-\rho)\right\}$$

$$\left\{s^2 + [(\mu_1 + \mu_2)/(1-\rho)]s + (\mu_1 + \mu_2)/(1-\rho)\right\}^{-1}$$

$$= \left\{\mu_1\mu_2/(1-\rho)\right\}\left\{(s - R_1)(s - R_2)\right\}^{-1},$$

Where

$$R_i = -(1-\rho)^{-1}\left\{(\mu_1 + \mu_2)/2 + (-1)^i\,[(\mu_1 - \mu_2)^2/4 + \rho\,\mu_1\mu_2]^{1/2}\right\}.$$

That is $R_i = -\theta_i/(1-\rho)$, with

$$\theta_i = \left\{(\mu_1 + \mu_2)/2 + (-1)^i \left[(\mu_1 - \mu_2)^2/4 + \rho\, \mu_1\mu_2 \right]^{1/2}\right\};\ i = 1,2.$$

Therefore

$$\tilde{f}_1(s) = \left\{\mu_1\mu_2/(1-\rho)\right\}\left\{(R_1 - R_2)^{-1}\left[(s - R_1)^{-1} - (s - R_2)^{-1}\right]\right\}$$

$$= \left\{\mu_1\mu_2(1-\rho)\right\}\left\{(\theta_2 - \theta_1)(1-\rho)\right\}^{-1}$$

$$\left[(s - R_1)^{-1} - (s - R_2)\right]^{-1} \qquad (3.3.1)$$

By inverting the Laplace transform given in Equation (3.3.1) we write

$$f(t) = \theta_1\theta_2 \left\{(\theta_2 - \theta_1)(1-\rho)\right\}^{-1}\left[\exp[+R_1 t] - \exp[+R_2 t]\right]$$

$$= \theta_1\theta_2 \left\{(\theta_2 - \theta_1)(1-\rho)\right\}^{-1}$$

$$\times \left[\exp([-\theta_1/(1-\rho)]\,t) - \exp([-\theta_2/(1-\rho)]t)\right]$$

$$(3.3.2)$$

By reparameterizing the Equation (3.3.2), we can write

$$f(t) = \xi\eta\,(\xi - \eta)^{-1}\left[\exp[-\eta t] - \exp[-\xi t]\right];\ t \ge 0,\ \xi > \eta$$

$$\text{Where } \eta = \theta_1/(1-\rho) \text{ and } \xi = \theta_2/(1-\rho). \qquad (3.3.3)$$

The Equation (3.3.3) is the density function of the hypoexponential distribution.

In order to compare the hypoexponential distribution with Marshall-Olkin BVED it is necessary to obtain the distribution of $T_1 + T_2$, Where $(T_1, T_2)^T$ is a random vector following BVED of Marshall-Olkin.

3.3.2. Relation with Marshall-Olkin BVED

In the following the relationship between hypoexponential distribution and BVED of Marshall-Olkin is established.

Theorem (3.3.2) Suppose that $(T_1, T_2)^T$ is a random vector follows BVED of Marshall-Olkin then the distribution of $T_1 + T_2$ is a weighted sum of three exponential distributions.

Proof : Let $\tilde{f}_2(s)$ be the Laplace transform corresponding to the distribution of $T_1 + T_2$. That is

$$\tilde{f}_2(s) = E\left\{\exp[-\, s(T_1 + T_2)]\right\}$$

The $\tilde{f}_2(s)$ is to be obtained by substituting $s_1 = s_2 = s$ in the Laplace transform of Marshall-Olkin BVED as given in Equation (2.2.6). Thus (2.4.24)

$$\tilde{f}_2(s) = \mu_1\, \mu_2\, [\,(\mu_1 + s)(\mu_2 + s)\,]^{-1}$$

$$\left\{1 + \rho\, s^2(\mu_1 + \mu_2)[\,\mu_1\mu_2\, (\mu_1 + \mu_2 + 2\,(1 + \rho)\, s)\,]^{-1}\right\}$$

$$= \mu_1\mu_2/(\mu_2 - \mu_1)[\,(\mu_1 + s)^{-1} - (\mu_2 + s)^{-1}]$$

$$+ \left\{\rho\, (\mu_1 + \mu_2)/[\,2(1 + \rho)]\right\}$$

$$\times \left\{ s^2 \,/\, [\,(\mu_1 + s)(\mu_2 + s)(\mu_3 + s)] \right\}, \qquad (3.3.4)$$

