CHAPTER - III

A COMPOSITE PROCEDURE FOR SEGMENTED AND NON SEGMENTED MATERIAL

3.1 Introduction:-

In the previous chapter, we studied the composite sampling procedure for "Bulk material" by considering a linear model. In this chapter, we consider two other models, given by Brown and Fisher(1972) and Rohde(1977). These models are based on the assumption that subsample values are weighted averages of increments in the composite sample, where weight are random. These models are useful when the population is either in segmented form or in nonsegmented form.

3.2 Composite Sampling Procedure for segmented populations>

This procedure was first introduced by Brown and Fisher (1972). The model for this procedure has parameters (N,J,K,I,r,1). That is in this procedure, first select J segments from a population of N segments. Then from each selected segment, take K increments. Mix these JK increments to form a composite sample. From this composite sample, select r subsamples and take one test on each subsample.

Let $y_1, y_2, \ldots y_r$ be observations on the r subsamples. Let x_{jk} be observation on the charactristic based on k^{th} increment of jth segment (j=1,2,...J, K= 1,2,...K). Let us assume that x_{jk} are such that

$$X_{jk} = \mu + b_j + c_{jk} \qquad (3.2.1)$$

Where μ is the mean charactristic of population, b_j is the deviation of mean of jth segment from μ such that $\sum_{j=1}^{N} b_j = 0$ and

$$\sum_{j=1}^{N} b_j^2 = N\,\sigma_b^2 \qquad (3.2.2)$$

and c_{jk} is random error associated with x_{jk} having mean zero and variance σ_c^2 .

With these assumptions .

$$E(x_{jk}) = E(\mu + b_j + c_{jk}) = \mu \qquad (3.2.3.(a))$$

$$Var(x_{jk}) = Var(\mu + b_j + c_{jk}) = \sigma_b^2 + \sigma_c^2 \qquad (3.2.3.(b))$$

$$Cov(x_{jk}, x_{j'k}) = E(x_{jk}x_{j'k}) - E(x_{jk})E(x_{j'k})$$
$$= E(\mu + b_j + c_{jk})(\mu + b_{j'} + c_{j'k}) - \mu^2$$
$$= \mu^2 + E(b_j b_{j'}) - \mu^2$$
$$= E(b_j b_{j'})$$

But we have $(\sum_j b_j)^2 = \sum_{j=1}^{m} b_j^2 + \sum_j \sum_{j \neq j'} b_j b_j'$

$$0 = N\sigma_b^2 + N(N-1)E(b_j b_j')$$
$$E(b_j b_j') = -\sigma_b^2 / N-1$$
$$\therefore cov(x_{jk}, x_{j'k}) = -\sigma_b^2/N-1$$

similarly

$$cov(x_{jk}, x_{jk'}) = +\sigma_b^2$$
$$cov(x_{jk}, x_{j'k'}) = -\sigma_b^2 / N-1$$

Hence $$cov(x_{jk}, x_{j'k'}) = \begin{cases} \sigma_b^2 & \text{if } k \neq k' \\ -\sigma_b^2 / N-1 & \text{if } j \neq j' \text{ or } JK \neq j'k \end{cases}$$

(3.2.2(c)

Let us assume that the amount of material in each increment is 'g' and in each subsample it is 'G'. If the composite sample is exhaustively subsampled in to R subsamples then.

$$JKg = RG$$

Let g_{ijk} be the amount of material in i^{th} subsample from k^{th} increment of jth segment then

$$\sum_{i=1}^{R} g_{ijk} = g \text{ and } \sum_{jk=1}^{JK} g_{ijk} = G \qquad (3.2.4)$$

If g is not same for all (jk), then we have to adjust G so that

$$JK\bar{g} = RG$$

Define $\alpha_{ijk} = g_{ijk}/G$

Then α_{ijk} is the proportion of $(jk)^{th}$ increment incorporated in i^{th} subsample. Here we assume that α_{ijk}'s to be independent among themselves and also independent of x_{jk}'s.

Let $E(\alpha_{jk}) = \mu_\alpha$ and $\text{var}(\alpha_{ijk}) = \sigma_\alpha^2$

As this model is based on randomly weighted averages, each increment has an equal probability of occurence and is 1/JK Therefore,

$$\begin{aligned} E(\alpha_{ijk}) &= \sum_{jk} \alpha_{ijk} P(\alpha_{ijk}) \\ &= \sum_{jk} \alpha_{ijk} \, 1/JK \\ &= 1/JK \end{aligned} \qquad (3.2.5)$$

Theorem (3.2.1)

If α_{ijk} is the proportion of $(jk)^{th}$ increment incorporated in i^{th} subsample then

$$\text{Cov}(\alpha_{ijk}, \alpha_{i'j'k'}) = \begin{cases} -\sigma_\alpha^2/(R-1) & \text{if } i \neq i' \\ -\sigma_\alpha^2/(JK-1) & \text{if } JK \neq j'k' \\ \sigma_\alpha^2/(R-1)(JK) & \text{if } i \neq i' \text{ and } JK \neq j'k' \end{cases}$$

