CHAPTER - IV

ESTIMATION FROM HYPOEXPONENTIAL DISTRBUTION

4.1 Introduction

In earlier chapter we have studied some important properties of hypoexponential distribution and its relationship with some standard bivariate exponential distributions. In this chapter we consider the problem of estimation of the scale parameters of the hypoexponential distribution. Section 4.2 deals with some standard definition and important results which are used in later sections of this chapter. In Section 4.3 we obtain the likelihood equations. However, the likelihood equations can not be solved analytically therefore we are in need of some alternative methods to obtain the solution of the likelihood equations. In Section 4.4 we use Newton-Raphson iterative method and method of Scoring to obtain the solution of the likelihood equations. In Section 4.5 numerical comparisons of maximum likelihood estimators by Newton-Raphson method and method of Scoring are discussed.

4.2 Preliminary results

Definition(4.2.1) Fisher Information $I(\theta)$ is given by

$$I(\theta) = E_{\theta}\left\{\partial \log f(x|\theta)/\partial\theta \right\}^{2} = - E_{\theta}\left\{\partial^{2} \log f(x|\theta)/\partial\theta^{2}\right\},$$

Where $f(x|\theta)$ is the probability density function of X and $\theta \in \Theta$ and it satisfies regularity conditions (we refer Zacks(1981)).

Remark : $I(\theta)$ is expected amount of the information about θ contained in single observation X. If $X_1, X_2, X_3, \ldots, X_m$ be random sample from distribution whose p.d.f. is $f(x|\theta)$. The expected amount of information about θ contained in a random sample of size m is denoted by $I_m(\theta)$ and we have, $I_m(\theta) = m\, I(\theta)$.

Theorem (4.2.1) let T be a random variable with p.d.f. given in the Equation-(3.2.1) then the Information matrix in single observation is given by

$$I(\eta,\ \xi) = m \begin{bmatrix} \dfrac{\xi(\xi - 2\eta)}{\eta^2(\xi - \eta)^2} + \eta\,\xi\,\phi_1 & (\xi - \eta)^{-2} - \eta\,\xi\,\phi_1 \\ & \\ (\xi - \eta)^{-2} - \eta\,\xi\,\phi_1 & \dfrac{\eta(\eta - 2\xi)}{\xi^2(\xi - \eta)^2} + \eta\,\xi\,\phi_1 \end{bmatrix}.$$

Where $\phi_1 = 2\,(\xi - \eta)^{-4}\,\zeta(3,\ \xi/(\xi - \eta))$

and $\zeta(3,\ \xi/(\xi - \eta))$ is a generalised Zeta function.

Proof : In order to find Information matrix, we have to find

$$E\left\{\partial^2 \log\ f(t;\ \eta,\ \xi)/\partial\eta^2\right\}, \qquad E\left\{\partial^2 \log\ f(t;\ \eta,\ \xi)/\partial\xi^2\right\},$$

and $E\left\{\partial^2 \log\ f(t;\ \eta,\ \xi)/\partial\eta\partial\xi\right\}$, where $f(t;\ \eta,\ \xi)$ is the probability density function of T as given in Equation-(3.2.1). Now taking logarithm of Equation-(3.2.1) on both sides we get

$$\log\ f(t;\ \eta,\ \xi) = \log\left\{\xi\eta\ (\xi - \eta)^{-1}\right\} + \left\{\log[\exp(-\ \eta t) - \exp(-\ \xi t)]\right\}. \qquad (4.2.1)$$

By differentiating Equation-(4.2.1) partially with respect to η and ξ respectively yields

$$\partial\ \log f(t;\ \eta,\ \xi)/\partial\eta = \left\{\xi[\ \eta(\xi - \eta)]^{-1}\right\} + \left\{\ t\ \exp(-\ \eta t)\ [\exp(-\ \eta t) - \exp(-\ \xi t)]^{-1}\right\} \qquad (4.2.2)$$

and

$$\partial\ \log f(t;\ \eta,\ \xi)/\partial\xi = -\left\{\eta[\ \xi(\xi - \eta)]^{-1}\right\} + \left\{\ t\ \exp(-\ \xi t)\ [\exp(-\ \eta t) - \exp(-\ \xi t)]^{-1}\right\}. \qquad (4.2.3)$$