Where $\mu_3 = (\mu_1 + \mu_2)/[\,2(1 + \rho)]$. Now consider

$$\left\{ s^2/[\,(\mu_1 + s)(\mu_2 + s)(\mu_3 + s)] \right\}$$

$$= A/(\mu_1 + s) + B/(\mu_2 + s) + C/(\mu_3 + s) \qquad (3.3.5)$$

That is

$$s^2 = A\, (\mu_2 + s)\, (\mu_3 + s) + B\, (\mu_1 + s)\, (\mu_3 + s)$$

$$+ C\, (\mu_1 + s)\, (\mu_2 + s) \qquad (3.3.6)$$

Substituting $s = -\mu_1$, $s = -\mu_2$ and $s = -\mu_3$ in the Equation (3.4.3) respectively we get

$$A = \mu_1^2/[(\mu_2 - \mu_1)(\mu_3 - \mu_1)], \quad B = -\mu_2^2/[(\mu_2 - \mu_1)(\mu_3 - \mu_2)],$$

and $$C = \mu_3^2/[(\mu_3 - \mu_1)(\mu_3 - \mu_2)]. \tag{3.3.7}$$

From Equations-(3.3.5),(3.3.6) and (3.3.7) we write

$$\begin{aligned}\tilde{f}_2(s) = {} & \mu_1\mu_2/(\mu_2 - \mu_1)[(\mu_1 + s)^{-1} - (\mu_2 + s)^{-1}] \\ & + \left\{\rho\, \mu_3\mu_1^2/[(\mu_2 - \mu_1)(\mu_3 - \mu_1)(\mu_1 + s)]\right\} \\ & - \left\{\rho\mu_3\mu_2^2/[(\mu_2 - \mu_1)(\mu_3 - \mu_2)(\mu_2 + s)]\right\} \\ & + \left\{\rho\, \mu_3^3/[(\mu_3 - \mu_1)(\mu_3 - \mu_2)(\mu_3 + s)]\right\}\end{aligned} \tag{3.3.8}$$

By inverting the Laplace transform given in Equation (3.3.8) we get

$$\begin{aligned}f(t) = {} & \left\{\mu_1\mu_2/(\mu_2 - \mu_1) + \rho\mu_3\mu_1^2/[(\mu_2 - \mu_1)(\mu_3 - \mu_1)]\right\} \exp(-\mu_1 t) \\ & - \left\{\mu_1\mu_2/(\mu_2 - \mu_1) + \rho\mu_3\mu_2^2/[(\mu_2 - \mu_1)(\mu_3 - \mu_2)]\right\} \exp(-\mu_2 t) \\ & + \left\{\rho\mu_3^3/[(\mu_3 - \mu_1)(\mu_3 - \mu_2)]\right\} \exp(-\mu_3 t), \quad t \geq 0.\end{aligned} \tag{3.3.9}$$

The Equation (3.3.9) yields a weighted sum of three exponentials.

Remark : By letting $\rho = 0$ in the Equation (3.3.9) we write

$$f(t) = \mu_1\mu_2/(\mu_2 - \mu_1)\left\{\exp(-\mu_1 t) - \exp(-\mu_2 t)\right\}, \; t \geq 0, \; \mu_2 > \mu_1. \tag{3.3.10}$$

Now it may be interesting to note that the Equation (3.4.10) is density of hypoexponential distribution with parameters μ_1 and μ_2.

Thus we say that the distribution of $T_1 + T_2$ under M-O model is hypoexponential distribution with parameter μ_1 and μ_2.