Proof: By using (3.2.4)

$$\begin{aligned} \text{Var}\left(\sum_{i=1}^{R} \alpha_{ijk}\right) &= \text{var}\left[\sum_{i=1}^{R} \text{var}\, g_{ijk} / G\right] \\ &= \text{var}\,(g/G) \\ &= 0 \end{aligned}$$

$$\text{Var}\left(\sum_{i=1}^{R} \alpha_{ijk}\right) = \text{var}\left[\sum_{i=1}^{R} \text{var}(\alpha_{ijk}) + \sum_{i} \sum_{i' \neq i} \text{cov}(\alpha_{ijk}, \alpha_{i'jk}) = 0\right.$$

$$R\,\sigma_\alpha^2 + R(R-1)\,\text{cov}(\alpha_{ijk}, \alpha_{i'jk}) = 0$$

$$\text{cov}(\alpha_{ijk}, \alpha_{i'jk}) = -\sigma_\alpha^2/R-1$$

Also we have

$$\sum_{jk=1}^{jk} \alpha_{ijk}=1$$

$$\text{var} \sum_{jk=1} \alpha_{ijk}=0$$

$$\sum_{jk=1}^{JK} \text{var}\ (\alpha_{ijk}) + \sum_{jk} \sum_{jk \neq jk'} \text{cov}\ (\alpha_{ijk}, \alpha_{ij'k'}) = 0$$

$$JK\ \sigma_\alpha^2 + JK\ (JK-1)\ \text{cov}\ (\alpha_{ijk}, \alpha_{ij'k'})=0$$

$$\text{cov}\ (\alpha_{ijk}, \alpha_{ij'k'}) = -\sigma_\alpha^2 / JK-1$$

Also

$$\sum_{i=1}^{R} \sum_{jk=1}^{JK} \alpha_{ijk}= R$$

Hence $\text{var}\ (\sum_i \sum_{jk} \alpha_{ijk}) = 0$

$$\sum_i \sum_{jk} \text{var}\ (\alpha_{ijk}) + \sum_i \sum_{jk} \sum_{i \neq i'} \text{cov}\ (\alpha_{ijk}, \alpha_{i'jk}) +$$

$$\sum_i \sum_{jk} \sum_{jk \neq j'k'} \text{cov}(\alpha_{ijk}, \alpha_{ij'k'}) + \sum_i \sum_{jk} \sum_{i \neq i'} \sum_{jk \neq j'k'} \text{cov}(\alpha_{ijk}, \alpha_{i'j'k'})=0$$

$$RJK\ \sigma_\alpha^2 + RJK(R-1)\ \text{cov}\ (\alpha_{ijk}, \alpha_{i'jk}) + RJK(JK-1)\ \text{cov}\ (\alpha_{ijk}, \alpha_{ij'k'}) + RJK(R-1)(JK-1)\text{cov}(\alpha_{ijk}, \alpha_{i'j'k'})=0$$

$$RJK\ \sigma_\alpha^2 + RJK(R-1)\ \left(\frac{-\sigma_\alpha^2}{(R-1)}\right) + RJK\ (JK-1)\left(\frac{-\sigma_\alpha^2}{JK-1}\right) + RJK(R-1)(JK-1)\cdot$$

$$\text{cov}\ (\alpha_{ijk}, \alpha_{i'j'k'}) = 0\ ,$$

$$\text{cov}\ (\alpha_{ijk}, \alpha_{i'j'k'}) = \sigma_\alpha^2 / (R-1)(JK-1)$$

Lemma (3.2.1)

If (U,U') is a pair of random variables such that $E(U)=E(U')$ and $\text{var}\ (U) = \text{var}\ (U')$ and (W,W') is another such pair independent of (U,U') then

$$\text{cov}(UW,U'W') = \text{cov}(U,U')\text{cov}(W,W') + E(U)^2\text{cov}(W,W') + E(W)^2\text{cov}\ (U,U')$$

Proof:

From conditional expectation definition

$$\text{cov}\,(UW, U'W') = E\,(\text{cov}(UW, U'W')/UU') + \text{cov}\,(E(UW, E(U'W')/UU')$$

$$= E(U\text{cov}(W, W')U') + \text{cov}(UE(W), U'E(W'))$$

$$= E(UU')\ \text{cov}\,(W, W') + E(W)\ \text{cov}\,(U, U')\ E\,(W')$$

$$= (\text{cov}(U, U') + E(U)\ E(U'))\ \text{cov}\,(W, W') + \text{cov}(U, U')\ E(W)^2$$

$$= \text{cov}\,(U, U')\ \text{cov}(W, W') + E(U)^2\ \text{cov}(W, W') + E(W)^2\ \text{cov}(U, U')$$

Corollari 3.2.1

If U=U' and W=W' then above lemma becomes

$$\text{var}\,(UW) = \text{Var}\,(U)\ \text{var}\,(W) + E(U)^2\ \text{var}(W) + E(W)^2\ \text{var}\,(U)$$

Theorem (3.2.2)