Again differentiate Equation-(4.2.2) and (4.2.3) with respect to η and ξ respectively we write

$$\partial^2 \log f(t;\ \eta,\ \xi)/\partial\eta^2 = -\left\{\xi(\xi - 2\eta)[\eta(\xi - \eta)]^{-2}\right\} - \left\{ t^2 \exp(-(\eta + \xi)t) [\exp(-\eta t) - \exp(-\xi t)]^{-2}\right\} \quad (4.2.4)$$

and

$$\partial^2 \log f(t;\ \eta,\ \xi)/\partial\xi^2 = -\left\{\eta(\eta - 2\xi)[\xi(\xi - \eta)]^{-2}\right\} - \left\{ t^2 \exp(-(\eta + \xi)t) [\exp(-\eta t) - \exp(-\xi t)]^{-2}\right\}. \quad (4.2.5)$$

Let us differentiate Equation-(4.2.2) with respect to ξ we get

$$\partial^2 \log f(t;\ \eta,\ \xi)/\partial\xi\partial\eta = \partial^2 \log f(t;\ \eta,\ \xi)/\partial\eta\partial\xi = -(\xi - \eta)^{-2} + \left\{ t^2 \exp(-(\eta + \xi)t) [\exp(-\eta t) - \exp(-\xi t)]^{-2}\right\} \quad (4.2.6)$$

First we obtain expectation of Equation-(4.2.4) as follows

$$E\left\{\partial^2 \log f(t;\ \eta,\ \xi)/\partial\eta^2\right\} = -\left\{\xi(\xi - 2\eta)[\eta(\xi - \eta)]^{-2}\right\} - E\left\{ T^2 \exp(-(\eta + \xi)T) [\exp(-\eta T) - \exp(-\xi T)]^{-2}\right\}. \quad (4.2.7)$$

Consider

$$E\left\{T^2 \exp(-(\eta + \xi)T) [\exp(-\eta T) - \exp(-\xi T)]^{-2}\right\}$$

$$= \int_0^\infty \left\{t^2 \exp(-(\eta + \xi)t) [\exp(-\eta t) - \exp(-\xi t)]^{-2}\right\} f(t;\ \eta,\ \xi)\, dt$$

$$= [\xi\eta(\xi - \eta)^{-1}] \int_0^\infty t^2 [\exp(\xi t) - \exp(\eta t)]^{-1}\, dt. \quad (4.2.8)$$

//

let $$I_1 = \int_0^{\infty} t^2 \, [\exp(\xi t) - \exp(\eta t)]^{-1} \, dt$$

$$= \int_0^{\infty} t^2 \exp(-\xi t) \left\{1 - \exp(-(\xi - \eta)t)\right\}^{-1} dt.$$

Setting $(\xi - \eta)t = y$; on simplification we get

$$I_1 = (\xi - \eta)^{-3} \int_0^{\infty} y^2 [\exp(-\xi(\xi - \eta)^{-1})y] [1 - \exp(-y)] \, dy \qquad (4.2.9)$$

To evaluate the integral in the Equation-(4.2.9) we define generalised Zeta function as follows.