3.3.3. Relation with Hawkes BVED

We have discussed hypoexponential distribution is related with Downton's BVED and also M-O BVED in Section-(3.3.1) and (3.3.2) respectively. On similar line in the following theorem we obtain relationship between hypoexponential distribution and Hawkes BVED.

Theorem(3.3.3) Suppose that $(T_1, T_2)^T$ is a random vector following Hawkes BVED, then the distribution of $T_1 + T_2$ is a weighted sum of four exponential distributions.

Proof : Let $\tilde{f}_s(s)$ be the Laplace transform corresponding to the distribution of $T_1 + T_2$. Therefore

$$\tilde{f}_s(s) = E\left\{ \exp[-s(T_1 + T_2)] \right\}.$$

The $\tilde{f}_s(s)$ is to be obtained by substituting $s_1 = s_2 = s$ in the Laplace transform of Hawkes BVED as given in Equation (2.2.6). Thus

$$\tilde{f}_s(s) = \mu_1 \mu_2 [(\mu_1 + s)(\mu_2 + s)]^{-1} + \mu_2 \mu_1 (P_{oo} - Q_1 Q_2) \left\{(\mu_1 + s)(\mu_2 + s) [(\mu_2 + P_2 s)(\mu_1 + P_2 s) - \mu_1 \mu_2 P_{oo}]\right\}^{-1}$$

$$= \left\{\mu_1 \mu_2 / (\mu_2 - \mu_1) [(\mu_1 + s)^{-1} - (\mu_2 + s)^{-1}]\right\} + \left\{K s^2 / [(\mu_1 + s)(\mu_2 + s)(as^2 + bs + c)]\right\}, \quad (3.3.11)$$

Where $K = \mu_2 \mu_1 (P_{oo} - Q_1 Q_2)$, $a = P_1 P_2$,

$b = \mu_1 P_2 + \mu_2 P_1$ and $c = \mu_1 \mu_2 (1 - P_{oo})$.

Consider

$$s^2/[(\mu_1 + s)(\mu_2 + s)(as^2 + bs + c)]$$

$$= s^2/[(\mu_1 + s)(\mu_2 + s)(s - \mu_3)(s - \mu_4)],$$

Where μ_3 and μ_4 are roots of the quadratic equation $a s^2 + b s + c = 0$. Then

$$s^2/[(\mu_1 + s)(\mu_2 + s)(s - \mu_3)(s - \mu_4)]$$

$$= A/(\mu_1 + s) + B/(\mu_2 + s) + C/(s - \mu_3) + D/(s - \mu_4) \quad (3.3.12)$$

$$s^2 = A(\mu_2 + s)(s - \mu_3)(s - \mu_4) + B(\mu_1 + s)(s - \mu_3)(s - \mu_4)$$

$$+ C(\mu_1 + s)(\mu_2 + s)(s - \mu_4) + D(\mu_1 + s)(\mu_2 + s)(s - \mu_3)$$

(3.3.13)

Substituting $s = -\mu_1$, $s = -\mu_2$, $s = -\mu_3$ and $s = -\mu_4$ in the Equation (3.5.3) respectively and on simplification yields

$$A = \mu_1^2/[(\mu_2 - \mu_1)(\mu_1 + \mu_3)(\mu_1 + \mu_4)],$$

$$B = -\mu_2^2/[(\mu_2 - \mu_1)(\mu_2 + \mu_3)(\mu_2 + \mu_4)],$$

$$C = \mu_3^2/[(\mu_2 + \mu_3)(\mu_1 + \mu_3)(\mu_3 - \mu_4)],$$

$$D = -\mu_4^2/[(\mu_1 + \mu_4)(\mu_2 + \mu_4)(\mu_3 - \mu_4)]. \quad (3.3.14)$$

From Equation (3.3.12),(3.3.13) and (3.3.14) we write

$$\tilde{f}_3(s) = \mu_1\mu_2/(\mu_2 - \mu_1)[(\mu_1 + s_1)^{-1} - (\mu_2 + s_2)^{-1}]$$

$$+ \left\{K\, \mu_1^2/[(\mu_2 - \mu_1)(\mu_3 + \mu_1)(\mu_1 + \mu_4)]\right\}(\mu_1 + s)^{-1}$$