If $y_1, y_2 \ldots y_r$ are observations on subsamples then

i) subsample mean $\bar{y}$ is an unbiased estimate of μ and

ii) $$\text{var}\,(\bar{y}) = \left[\frac{JK^2N(J-1)(R-r)}{(JK-1)(R-1)(N-1)}\ \sigma^2\alpha\sigma_b^2 + \frac{JK(R-r)}{(R-1)}\ \sigma^2\alpha\sigma^2 c + \frac{r(N-J)}{J(N-1)}\ \sigma_b^2\ \sigma_c^2 + \frac{r}{JK}\ \sigma_c^2 \right]$$

Proof:

As $y_1, y_2, \ldots y_r$ are r observations on subsamples then we can write any observation y_i as

$$y_i = \sum_j \sum_k \alpha_{ijk}\ x_{jk} \qquad i = 1, 2, \ldots r$$

and its mean is given by

$$\bar{y} = 1/r \sum_i \sum_j \sum_k \alpha_{ijk}\ x_{jk}$$

therefore

$$E(\bar{y}) = 1/r \sum_i \sum_j \sum_k E(\alpha_{ijk}\ x_{jk})$$

$$= 1/r \sum_i \sum_j \sum_k E(\alpha_{ijk}) E(x_{jk})$$

$$= 1/r \sum_i \sum_j \sum_k \mu_\alpha\ \mu$$

$$= \mu \qquad \therefore\ \mu_\alpha = 1/JK \qquad [3.2.7]$$

Now $\quad var(\bar{y}) = var\left[= 1/r \sum_i \sum_j \sum_k \alpha_{ijk}\, x_{jk}\right]$

$$=1/r^2 \left[\sum_i \sum_j \sum_k var\,(\alpha_{ijk}x_{jk}) + \sum_{ijk} \sum_{i'\neq i} cov\,(\alpha_{ijk}\, x_{jk}, \alpha_{i'jk}x_{jk}) \right]$$

$$+ \sum_{ijk} \sum_{j'\neq j} cov\,(\alpha_{ijk}x_{jk}, \alpha_{ij'k}x_{j'k}) + \sum_{ijk} \sum_{k'\neq k} cov\,(\alpha_{ijk}x_{jk}, x_{jk'})$$

$$+ \sum_{ijk} \sum_{i'\neq i} \sum_{j'\neq j} cov\,(\alpha_{ijk}x_{jk}, \alpha_{i'jk}x_{jk}) + \sum_{ijk} \sum_{i'\neq i} \sum_{k'\neq k}$$

$$cov\,(\alpha_{ijk}x_{jk}\alpha_{i'jk}\, x_{jk'}) + \sum_{ijk} \sum_{j'\neq j} \sum_{k'\neq k} cov\,(\alpha_{ijk}x_{jk}, \alpha_{ij'k}x_{j'k'})$$

$$+ \sum_{ijk} \sum_{i'\neq i} \sum_{j'\neq j} \sum_{k'\neq k} cov\,(\alpha_{ijk}x_{jk}\alpha_{i'j'k'}\, x_{j'k'}) \qquad [3.2.8]$$

Now by using Lemma (3.2.1) and equations (3.2.3) (3.2.5),(3.2.6) we get.

$$var\,(\alpha_{ijk}\, x_{jk}) = var\,(\alpha_{ijk})\, var(x_{jk}) + E(\alpha_{ijk})^2 var(x_{jk})$$
$$+ E(x_{jk})^2 var\,(\alpha_{ijk})$$
$$= \sigma^2\alpha(\sigma_b^2+\sigma_c^2) + \mu_\alpha^2(\sigma_b^2+\sigma_c^2) + \mu^2\sigma_\alpha^2$$

$$= \sigma^2\alpha\, \sigma_b^2+\sigma_c^2\, \sigma_\alpha^2 + 1/J^2K^2\, (\sigma_b^2+\sigma_c^2) + \mu^2\sigma_\alpha^2 \qquad [3.2.8(a)]$$

$$cov(\alpha_{ijk}x_{jk}, \alpha_{i'jk}x_{jk}) = cov(\alpha_{ijk}, \alpha_{i'jk})\, cov(x_{jk}, x_{jk}) + E(x_{jk})^2$$

$$cov(\alpha_{ijk}\alpha_{i'jk}, \alpha_{ij'k}) + E(\alpha_{ijk})^2\, cov\,(x_{jk}, x_{jk})$$
$$= \left(-\frac{\sigma_\alpha^2}{(R-1)}\right)(\sigma_b^2+\sigma_c^2) + \mu_\alpha^2(\sigma_b^2+\sigma_c^2) + \mu^2\left(\frac{-\sigma_\alpha^2}{(R-1)}\right)$$
$$= -\frac{\sigma_\alpha^2\, \sigma_b^2}{R-1} - \frac{\sigma_\alpha^2\sigma_c^2}{R-1} + \frac{1}{J^2K^2}(\sigma_b^2 + \sigma_c^2) - \frac{\mu^2\sigma_\alpha^2}{R-1} \qquad [3.2.8(b)]$$

$$cov(\alpha_{ijk}x_{jk}, \alpha_{ij'k}x_{j'k}) = cov\,(\alpha_{ijk}, \alpha_{ij'k})\, cov(x_{jk}, x_{j'k})$$
$$+ E(\alpha_{ijk})^2\, cov\,(x_{jk}, x_{j'k}) + E(x_{jk})^2\, cov\,(\alpha_{ijk}, \alpha_{ij'k})$$