Definition(4.2.2) For $s > 1$ we write

$\zeta(s, v) = \sum_{n=0}^{\infty} (\nu + n)^{-s}, \nu \neq 0, 1, 2, \ldots$. It satisfies the functional equation and

$$\gamma(s, \nu) = \gamma(s, m' + \nu) + \sum_{n=0}^{m'-1} (\nu + n)^{-s}, \quad m' = 1, 2, 3, \ldots$$

For real $s > 0$ and real $\nu > 0$ we know that

$$\Gamma z = s^z \int_0^{\infty} y^{z-1} \exp(-sy) \, dy$$

and

$$(\nu + n)^{-s} \, \Gamma s = \int_0^{\infty} y^{s-1} \exp(-[\nu + n]y) \, dy,$$

That is

$$\Gamma s \; \zeta(s, \nu) = \int_0^{\infty} y^{s-1} \exp(-\nu y) \, [1 - \exp(-y)]^{-1} dy. \qquad (4.2.10)$$

By comparing the right hand side of the Equations-(4.2.9) and (4.2.10) we write

$$I_1 = (\xi - \eta)^{-3}\, \Gamma 3\, \zeta(3,\ \xi/(\xi - \eta)),$$

$$= 2\ (\xi - \eta)^{-3}\, \zeta(3,\ \xi/(\xi - \eta)).$$

Substituting value of the integral I_1 in Equation-(4.2.8) and using corresponding expression for expectation in Equation-(4.2.7) yields

$$E\left\{\partial^2 \log\ f(t;\ \eta,\ \xi)/\partial\eta^2\right\}$$

$$= -\left\{\xi(\xi - 2\eta)[\eta(\xi - \eta)]^{-2}\right\} - \left\{2(\xi-\eta)^{-4}\ \eta\xi\ \zeta(3,\ \xi/(\xi - \eta))\right\}$$

$$= -\left\{\xi(\xi - 2\eta)[\eta(\xi - \eta)]^{-2}\right\} - \eta\xi\ \phi_1, \qquad (4.2.11)$$

where $\phi_1 = 2\ (\xi-\eta)^{-4}\ \eta\xi\ \ \zeta(3,\ \xi/(\xi - \eta))$.

Similarly we obtain the expectations of the Equation(4.2.5) and (4.2.6) as

$$E\left\{\partial^2 \log f(t;\ \eta,\ \xi)/\partial\xi^2\right\} = -\left\{\eta(\eta - 2\xi)\ [\xi(\xi - \eta)]^{-2}\right\} - \eta\xi\ \phi_1 \qquad (4.2.12)$$

and

$$E\left\{\partial^2 \log f(t;\ \eta,\ \xi)/\partial\eta\partial\xi\right\} = E\left\{\partial^2 \log f(t;\ \eta,\ \xi)/\partial\xi\partial\eta\right\}$$

$$= -\ (\xi - \eta)^{-2} + \eta\xi\ \phi_1 \qquad (4.2.13)$$

respectively.

Now Information matrix is given by

$$I(\eta,\ \xi) = \begin{bmatrix} -E\ [\partial^2 \log f(t;\eta,\xi)/\partial\eta^2] & -E\ [\partial^2 \log f(t;\eta,\xi)/\partial\eta\partial\xi] \\ & \\ -E\ [\partial^2 \log f(t;\eta,\xi)/\partial\xi\partial\eta] & -E\ [\partial^2 \log f(t;\eta,\xi)/\partial\xi^2] \end{bmatrix}.$$

Substituting expressions for all these expectations as obtained in Equations-(4.2.11), (4.2.12) and (4.2.13) we get

$$
I(\eta, \xi) = \begin{bmatrix} \dfrac{\xi(\xi - 2\eta)}{\eta^2(\xi - \eta)^2} + \eta\,\xi\,\phi_1 & (\xi - \eta)^{-2} - \eta\,\xi\,\phi_1 \\ \\ (\xi - \eta)^{-2} - \eta\,\xi\,\phi_1 & \dfrac{\eta(\eta - 2\xi)}{\xi^2(\xi - \eta)^2} + \eta\,\xi\,\phi_1 \end{bmatrix}. \tag{4.2.14}
$$
□

Corollary(4.2.1) If T_1, T_2, ..., T_m are independent and identically distributed random variables with distribution given by Equation-(3.2.1). Then Information matrix corresponding to a sample of size m is given by Zacks(1981) as