$$- \left\{K\, \mu_2^2/[(\mu_2 - \mu_1)(\mu_3 + \mu_2)(\mu_2 + \mu_4)]\right\}(\mu_2 + s)^{-1}$$

$$+ \left\{K\, \mu_3^3/[(\mu_3 + \mu_1)(\mu_3 + \mu_2)(\mu_3 - \mu_4)]\right\}(s - \mu_3)^{-1}$$

$$- \left\{K\, \mu_4^2/[(\mu_4 + \mu_1)(\mu_4 + \mu_2)(\mu_3 - \mu_4)]\right\}(s - \mu_4)^{-1}. \quad (3.3.15)$$

By inverting the Laplace transform given in Equation (3.3.15) We get

$$f(t) = (\mu_2 - \mu_1)^{-1}\left\{\mu_1\mu_2 + K\,\mu_1^2\,[(\mu_3 + \mu_1)(\mu_1 + \mu_4)]^{-1}\right\}\exp(-\mu_1 t)$$

$$(\mu_2 - \mu_1)^{-1}\left\{\mu_1\mu_2 + K\,\mu_2^2\,[(\mu_3 + \mu_2)(\mu_2 + \mu_4)]^{-1}\right\}\exp(-\mu_2 t)$$

$$+ \left\{K\mu_3^2\,[(\mu_3 + \mu_1)(\mu_3 + \mu_2)(\mu_3 - \mu_4)]^{-1}\right\}\exp(\mu_3 t)$$

$$- \left\{K\,\mu_4^2\,[(\mu_3 + \mu_1)(\mu_3 + \mu_2)(\mu_3 - \mu_4)]^{-1}\right\}\exp(\mu_4 t),$$

(3.3.16)

Where $\mu_3 = -\left\{(\mu_1 P_2 + \mu_2 P_1)\right.$

$$\left. - ((\mu_1 P_2 - \mu_2 P_1)^2 + 4\,\mu_1\mu_2 P_1 P_2 P_{oo})^{1/2}\right\} / (2\,P_1 P_2)$$

and $\mu_4 = -\left\{(\mu_1 P_2 + \mu_2 P_1)\right.$

$$\left. + ((\mu_1 P_2 - \mu_2 P_1)^2 + 4\,\mu_1\mu_2 P_1 P_2 P_{oo})^{1/2}\right\} / (2\,P_1 P_2).$$

The Equation (3.5.6) yields a weighted sum of four exponential variates.

□

3.4. Model sampling from Hypoexponential distribution

In order to obtain an observation on Hypoexponential distribution, we proceed as follows.

First we have to obtain the observation on exponential distribution with parameter η. We note that If U_1 has a uniform distribution over (0, 1) then we find the distribution of $T_1 = -\log(U_1)/\eta$.

Thus $$F_{T_1}(t_1) = \begin{cases} P_r(T_1 \le t_1);\ t_1 \ge 0 \\ 0 \qquad ;\ \text{Otherwise.} \end{cases}$$

$$= \begin{cases} P_r(- \log(U_1)/\eta \le t_1); & t_1 \ge 0 \\ 0 & ; \text{Otherwise} \end{cases}$$

$$= \begin{cases} P_r(- \log(U_1) \le \eta\ t_1); & t_1 \ge 0 \\ 0 & ; \text{Otherwise} \end{cases}$$

$$= \begin{cases} P_r(U_1 \ge \exp(- \eta\ t_1)); & t_1 \ge 0 \\ 0 & ; \text{Otherwise} \end{cases}$$

$$= \begin{cases} 1 - \exp(- \eta\ t_1); & t_1 \ge 0 \\ 0 & ; \text{Otherwise} \end{cases} \qquad (3.6.1)$$

Hence the distribution of T_1 is exponential distribution with parameter η.