$$= \left(\frac{\sigma^2_\alpha}{JK-1} \right) \left(\frac{-\sigma^2_b}{N-1} \right) + \mu^2_\alpha \left(\frac{-\sigma^2_b}{N-1} \right) + \mu^2 \left(\frac{-\sigma^2_\alpha}{JK-1} \right)$$

$$= \frac{\sigma^2_\alpha \sigma^2_b}{(JK-1)(R-1)} - \frac{\sigma^2_b}{J^2K^2(N-1)} - \frac{\mu^2(-\sigma^2_\alpha)}{JK-1} \qquad [3.2.8(c)]$$

Similary we find

$$\mathrm{cov}(\alpha_{ijk}, x_{jk}, \alpha_{ijk'} x_{jk'}) = \frac{-\sigma^2_\alpha \sigma^2_b}{JK-1} + \frac{\sigma^2_b}{J^2K^2} - \frac{\mu^2\sigma^2_\alpha}{(JK-1)} \qquad [3.2.8(d)]$$

$$\mathrm{cov}(\alpha_{ijk} x_{jk}, \alpha_{i'j'k} x_{j'k}) = \frac{-\sigma^2_\alpha \sigma^2_b}{(N-1)(R-1)(JK-1)} - \frac{\sigma^2_b}{J^2K^2(R-1)} + \frac{\mu^2\sigma^2_\alpha}{(R-1)(JK-1)}$$

$$[\ 3.2.8(e)]$$

$$\mathrm{cov}\ (\alpha_{ijk} x_{jk}, \alpha_{i'jk'} x_{jk'}) = \frac{\sigma^2_\alpha \sigma^2_b}{(R-1)(JK-1)} + \frac{\sigma^2_\alpha}{J^2K^2} + \frac{\mu^2\sigma^2_\alpha}{(R-1)(JK-1)}$$

$$[3.2.8(f)]$$

$$\mathrm{cov}\ (\alpha_{ijk} x_{jk}, \alpha_{ij'k'} x_{j'k'}) = \frac{\sigma^2_\alpha \sigma^2_b}{(N-1)(JK-1)} - \frac{\sigma^2_b}{J^2K^2(N-1)} - \frac{\mu^2\sigma^2_\alpha}{(JK-1)}$$

$$[3.2.8(g)]$$

$$\mathrm{cov}\ (\alpha_{ijk} x_{jk}, \alpha_{i'j'k'} x_{j'k'}) = \frac{-\sigma^2_\alpha \sigma^2_b}{(H-1)(R-1)(JK-1)} - \frac{\sigma^2_b}{J^2K^2(N-1)} + \frac{\mu^2 \sigma^2_\alpha}{(R-1)(JK-1)}$$

$$[3.2.8(h)]$$

Substituting these values in equation (3.2.8) we get

$$\mathrm{var}\ (\bar{y}) = \frac{1}{r^2} \left[rJK \left(\sigma^2_\alpha \sigma^2_b + \sigma^2_\alpha \sigma^2_c + \frac{(\sigma^2_b + \sigma^2_c)}{J^2 K^2} + \mu^2 \sigma^2_\alpha \right) \right.$$

$$+ rJK(r-1) \left[\frac{-\sigma^2_\alpha \sigma^2_b}{R-1} - \frac{\sigma^2_\alpha \sigma^2_c}{R-1} + \frac{\sigma^2_b + \sigma^2_c}{J^2K^2} - \frac{\mu^2\sigma^2_\alpha}{R-1} \right]$$

$$+ rJK(J-1) \left[\frac{\sigma^2_\alpha \sigma^2_b}{(N-1)(JK-1)} - \frac{\sigma^2_b}{(N-1)J^2K^2} - \frac{\mu^2 \sigma^2_\alpha}{JK-1} \right]$$

$$+rJK(k-1)\left[\frac{-\sigma_\alpha^2\sigma_b^2}{JK-1}+\frac{\sigma_b^2}{J^2K^2}-\frac{\mu^2\sigma_\alpha^2}{JK-1}\right]$$

$$+\ rJK(r-1)\ (J-1)\left[\frac{-\sigma_\alpha^2\ \sigma_b^2}{(N-1)(R-1)(J-1)}-\frac{\sigma_b^2}{(N-1)J^2K^2}+\frac{\mu^2\ \sigma_\alpha^2}{(JK-1)(R-1)}\right]$$

$$+\ rJK(r-1)\ (K-1)\left[\frac{\sigma_\alpha^2\ \sigma_b^2}{(R-1)(JK-1)}+\frac{\sigma_b^2}{J^2K^2}+\frac{\mu^2\ \sigma_\alpha^2}{(JK-1)(R-1)}\right]$$

$$+\ rJK(J-1)\ (K-1)\left[\frac{\sigma_\alpha^2\ \sigma_b^2}{(N-1)(JK-1)}-\frac{\sigma_b^2}{(N-1)J^2K^2}-\frac{\mu^2\ \sigma_\alpha^2}{(JK-1)}\right]$$

$$+\ rJK(r-1)(J-1)(K-1)\left[\frac{-\ \sigma_\alpha^2\ \sigma_b^2}{(N-1)(R-1)(JK-1)}-\frac{\sigma_\alpha^2}{J^2K^2(N-1)}+\frac{\mu^2\ \sigma_\alpha^2}{(R-1)(JK-1)}\right]$$