$I_m(\eta, \xi) = m\, I(\eta, \xi)$ so that

$$
I(\eta, \xi) = m \begin{bmatrix} \dfrac{\xi(\xi - 2\eta)}{\eta^2(\xi - \eta)^2} + \eta\,\xi\,\phi_1 & (\xi - \eta)^{-2} - \eta\,\xi\,\phi_1 \\ \\ (\xi - \eta)^{-2} - \eta\,\xi\,\phi_1 & \dfrac{\eta(\eta - 2\xi)}{\xi^2(\xi - \eta)^2} + \eta\,\xi\,\phi_1 \end{bmatrix}. \tag{4.2.15}
$$

4.3 Maximum likelihood estimation (M.L.E.) for the parameters of Hypoexponential distribution

Let T_1, T_2, ..., T_m are independent and identically distributed random variables with distribution function $F(t,\theta)$ (either continuous or discrete) which depends on a unknown parameter θ.. Let Θ be the parametric space which is assumed to be subset of $\mathbb{R}^k$,the k dimensional euclidean space. Let $f(t,\theta)$ be the probability density function or probability mass function corresponding to $F(t,\theta)$. Then $f(t_1, t_2, ..., t_m; \theta)$ when $(t_1, t_2, ..., t_m)$ is fixed gives likelihood of observating $(t_1, t_2, ..., t_m)$ at θ.

Thus for fixed $(t_1, t_2, \ldots, t_m)$, $f(t_1, t_2, \ldots, t_m; \theta)$ is a function of θ and is called the likelihood function. The likelihood function is denoted by $L(\theta \mid t_1, t_2, \ldots, t_m)$ and is given by

$$L(\theta|t_1, t_2, \ldots, t_m) = f(t_1, t_2, \ldots, t_m; \theta)$$
$$= \prod_{i=1}^{m} f(t_i, \theta).$$

Suppose that there exist a value of θ; say $\hat{\theta} \in \Theta$ such that

$$L(\hat{\theta}|t_1, t_2, \ldots, t_m) = \sup_{\theta \in \Theta} L(\theta|t_1, t_2, \ldots, t_m)$$

That is $\hat{\theta}$ maximises likelihood function at $(t_1, t_2, \ldots, t_m)$. Note that $\hat{\theta}$ depends on $(t_1, t_2, \ldots, t_m)$ thus infact $\hat{\theta} = \hat{\theta}(t_1, t_2, \ldots, t_m)$. Also $\hat{\theta}$ may not exist for every $(t_1, t_2, \ldots, t_m) \in \Omega$; where Ω is sample space. If it exists for every $(t_1, t_2, \ldots, t_m) \in \Omega$ then we say that $\hat{\theta} : \Omega \longrightarrow \theta$ is M.L.E. of θ (provided that $\hat{\theta}$ is random variable).

In order to obtain the maximum likelihood estimates of the parameters of hypoexponential distribution, first we obtain the likelihood equations for the probability density function given in the Equation-(3.2.1).

Theorem(4.3.1) If $T_1, T_2, \ldots, T_m$ are independent and identically distributed random variables with probability density function given in the Equation-(3.2.1). Then maximum likelihood estimators $(\hat{\eta}, \hat{\xi})$ of (η, ξ) is given by the solution of the two equations.

$$m\,\eta\,[\xi(\xi-\eta)]^{-1} = \sum_{i=1}^{m}\left\{t_i \exp(-\xi t_i)[\exp(-\eta t_i) - \exp(-\xi t_i)]^{-1}\right\} \quad (4.3.1)$$

and

$$m\,\xi\,[\eta(\xi-\eta)]^{-1} = \sum_{i=1}^{m}\left\{t_i \exp(-\eta t_i)[\exp(-\eta t_i) - \exp(-\xi t_i)]^{-1}\right\} \quad (4.3.2)$$