On similar lines we obtain another observation on exponential distribution with parameter ξ. In this case we also note that If U_2 has a uniform distribution over (0, 1) then we find the distribution of $T_2 = - \log(U_2)/\xi$. On the same lines of Equation (3.6.1) we get

$$F_{T_2}(t_2) = \begin{cases} 1 - \exp(- \xi\ t_2); & t_2 \ge 0 \\ 0 & ; \text{Otherwise.} \end{cases} \qquad (3.6.2)$$

Now using the reproductive property of hypoexponential distribution we obtain one observation on hypoexponential distribution by considering $T = [- \log(U_1)/\eta\ ] + [- \log(U_2)/\xi\ ]$ as follows.

Step-1. Generate m pairs of uniform random numbers over (0, 1).

Step-2. Compute

$$T_i = \left\{[- \log(U_1)/\eta\ ] + [- \log(U_2)/\xi\ ]\right\} \qquad \text{for } i = 1, 2, \ldots, m.$$

Then the distribution of T_i; $i = 1, 2, \ldots, m$ is hypoexponential distribution with parameters η and ξ.

Using the following computer program in BASIC we generate observations on hypoexponential distribution for parameters $\eta = 1$ and $\xi = 1$

Computer program in BASIC

To generate observations from hypoexponential distribution with $\eta = 1$ and $\xi = 1$ we use the following program.

```
 5  REM MODEL SAMPLING FROM HYPOEXPONENTIAL DISTRIBUTION
15  A0 = 1:B0 = 1
20  LPRINT TAB(35);"TABLE 3.1"
22  LPRINT
24  LPRINT
25  LPRINT CHR$(15)
30  WIDTH LPRINT 144
35  FOR I = 1 TO 45
45  FOR J = 1 TO 12
50  RANDOMIZE 107 + J + I
55  U1 = RND
60  T0 = - LOG( U1 )/A0
65  U2 = RND
70  T1 = - LOG( U2 )/B0
75  T = T0 + T1
85  LPRINT USING "  ####.####"; T ;
95  NEXT J
96  NEXT I
98  END
```

The Table 3.1 gives obervations from hypoexponential distribution with parameter $\eta = 1$ and $\xi = 1$ using above computer program in BASIC