After little simplification we get

$$var(\bar{y})=1/r\left[\frac{JK^2N(J-1)(R-r)}{(JK-1)(R-1)(N-1}\ \sigma_\alpha^2\ \sigma_b^2+\frac{JK(R-r)}{R-1}\ \sigma_\alpha^2\sigma_c^2\right.$$

$$\left.+\ \frac{r(N-J)}{J(N-1)}\ \sigma_b^2+\frac{r}{JK}\ \sigma_c^2\right] \qquad [3.2.9]$$

Corollari:3.2.2.(a)

If population contains infinite number of lots that is if $N \to \infty$ then

$$var(\bar{y})=1/r\left[\frac{JK^2(J-1)(R-r)}{(JK-1)(R-1)}\ \sigma_\alpha^2\sigma_b^2+\frac{JK(R-r)}{R-1}\ \sigma_\alpha^2\sigma_c^2+r/J\ \sigma_b^2+\frac{r}{JK}\ \sigma_c^2\right]$$

Corollari 3.2.2(b)

If we select increments from each segment of lot that is if N=J then

$$var\ (\bar{y})=\frac{1}{r}\left[\frac{N^2K^2(R-r)}{(JK-1)(R-1)}\ \sigma_\alpha^2\ \sigma_b^2+\frac{NK\ (R-r)}{(R-1)}\ \sigma_\alpha^2\sigma_c^2+\frac{r}{NK}\ \sigma_c^2\right]$$

Here variation due to segment vanishes.

corollari 3.2.2(c)

If composite sample is exhaustively subsampled that is if R=r then

$$\text{var}(\bar{y}) = \frac{N-J}{J(N-1)}\sigma_b^2 + \frac{\sigma_c^2}{JK} \qquad [3.2.9(c)]$$

Here variation due to proportion vanishes.

Corollari 3.2.2(d)

If r= R and N = J then

$$\text{var}(\bar{y}) = \sigma_c^2/(NK) \qquad [3.2.9(d)]$$

In the general formula for var($\bar{y}$) (3.2.9) we see that it involves three unknown quantities σ_b^2, σ_c^2 and σ_α^2. Estimates of σ_b^2 and σ_c^2 can be obtained from laboratory sample. Hence only unknown is σ_α^2 ; the variation due to proportions. To obtain estimate of σ_α^2 we proceed as follows.

Suppose there are s balls each, of colour $c_1, c_2 \ldots c_{jk}$. Mix these JKS balls to form a composite sample. Select 'm' balls. Then probability of selecting x_{il} balls of colour 'c_l' (l=1,2,3,..JK) is given by

$$P(x_{i1}, x_{i2}, \ldots x_{ijk}) = \frac{\binom{S}{x_{i1}}\binom{S}{x_{i2}} \cdots \binom{S}{x_{ijk}}}{\binom{JKS}{m}} \qquad [3.2.10]$$

The marginal probability mass function for the number of balls of any colour drawn in the sample is

$$P(x_{il}) = \frac{\binom{S}{x_{il}}\binom{JKS-S}{M-x_{il}}}{\binom{JKS}{M}}, \; l = 1,2,\ldots,JK \qquad [3.2.11]$$

Which is probability mass function of hypergeometric distribution.

The moments of hypergeometric distribution are

$$E(x_{il}) = \frac{M}{JKS} S$$

$$= \frac{M}{JK}$$

and

$$\text{var}\ (x_{i1}) = \frac{MS}{J^2K^2S^2(JKS-1)}\ (JKS-S)\ (JKS-M)$$

$$= \frac{M(JKS-S)(JKS-M)}{J^2K^2S^2(JKS-1)}$$

By definition of proportion $\alpha_{ijk} = \frac{x_{ijk}}{M}$,
the proportion of balls of colour c_1 in sample.

$$E(\alpha_{ijk}) = E\left(\frac{x_{ijk}}{M}\right)$$

$$= 1/M\ E\ (x_{ijk})$$

$$= 1/M \cdot M/JK$$

$$E\ (\alpha_{ijk}) = 1/JK \quad \text{and} \qquad [3.2.12]$$

$$\text{var}\ (\alpha_{ijk}) = \text{var}\left[\frac{x_{ijk}}{M}\right]$$

$$= 1/M^2\ \text{var}\left[x_{ijk}\right]$$

$$= 1/M^2\left[\frac{M(JKS-S)\ (JKS-M)}{J^2K^2S\ (JKS-1)}\right]$$

$$= \frac{(JK-1)\ (JKS-M)}{MJ^2K^2(JKS-1)}$$

$$\therefore\ \sigma_\alpha^2 = \frac{(JK-1)\ (JKS-M)}{MJ^2\ K^2(JKS-1)} \qquad [3.2.13]$$