Proof : Given that $T_1, T_2, \ldots, T_m$ are i.i.d. distributed random variables from hypoexponential distribution with parameters η and ξ having p.d.f. given in the Equation-(3.2.1). The joint p.d.f. of $T_1, T_2, \ldots, T_m$ is

$$L(\eta, \xi\,;\underline{t}) = [\eta\xi(\xi-\eta)^{-1}]^{m}\,[\exp(-\eta t) - \exp(-\xi t)]^{m}.$$

Then the log likelihood function is

$$\log L(\eta, \xi\,;\underline{t}) = m\log[\eta\xi(\xi-\eta)^{-1}] + \sum_{i=1}^{m}\log[\exp(-\eta t_i) - \exp(-\xi t_i)] \quad (4.3.4)$$

Differentiating partially with respect to η and ξ respectively we get

$$\partial\log L(\eta, \xi, \underline{t})/\partial\eta = m\,\xi\,[\eta(\xi-\eta)]^{-1} - \sum_{i=1}^{m}\left\{t_i \exp(-\eta\, t_i)[\exp(-\eta t_i) - \exp(-\xi t_i)]^{-1}\right\} \quad (4.3.5)$$

and

$$\partial\log L(t;\,\eta, \xi)/\partial\xi = -\,m\,\eta\,[\xi(\xi-\eta)]^{-1} + \sum_{i=1}^{m}\left\{t_i \exp(-\xi t_i)[\exp(-\eta t_i) - \exp(-\xi t_i)]^{-1}\right\}. \quad (4.3.6)$$

The second order partial differentiation of Equation-(4.3.4) with respect to η and ξ respectively gives

$\partial^2 \log L(t;\ \eta,\ \xi)/\partial\eta^2$

$$= -m\left\{\xi(\xi-2\eta)[\eta(\xi-\eta)]^{-2}\right\} - \sum_{i=1}^{m}\left\{t_i^2 \exp(-(\eta+\xi)t_i)\right.$$

$$\left.[\exp(-\eta t_i) - \exp(-\xi t_i)]^{-2}\right\} \quad (4.3.7)$$

and

$\partial^2 \log L(t;\ \eta,\ \xi)/\partial\xi^2$

$$= -m\left\{\eta(\eta-2\xi)[\xi(\xi-\eta)]^{-2}\right\} - \sum_{i=}^{m}\left\{t_i^2 \exp(-(\eta+\xi)t_i)\right.$$

$$\left.[\exp(-\eta\ t_i) - \exp(-\xi\ t_i)]^{-2}\right\}. \quad (4.3.8)$$

We observed in Equations-(4.3.7) and (4.3.8) that the value of these equations are negative. by equating (4.3.5) and (4.3.6) to zero we get

$$m\ \xi\ [\eta(\xi-\eta)]^{-1}$$

$$= \sum_{i=1}^{m}\left\{t_i \exp(-\eta t_i)[\exp(-\eta t_i) - \exp(-\xi t_i)]^{-1}\right\} \quad (4.3.9)$$

and

$$m\ \eta\ [\xi(\xi-\eta)]^{-1}$$

$$= \sum_{i=1}^{m}\left\{t_i \exp(-\xi t_i)[\exp(-\eta t_i) - \exp(-\xi t_i)]^{-1}\right\}. \quad (4.3.10)$$

Hence the m.l.e.'s are the solution of the Equations-(4.3.9) and (4.3.10).

Note that both the equations are not possible to solve analytically therefore we have to solve these two simultaneous equations. In order to obtain m.l.e.'s of parameters η and ξ in the following we have used iterative methods namely Newton-Raphson method and method of Scoring.

4.4 Newton-Raphson and Scoring Iterative method

In statistical problem solution of maximum likelihood estimates (m.l.e.'s) are essential. That is, to obtain certain points $\hat{\theta} = (\theta_1, \theta_2, \ldots, \theta_r)^T$ such that a log likelihood (log L) function is maximized.