TABLE 3.1

0.1608	6.8756	1.1884	2.0396	1.7901	1.6730	1.8602	1.6423	0.9297	0.6815	1.4204
3.9266	1.8011	4.6471	4.2359	1.1877	1.8595	2.0733	0.2041	0.9123	2.1182	0.3440
0.6041	1.9054	0.9014	0.7746	0.5594	2.2160	1.0121	2.1306	1.5185	0.3809	2.6896
0.5930	1.1062	4.0369	1.2704	0.3829	3.8450	1.8105	5.9105	1.4284	1.5607	0.9926
5.1649	1.6273	1.5412	4.7965	0.0917	0.6276	1.8780	1.4444	1.7666	2.8622	3.4149
2.1576	0.0232	1.5243	1.9986	1.3696	0.9918	0.6717	1.9596	1.2775	3.9197	4.4461
0.5443	0.1642	0.8081	0.7899	2.8105	1.7434	6.0214	4.4778	0.8699	1.4601	0.9360
4.9724	0.7657	1.6369	0.9095	3.5352	0.3468	6.3645	2.6320	1.0582	5.4987	2.0057
2.1054	1.2138	0.8499	1.0526	1.4645	2.6878	0.6213	2.7994	0.8033	1.5347	1.1885
1.6257	5.9092	0.6300	1.8646	2.1922	4.6917	0.7960	5.5333	1.1062	0.8249	1.6575
2.6409	1.2813	0.3081	2.0639	1.7651	2.6934	0.4924	0.2755	2.6143	0.5184	0.0681
1.2979	1.2314	1.0771	2.0778	2.3470	0.1054	4.3368	1.3980	4.2698	4.6227	3.0175
2.5486	2.6617	0.9430	2.1337	3.1027	0.6075	3.5297	0.1423	0.9888	1.5285	1.9459
2.6043	3.3058	1.2029	1.5806	3.4952	1.4610	2.7874	0.8069	2.4936	3.1393	6.9393
1.6156	2.9604	1.4677	3.9103	1.0771	0.2816	3.2903	4.4128	2.5549	2.4679	2.2836
1.9493	1.2362	4.5436	0.6408	2.7738	4.1159	1.7344	0.4338	1.3519	3.0834	0.3266
7.1921	1.0327	1.9923	1.1310	1.8744	1.4801	0.2324	1.8899	2.1915	3.1755	3.7913
2.7614	2.0460	0.8209	2.9949	0.9128	0.9277	1.5269	0.6571	0.9627	0.9654	1.2748
2.2997	0.8970	0.8899	1.7613	0.6261	3.3946	3.0897	1.6226	1.0181	3.1566	1.8885
0.7827	1.1516	0.8743	2.4340	1.2910	0.4326	4.5387	4.1114	1.5020	1.3450	1.3485
2.4924	2.7542	1.4642	4.7869	1.2339	0.4467	0.9063	2.3232	0.3856	1.9100	3.7126
0.6257	1.5915	8.3197	1.6464	2.3113	4.1879	1.1760	1.3291	3.7008	4.5426	4.1820
2.7201	5.3134	0.5893	0.9975	4.3273	0.6866	0.5514	2.7528	1.7727	1.1806	0.9549
1.2671	2.7530	2.7774	1.3505	0.2809	0.3629	4.1473	1.7913	0.6309	5.2859	1.4064
2.7100	0.7824	2.1826	2.7890	2.2106	3.2702	0.5838	1.4253	0.7497	2.0719	0.2073
2.3876	2.8351	1.0234	1.7257	5.4161	1.8586	2.4803	0.6851	0.8409	2.6654	2.3442
0.4773	1.9861	0.7708	1.8874	2.8805	1.4188	4.6982	0.4460	1.6294	1.5551	3.5070
1.5735	2.9109	0.9235	0.5678	0.6020	1.0462	1.0415	0.1663	1.8371	6.4758	2.0225
0.3131	2.7815	2.1727	0.3501	0.8175	0.9637	0.9544	0.7935	1.6260	1.8777	3.6491
2.5076	8.1781	3.6261	3.8432	2.6410	2.0651	2.3385	0.7491	4.5080	1.1885	0.4310
2.1037	0.7109	0.8058	1.3057	1.4180	2.0687	2.9028	1.0160	1.3466	2.3217	2.6379
2.4646	0.6539	2.2004	2.4776	2.4479	1.2663	1.9616	1.0666	2.2375	2.0142	0.1406
6.4650	2.9158	1.3674	1.9912	2.0298	5.7207	2.4254	4.6985	0.4732	2.4461	3.4513
3.2958	0.2679	1.0722	2.5642	2.5525	4.7199	1.9233	1.7114	0.9568	1.5056	1.2712
0.0959	1.5672	1.6368	2.7679	4.1358	9.8663	0.4582	0.6532	2.6362	0.7076	3.7479
0.4355	0.5021	2.3174	0.7930	3.7503	1.6599	0.7445	0.6996	1.2604	0.4302	3.7989
2.6785	1.1702	0.8544	3.4399	3.7755	1.6717	2.4650	0.6773	1.9236	2.3409	1.1597
1.1357	0.1869	1.2979	4.3363	1.1603	2.2156	0.5010	1.2206	3.9745	2.2648	1.1632
5.6394	1.6498	1.0729	0.6183	3.0767	0.4597	1.1832	3.4741	2.9859	1.9240	4.4597
2.3391	1.9350	1.7839	2.5560	3.5693	1.7785	4.2165	0.4367	2.0346	3.6763	0.6119
7.8984	0.9360	2.6215	1.4515	0.7909	2.3059	1.9370	2.2822	1.1108	0.1333	1.0891
1.7444	0.3485	2.7286	1.3199	0.9105	2.2923	0.6274	1.7321	1.9822	1.2797	2.8146
2.0562	1.8804	1.2942	4.6155	5.6595	1.3035	2.5007	0.7469	1.4594	0.3421	1.8745
3.0735	0.4118	0.9180	1.9635	1.4010	2.0245	1.4117	0.5533	1.6576	2.5086	1.3921
0.9417	3.3074	2.4913	2.8982	1.5487	1.4587	4.5362	1.1902	0.4229	3.1788	0.7872
1.5718	3.0231	2.7265	1.2037	0.9428	0.8603	1.6700	1.5395	2.1255	1.6293	2.3667
1.1141	1.4713	5.3393	4.1773	2.9432	1.0700	2.2424	1.2479	2.6830	1.1028	5.0913
0.3737	1.6038	2.4935	0.4308	0.9434	0.5726	3.8534	1.1446	1.9938	2.1764	1.2028
0.4515	1.5312	2.1991	4.0782	1.6655	9.2936	3.1790	2.3269	4.4096	2.3332	0.6407
1.0359										