If we assume that each segment is sampled that is if N=J then

$$\sigma_\alpha^2 = \frac{(NK-1)\ (NKS-M)}{MN^2\ K^2(NKS-1)} \qquad [3.2.13]$$

Using this value of σ_α^2 in equation (3.2.9(b)) we get,

$$\text{var}\ (\bar{y}) = \frac{1}{r}\left[\frac{N^2K^2(R-r)}{(NK-1)(R-1)} \cdot \frac{(NK-1)(NKS-M)}{MN^2K^2(NKS-1)}\ \sigma_b^2 + \frac{NK(R-r)}{(R-1)}\right.$$

$$\left.\frac{(NKS-M)\ (NK-1)}{MN^2K^2\ (NKS-1)}\ \sigma_c^2 + \frac{r}{NK}\ \sigma_c^2\right]$$

$$= \frac{1}{r}\left[\frac{(R-r)\ (NKS-M)}{(NKS-1)(R-1)}\ \sigma_b^2 + \frac{(R-r)(NK-1)(NKS-M)}{MNK\ (NKS-1)(R-1)}\ \sigma_c^2 + \frac{r}{NK}\ \sigma_c^2\right]$$

If S=M and R = NK then

$$\text{var}(\bar{y}) = \frac{1}{r}\left[\frac{R-r}{(RS-1)}\sigma_b^2 + \frac{R-r}{(RS-1)}\frac{(R-1)}{R}\sigma_c^2 + \frac{r}{R}\sigma_c^2\right]$$

$$\approx 1/r\left[\frac{R-r}{RS-1}(\sigma_b^2 + \sigma_c^2) + \frac{r}{R}\sigma_c^2\right]; \text{ since } \frac{R-1}{R} \leq 1$$

Thus value of S is seen to be critical in determining the effect of σ_α^2

This is an under estimate because of the approximation.
It we take s --→ ∞ then

$$\text{var}(\bar{y}) = \frac{\sigma_c^2}{NK}$$

which is same as 3.2.9 (d)

3.3 Composite sampling procedure for nonsegmented Population.

In this section, the result of Brown and Fisher is extended for non segmented population, such as soil, lake, coal etc. Here we form more than one composite sample so that variation between as well as within composite samples can be studied.

The mathematical model for such population was first introduced by Rohde (1976) with parameters (∞, n 1, r, s, k). In this procedure, select n samples (places) from non segmented population by any sampling device. Mix these n increments to form r composite samples; take s subsamples from each composite sample and take k tests on each subsample, so that there are in all r s k tests.

Let $y_1, y_2, \ldots y_n$ be observations on n increments. Mix these increments to form one composite sample. Let Z be observation on the composite sample. Then

$$Z = \sum_{i=1}^{n} \alpha_i y_i$$
$$= \underset{\sim}{\alpha}' \underset{\sim}{y}$$

where α_i is the proportion of ith increment incorporated in Z and $\underset{\sim}{\alpha}$ is proportion vector.
that is $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \ldots \alpha_n)'$ and $\underset{\sim}{y} = (y_1, y_2, \ldots y_n)'$

Let us assume that $y_i (i=1,2,..n)$ have joint distribution with

$E(\underset{\sim}{y}) = \underset{\sim}{\mu}$ and and var $(\underset{\sim}{y}) = V$ [3.3.2]

Also α_i's are independent and independent of y_i's with

$E(\underset{\sim}{\alpha}) = \underset{\sim}{\mu}_{\alpha}$ and var $(\underset{\sim}{\alpha}) = V_c$ [3.3.3]

Result 3.3.1

Let $\underset{\sim}{1} = (1,1,..1)'_{1xn}$

as $\underset{\sim}{1}' \underset{\sim}{\alpha} = 1$

then $\underset{\sim}{1}' \mu_c = 1$ [3.3.4(a)]

Also

$$\underset{\sim}{1}' V_c \underset{\sim}{1} = \sum_{i=1}^{n} \text{var}(\alpha_i)$$
$$= \text{var}\left[\sum_{i=1}^{n} \alpha_i\right]$$
$$= \text{var}(1)$$
$$=0$$

$\underset{\sim}{1}' V_c \underset{\sim}{1} = 0$ [3.3.4(b)]

Lemma (3.3.1)

If $Z = \underset{\sim}{\alpha}' \underset{\sim}{y}$ where $\underset{\sim}{\alpha}$ and $\underset{\sim}{y}$ are vectors satisfying (3.3.2),(3.3.3) and (3.3.4) then,

a) $E(Z) = \underset{\sim}{\mu}_c' \underset{\sim}{\mu}$

b) var $(Z) = \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c + \underset{\sim}{\mu}' V \underset{\sim}{\mu} + \text{tr}(V_c V)$

and if $w = \underset{\sim}{\beta}' \underset{\sim}{y}$ where $\underset{\sim}{\beta}$ is also vector of compositing proportions then

c) cov $(w,Z) = \underset{\sim}{\mu}' C_{\alpha\beta} \underset{\sim}{\mu} + \underset{\sim}{\mu}_{c\alpha}' V \underset{\sim}{\mu}_{c\alpha} + \text{tr}(C_{\alpha\beta} V)$