In a few situations m.l.e.'s can be found analytically. But there are some situations for which, we can not find solution of likelihood equations analytically. Therefore, we use some iterative methods which are available in statistical literature. One of them is Newton-Raphson Iterative method. The procedure for obtaining $\hat{\theta}$ for which log L is maximum by Newton-Raphson Iterative method is given below.

Let $L(\theta_1, \theta_2, \ldots, \theta_r) = L(\theta)$ be the likelihood function defined on the parameter space Ω. Consider the situations in which point $\hat{\theta}$ at which $L(\theta)$, and so $\log L(\theta)$, is maximum and satisfies the likelihood equations

$$Q_i(\theta) = \partial \log L(\theta)/\partial\theta_i = 0, \; i = 1,2,\ldots r \qquad (4.4.1)$$

and $Q(\theta) = [Q_1(\theta), Q_2(\theta), \ldots, Q_r(\theta)]^T$, the r x 1 vector is called the score vector at θ.

Suppose that θ_o is a initial guess at $\hat{\theta}$ and expand each of the functions $Q_i(\theta)$ in a Taylor series about θ_o. The first order expansion gives

$$Q(\theta) = Q(\theta_o) + R(\theta_o)\,[\theta - \theta_o], \qquad (4.4.2)$$

Where $R(\theta)$ is the r x r matrix with entries

$$R_{ij}(\theta) = \partial^2 \log L(\theta)/\partial\theta_i\partial\theta_j$$

Since $\hat{\theta}$ satisfies $Q(\hat{\theta}) = 0$, The Equation-(4.4.2) gives the approximation

$$\hat{\theta} = \theta_o - Q(\hat{\theta}_o)/R(\hat{\theta}_o). \qquad (4.4.3)$$

To obtain $\hat{\theta}$ we use Equation-(4.4.3) iteratively. The right hand side of Equation-(4.4.3) gives a second approximation θ_1 to $\hat{\theta}$; this is in turn inserted in the right hand side of (4.4.3) to give a third approximation, and so on. This method is called Newton-Raphson method, provided that log $L(\theta)$ is well behaved at $\hat{\theta}$ and the initial value θ_o is chosen appropriately the sequence of approximations generated will converge to $\hat{\theta}$.

The Scoring method has convergence property similar to Newton-Raphson iteration and gives a simpler looking algorithms than Newton-Raphson method. The method of Scoring; we can describe as follows.

In Equation-(4.4.3) if we substitute $-I(\theta)$ for $R(\theta)$, the negative of the expected information matrix given in definition(4.2.15), which is a minor adjustment to the Newton-Raphson method the resulting procedure is known as the " Scoring method ". The Scoring method also yields the estimated value of θ. To develop the Computer program in FORTRAN-77 for Newton-Raphson method one has to find $R^{-1}(\theta_o)$ and $Q(\theta_o)$. Now we differentiate the Equation-(4.3.5) with respect to ξ we get

$$\partial^2 \log L(\eta, \xi, \underline{t})/\partial\xi\partial\eta = \partial^2 \log L(\eta, \xi, \underline{t})/\partial\eta\partial\xi$$

$$= -m(\xi - \eta)^{-1} + \sum_{i=1}^{m} \Big\{ t_i^2 \exp(-(\eta + \xi)t_i) [\exp(-\eta t_i) - \exp(-\xi t_i)]^{-2} \Big\},$$

(4.4.4)

If we write

$$M1 = \xi\ [\eta\ (\xi - \eta)]^{-1}, \qquad M2 = \eta\ [\xi\ (\xi - \eta\ )]^{-1},$$

$$M3 = \left\{\xi\ (\xi - 2\eta)[\eta\ (\xi - \eta\ )]^{-2}\right\},$$

$$M4 = \left\{\ \eta\ (\eta - 2\xi)[\xi\ (\xi - \eta\ )]^{-2}\right\}, \qquad M5 = (\xi - \eta)^{-2},$$