Table (3.1) Continue

2.4551	0.6848	0.8389	0.5208	0.9227	4.7750	5.4842	0.9015	2.2457	1.8628	0.6448
3.1154	3.6237	2.6701	1.2733	3.3139	1.4053	7.9131	1.0120	2.1960	0.8304	1.2964
1.5818	0.5584	0.5233	1.4888	1.6335	2.2197	2.3877	6.9343	3.6617	1.3825	1.2351
1.5894	0.1109	0.2417	2.6183	1.2854	4.5417	5.9085	0.2924	0.4166	0.4871	2.3487
0.4600	1.6135	1.3817	1.1268	0.8514	1.7020	1.1185	0.7078	0.7573	0.1033	1.7893
0.9335	2.0549	1.3540	1.1579	0.2776	3.2039	1.7902	0.7002	3.2024	0.3241	0.2927
1.5129	2.5595	0.7639	2.0236	2.3921	2.2034	0.9530	1.4713	2.1252	0.8097	2.2557
0.3210	2.1174	1.4467	2.2319	2.5354	0.3802	0.3181	4.0590	2.7392	0.5448	5.4793
1.2198	4.0429	0.7540	2.8682	0.3769	2.6247	1.8599	1.4096	2.0293	0.5941	2.8885
10.1686	1.5486	0.5865	5.1060	2.1900	1.3011	2.1968	2.2575	1.2735	2.3837	0.6072
1.1533	0.2167	0.2531	0.8743	2.5493	3.3855	2.5136	3.9101	0.7137	3.2682	0.8273
4.4220	2.1271	4.9895	1.2213	0.9304	2.3906	1.5599	1.9948	1.4091	0.7703	0.3355
2.3684	0.8336	0.8728	2.3998	1.1860	0.7190	1.1158	4.7650	1.2229	1.6006	0.9998
1.9616	0.9593	1.0773	3.9639	4.4932	1.3966	1.4278	3.4720	2.4783	2.5283	1.5264
1.0728	1.3302	4.4194	1.1044	1.9965	3.8977	1.0819	2.1436	0.5512	1.9479	2.6884
0.8615	1.0491	0.8410	1.8610	0.9003	2.5033	4.3303	1.0371	2.3798	3.3172	1.6018
0.4870	3.1020	2.1398	3.4398	5.0052	1.4456	1.3600	0.8735	0.2232	1.7650	0.4866
1.4533	0.9832	3.1386	0.2676	0.6685	0.8182	0.7386	1.9401	4.3624	3.3817	3.2389
1.1967	2.6056	0.7135	0.8302	0.3229	0.3061	0.2722	1.9166	1.0641	4.6045	1.8355
0.5772	0.8374	0.6551	1.7587	3.9279	2.2841	3.0228	1.7688	1.6002	3.4323	4.1332
2.0770	1.2182	1.3175	0.4588	3.2832	1.8008	3.5278	1.2782	3.0688	1.1116	0.6589
3.0574	0.8557	3.4081	1.9141	2.0803	0.6399	0.5889	4.0289	1.3639	1.8263	0.3594
1.1248	0.4317	2.4197	0.7792	0.4040	2.3543	3.2922	5.1807	5.0879	3.2462	2.9860
3.5471	1.4171	2.5159	1.1495	2.0668	2.5515	1.2814	1.2013	2.1773	0.9332	1.1165
2.9327	2.5991	1.5625	1.4872	1.7759	2.9346	2.6905	4.0150	2.3030	1.0224	1.4693
1.1283	1.7269	0.5308	1.6533	0.9076	5.5711	1.8057	2.3301	4.6906	3.1392	3.9073
1.7986	1.3220	0.4220	2.6274	0.3406	3.8705	0.5791	2.6959	5.0891	0.8385	5.5105