Proof:-

We prove this lemma by using conditional expectation as in section (3.2)

a) $E(Z) = E(\underset{\sim}{\alpha}' \underset{\sim}{x})$

$= E[E(\underset{\sim}{\alpha}' \underset{\sim}{x})/\underset{\sim}{x}]$

$= E[\underset{\sim}{\mu}_c' \underset{\sim}{x}]$

$= \underset{\sim}{\mu}_c' \underset{\sim}{\mu}$

b) $var(Z) = var(\underset{\sim}{\alpha}'\underset{\sim}{x})$

$= E[var(\underset{\sim}{\alpha}'\underset{\sim}{x})/\underset{\sim}{x}] + var[E(\underset{\sim}{\alpha}'\underset{\sim}{x})/\underset{\sim}{x}]$

$= E[\underset{\sim}{x}' V_c \underset{\sim}{x}] + var[\underset{\sim}{\mu}_c' \underset{\sim}{x}]$

$= E[tr(\underset{\sim}{x}' V_c \underset{\sim}{x})] + [\underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c]$

$= tr[E(\underset{\sim}{x}' \underset{\sim}{x})V_c] + \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c$

$= tr[(V + \underset{\sim}{\mu}'\underset{\sim}{\mu}) V_c] + \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c$

$= tr[(V V_c) + tr(\underset{\sim}{\mu}'\underset{\sim}{\mu} V_c) + \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c$

$= tr[(V V_c) + \underset{\sim}{\mu}' V_c \underset{\sim}{\mu} + \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c$

$\therefore var(Z) = \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c + \underset{\sim}{\mu}' V_c \underset{\sim}{\mu} + tr(V V_c)$

c) Let $cov(\underset{\sim}{\alpha}, \underset{\sim}{\beta}) = \mathbb{C}_{\alpha\beta}$

$E(\underset{\sim}{\alpha}) = \underset{\sim}{\mu}_{c\alpha}$

$E(\underset{\sim}{\beta}) = \underset{\sim}{\mu}_{c\beta}$

Then

$cov(w, Z) = cov(\underset{\sim}{\alpha}' \underset{\sim}{x}, \underset{\sim}{\beta}'\underset{\sim}{x})$

$= E(cov(\underset{\sim}{\alpha}'\underset{\sim}{x}, \underset{\sim}{\beta}'\underset{\sim}{x})/\underset{\sim}{x}) + cov(E(\underset{\sim}{\alpha}'\underset{\sim}{x}/\underset{\sim}{x}), E(\underset{\sim}{\beta}'\underset{\sim}{x}/\underset{\sim}{x}))$

$= E(\underset{\sim}{x}' cov(\underset{\sim}{\alpha}', \underset{\sim}{\beta}')\underset{\sim}{x}) + cov(\underset{\sim}{\mu}_{c\alpha}' \underset{\sim}{x}, \underset{\sim}{\mu}_{c\beta}' \underset{\sim}{x})$

$= E[(\underset{\sim}{x}'\underset{\sim}{x})\mathbb{C}_{\alpha\beta}] + \underset{\sim}{\mu}_{c\alpha}' V \underset{\sim}{\mu}_{c\beta}$

$= E\, tr(\underset{\sim}{x}'\underset{\sim}{x})\mathbb{C}_{\alpha\beta}] + \underset{\sim}{\mu}_{c\alpha}' V \underset{\sim}{\mu}_{c\beta}$

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$$= tr[E(\underset{\sim}{\chi}'\underset{\sim}{\chi})\mathbb{C}_{\alpha\beta}] + \underset{\sim}{\mu}'_{c\alpha} V \underset{\sim}{\mu}_{c\beta}$$

$$= tr\ [V + \underset{\sim}{\mu}'\underset{\sim}{\mu})\mathbb{C}_{\alpha\beta}] + \underset{\sim}{\mu}'_{c\alpha} V \underset{\sim}{\mu}_{c\beta}$$

$$= tr\ (V\mathbb{C}_{\alpha\beta}) + \underset{\sim}{\mu}'\mathbb{C}_{\alpha\beta}\ \underset{\sim}{\mu} + \underset{\sim}{\mu}'_{c\alpha} V \underset{\sim}{\mu}_{c\beta}$$

Hence

$$cov\ (w,z) = \underset{\sim}{\mu}'_{c\alpha} V \underset{\sim}{\mu}_{c\beta} + \underset{\sim}{\mu}' C_{\alpha\beta}\ \underset{\sim}{\mu} + tr\ (V, C_{\alpha\beta})$$

If we assume that each proportion of compositing vector has same expectation, same variance σ_c^2 and different proportions are equally corelated that is $cov(\alpha_i, \alpha_{i'}) = \sigma$ for $i \neq i'$