$$S1 = \sum_{i=1}^{m}\left\{t_i\ \exp(-\ \eta t_i)\ [\exp(-\ \eta\ t_i) - \exp(-\ \xi t_i)]^{-1}\right\},$$

$$S2 = \sum_{i=1}^{m}\left\{t_i\ \exp(-\ \xi t_i)\ [\exp(-\ \eta t_i) - \exp(-\ \xi t_i)]^{-1}\right\},$$

$$S3 = \sum_{i=1}^{m}\left\{t_i^2\ \exp(-\ (\eta + \xi)t_i)\ [\exp(-\ \eta t_i) - \exp(-\ \xi t_i)]^{-2}\right\},$$

then Equations-(4.3.5),(4.3.6),(4.3.7),(4.3.8), and (4.4.4) becomes

$$\partial \log\ L(\eta,\ \xi, \underline{t})/\partial\eta = m\ M1 - S1, \tag{4.4.5}$$

$$\partial \log\ L(t;\ \eta,\ \xi)/\partial\xi = -\ m\ M2 + S2, \tag{4.4.6}$$

$$\partial^2 \log\ L(t;\ \eta,\ \xi)/\partial\eta^2 = -\ (m\ M3 + S3), \tag{4.4.7}$$

$$\partial^2 \log\ L(t;\ \eta,\ \xi)/\partial\xi^2 = -(m\ M4 + S3), \tag{4.4.8}$$

$$\partial^2 \log\ L(\eta,\ \xi,\ \underline{t})/\partial\xi\partial\eta = \partial^2 \log\ L(\eta,\ \xi,\ \underline{t})/\partial\eta\partial\xi$$

$$= -\ m\ M5 + S3\ . \tag{4.4.9}$$

Therefore

$$Q(\theta) = \begin{bmatrix} \partial \log L(\eta,\xi;\underline{t})/\partial\eta \\ \\ \partial \log L(\eta,\xi;\underline{t})/\partial\xi \end{bmatrix} = \begin{bmatrix} m\ M1 - S1 \\ \\ S2 - m\ M2 \end{bmatrix} \tag{4.4.10}$$

and

$$R(\theta) = \begin{bmatrix} [\ \partial^2 \log L(\eta,\ \xi;\ \underline{t})/\partial\eta^2] & [\ \partial^2 \log L(\eta,\ \xi;\ \underline{t})/\partial\eta\partial\xi] \\ \\ [\ \partial^2 \log L(\eta,\ \xi;\ \underline{t})/\partial\xi\partial\eta] & [\ \partial^2 \log L(\eta,\ \xi;\ \underline{t})/\partial\xi^2] \end{bmatrix}$$

$$= \begin{bmatrix} -[m\,M3 + S3] & [S3 - m\,M5] \\ [S3 - m\,M5] & -[m\,M4 + S3] \end{bmatrix} \qquad (4.4.11)$$

Hence the Inverse of $R(\theta)$ is given by

$$R^{-1}(\theta) = 1\,/DET \begin{bmatrix} -[m\,M4 + S3] & [m\,M5 - S3] \\ [m\,M5 - S3] & -[m\,M3 + S3] \end{bmatrix} \qquad (4.4.12)$$

Where $DET = m^2\,(M3\,M4 - M5^2) + m\,S3\,(M3 + M4 - 2\,M5)$. Using Equation-(4.4.10) and (4.4.12) we develop the computer program in FORTRAN-77 to obtain $\hat{\eta}$ and $\hat{\xi}$ from Newton-Raphson method and method of Scoring will be supplied in Appendix-I(A).

In order to obtain m.l.e.'s of η and ξ, generate m observations from hypoexponential distribution with known parameter values η and ξ. By using the computer program for these iterative procedures, first we need to obtain initial values of η and ξ and it is denoted by $\hat{\eta}_0$ and $\hat{\xi}_0$ by the method of moments. Let $t_1, t_2, \ldots, t_m$ be a random sample of size m drawn from a population whose density function given in Equation-(3.2.1). Now from Equation-(3.2.5) we write

$$\bar{t} = 1/\eta + 1/\xi \qquad (4.4.13)$$

and $$S_t^2 = 1/\eta^2 + 1/\xi^2, \qquad (4.4.14)$$

where $\bar{t} = \sum_{i=1}^{m} t_i/m$ and $S_t^2 = \sum_{i=1}^{m} (t_i - \bar{t})^2/(m-1)$.