1.6246	1.9759	2.9085	2.1763	2.0292	0.5515	0.8772	1.4644	1.0924	4.3370	1.2289
2.4428	1.2681	2.1666	2.1392	3.0697	0.5206	0.6387	0.4137	1.1846	1.5330	1.1164
1.5015	1.6742	0.4642	0.5588	0.1806	1.8225	5.8345	3.2225	6.9987	3.3914	2.4260
1.0740	0.4901	2.8477	4.7928	0.7889	1.7144	0.1384	3.0780	1.4224	1.4671	3.8934
1.3872	2.1030	5.0654	2.2305	1.2383	0.8863	1.0779	0.5298	1.1762	7.4913	1.4272
1.0458	2.2592	3.5346	1.7071	1.7239	0.6467	1.9522	2.1572	2.2167	0.0725	2.1304
0.2981	2.0162	0.7919	2.9100	1.3079	0.3284	2.2498	1.9041	2.7485	1.1836	3.8975
1.3181	1.4149	1.0398	1.2977	1.2433	1.0815	2.0814	2.3620	0.1230	4.0431	2.1936
1.7867	1.9907	1.2653	2.5744	2.8228	0.9700	2.1929	3.0518	0.6260	3.3431	0.1610
0.2446	0.5802	2.4690	1.3570	3.6918	1.2518	1.6323	3.3832	1.5246	3.1896	0.8482
2.6535	1.6268	3.7234	2.9782	3.2108	1.4584	3.6560	1.1436	0.3012	4.3281	4.2575
2.5668	2.4895	0.7428	2.1610	7.8094	2.1703	0.6642	2.9780	0.4093	1.8399	0.4496
1.3683	3.1568	6.0423	1.5370	0.2126	0.8800	0.2789	1.8906	1.5396	0.2521	1.9234
2.1750	3.9002	4.2197	3.2158	2.4120	3.6156	1.1716	2.2285	0.9834	1.5267	0.6874
0.9727	1.0050	1.3614	2.2682	0.9462	2.5930	2.1075	1.3440	1.2401	3.2596	1.6074
1.0685	3.2681	1.9384	0.8047	1.1708	0.8966	1.2656	0.3716	1.4576	2.1323	4.4161
1.5684	1.4349	1.4115	2.7601	3.2526	1.5009	0.9649	0.2807	4.7512	5.7081	3.2124
0.4036	1.9018	3.8418	0.6491	1.5847	6.0795	1.7159	2.3535	1.7578	1.7004	0.3913
3.6487	4.1931	5.5873	2.6355	0.9004	0.6172	1.0216	5.9753	0.8017	1.5174	1.2266
0.4913	0.2697	0.9995	1.3323	2.6996	2.8212	1.3813	0.2979	0.4140	2.2769	3.4475
7.4679	2.3293	4.5940	2.7846	0.8205	2.1926	2.7395	2.2359	3.4297	1.1680	2.4620
2.9856	2.5016	3.5769	2.7225	3.2894	1.0278	1.7719	6.0897	2.2710	3.9136	1.5156
2.0262										

The computer program in FORTRAM-77 in Appendix-I(B) is used for obtaining the Theorotical and Sample Mean, variance for sample sizes 100, 200, 500 are observed as follows

Sample Size	M E A N		V A R I A N C E	
	Theorotical	Sample	Theorotical	Sample
100	1.8485	2.0000	1.5205	2.0000
200	2.0151	2.0000	2.2017	2.0000
500	2.0229	2.0000	2.1658	2.0000

■■■0■■■