Therefore $\bar{t}^2 = 1/\eta^2 + 1/\xi^2 + 2/[\eta\xi]$.

That is $\bar{t}^2 = S_t^2 + 2/[\eta\xi]$.

This implies $\eta^{-1} = 2\,[\xi(\bar{t}^2 - S_t^2)]^{-1}$.

Substituting in (4.4.13) we get

$$(\bar{t}^2 - S_t^2)\, \xi^2 - 2\, \xi\, \bar{t} + 2 = 0,$$

which is a quadratic equation in ξ. On Simplification we write

$$\xi = (\bar{t} + [2\, S_t^2 - \bar{t}^2]^{1/2})/(\bar{t}^2 - S_t^2)$$

$$= 2\, (\bar{t} \mp [2\, S_t^2 - \bar{t}^2]^{1/2})^{-1}.$$

Again substituting in Equation-(4.4.13) we get

$$\eta = 2\, (\bar{t} \pm [2\, S_t^2 - \bar{t}^2]^{1/2})^{-1}.$$

Hence required estimates of η and ξ by method of moments is given by

$$\hat{\eta}_o = 2\, (\bar{t} \pm [2\, S_t^2 - \bar{t}^2]^{1/2})^{-1} \qquad (4.4.15)$$

and

$$\hat{\xi}_o = 2\, (\bar{t} \mp [2\, S_t^2 - \bar{t}^2]^{1/2})^{-1}. \qquad (4.4.16)$$

Now if $\bar{t} > [2\, S_t^2 - \bar{t}^2]^{1/2}$, we substitute each pair $(\hat{\eta}_o, \hat{\xi}_o)$ from Equations-(4.4.15) and (4.4.16) into the likelihood function $L(\eta, \xi;\, \underline{t})$ and choose as our starting values of that pair which gives the likelihood and if $\bar{t} < [2\, S_t^2 - \bar{t}^2]^{1/2}$, we use $\hat{\xi}_o = 2/\bar{t}$, the successive pairs $(\hat{\eta}_1, \hat{\xi}_1), (\hat{\eta}_2, \hat{\xi}_2), \ldots, (\hat{\eta}_m, \hat{\xi}_m)$ are then obtained iteratively by either Newton-Raphson method or method of Scoring.

4.5 Numerical comparisons of the m.l.e.'s by Newton-Raphson method and method of Scoring.

In the following the numerical comparisons of m.l.e.'s of η and ξ by using Newton-Raphson method and method of Scoring are obtained. We performed a Monte Carlo study of the two iterative procedures. We obtained three different estimates of η and ξ by repeating the process 100 times for different sample sizes from the following methods

i) the method of moments estimates

ii) the Scoring estimates

and iii) the Newton-Raphson estimates.

The results are given in the following Table.

TABLE (4.5.1)

Estimates of $\eta = 1$, $\xi = 1$. Method of moments,Newton-raphson and Scoring for 100 samples of m = 200 and m = 500 respectively.

Average sample statistics.

For m = 200

	Method of moments	Scoring	Newton-Raphson
$\hat{\eta}$	0.7964075	0.8002702	0.7974517
$\hat{\xi}$	1.414065	1.295997	1.411505

For m = 500

	Method of moments	Scoring	Newton-Raphson
$\hat{\eta}$	0.8258979	0.8285895	0.8262493
$\hat{\xi}$	1.319038	1.278176	1.318527

We observed that the method of moments estimates are not too unreasonable except for the bias in the estimates for both the Newton-Raphson method and method of Scoring.

===OO===