Then

$$\underset{\sim}{1}'\underset{\sim}{\mu}_c = 1$$

That is $\Sigma\ \mu_c = 1$

$$\mu_c = 1/n \qquad [3.3.5(a)]$$

Also

$$\underset{\sim}{1}'\ V_c \underset{\sim}{1} = 0$$

That is $[\ 1,1,1,\ldots,1]$

$$\begin{bmatrix} \sigma_c^2 & \sigma & \sigma & \ldots & \sigma \\ \sigma & \sigma_c^2 & \sigma & \ldots & \sigma \\ \vdots & & & & \\ \sigma & \sigma & \ldots\ldots & \sigma & \sigma_c^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0$$

$$n\ \sigma_c^2 + n\ (n-1)\ \sigma = 0$$

$$\sigma_c^2 = -(n-1)\ \sigma$$

Hence correlation coefficient $\rho = \dfrac{cov\ (\underset{\sim}{\alpha}_i,\ \underset{\sim}{\alpha}_i')}{\sqrt{var(\underset{\sim}{\alpha}_i)}\ \sqrt{var(\underset{\sim}{\alpha}_i')}}$

$$= \frac{\sigma}{-(n-1)\sigma}$$

$$\rho = -\frac{1}{(\ n-1)} \qquad [3.3.5(b)]$$

Also

$$V_c = \begin{bmatrix} \sigma_c^2 & \sigma & \sigma & \ldots & \sigma \\ \sigma & \sigma_c^2 & \sigma & \ldots & \sigma \\ \vdots & & & & \\ \sigma & \sigma & \sigma & \ldots & \sigma_c^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_c^2 & -\frac{\sigma_c^2}{n-1} & \cdots & -\frac{\sigma_c^2}{n-1} \\ -\frac{\sigma_c^2}{n-1} & \sigma_c^2 & \cdots & -\frac{\sigma_c^2}{n-1} \\ \vdots & & \ddots & \vdots \\ -\frac{\sigma_c^2}{n-1} & \cdots & & \sigma_c^2 \end{bmatrix}$$

$$= \sigma_c^2 \begin{bmatrix} 1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 1 & \cdots & -\frac{1}{n-1} \\ \vdots & & \ddots & \vdots \\ -\frac{1}{n-1} & \cdots & & 1 \end{bmatrix}$$

$$= \sigma_c^2 \left[\frac{n}{n-1} I_n - \frac{n}{n-1} E_{nn} \right]$$

Where I_n is nxn identity matrix and E_{nn} is n x n matrix of 1's

The above result can be verified by using Dirichlet distribution. In the following we define Dirichlet distribution and study its moments.

Definition (3.3.1)

The set of variables $(\alpha_1, \alpha_2, \ldots \alpha_n)$ are said to have Dirichlet distribution with parameters $(l_1, l_2, \ldots l_n)$ if its joint density is given by

$$f(\alpha_1, \alpha_2, \ldots \alpha_n) = \begin{cases} \dfrac{\Gamma(\Sigma l_i)}{\Pi \Gamma l_i} \prod_{i=1}^{n} \alpha_i^{l_i - 1} & 0 \le \alpha_i \le 1, \ \Sigma \alpha_i = 1 \\ & \text{and } l_i \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad [3.3.6]$$

The mixed moments of Dirichlet distribution are given by,

$$\mu_{r_1, \ldots r_n} = E\left(\prod_{i=1}^{n} \alpha_i^{r_i} \right)$$

$$= \int_{r_1} \int_{r_2} \cdots \int_{r_n} \Pi \alpha_i^{r_i} \frac{\Gamma(\Sigma l_i)}{\Pi \Gamma l_i} \Pi \alpha_i^{l_i - 1} \prod_{i=1}^{n} d\alpha_i$$

$$= \frac{\Gamma(\Sigma l_i)}{\Pi \Gamma l_i} \times \frac{\Pi \Gamma(l_i + r_i)}{\Gamma(\sum^{n} l_i + r_i)}$$

In particular if $r_{i=1}$ $\quad i = 1,2,\ldots n$

$$\mu_i = E(\alpha_i) = \frac{\Gamma(\Sigma l_i)}{\Pi\ \Gamma l_i} \quad \frac{\Pi\ \Gamma(l_i+1)}{\Gamma(\Sigma l_i + 1)}$$

$$= \frac{l_i}{\Sigma l_i}$$

If $r_i = 2$, $i = 1,2,\ldots n$

$$E(\alpha_i^2) = \frac{\Gamma \Sigma l_i}{\Pi\ \Gamma l_i} \times \frac{\Pi\ \Gamma l_i+2}{\Gamma(\Sigma l_i+2)}$$

$$\frac{l_i(l_i+1)}{\Sigma l_i\ (\Sigma l_i+1)}$$

similarly for $i \neq i'$

$$E(\alpha_i\ \alpha_i') = \frac{l_i l_i'}{\Sigma\ l_i(\Sigma l_i+1)}$$

Hence $\text{var}\ (\alpha_i) = E(\alpha_i^2) - E\ (\alpha_i)^2$

$$= \frac{l_i(l_i+1)}{\Sigma l_i(\Sigma l_i+1)} - \left(\frac{l_i}{\Sigma l_i}\right)^2$$

$$= \frac{l_i\ \Sigma\ l_i - l_i^2}{(\Sigma l_i)^2(\Sigma l_i+1)}$$

and

$$\text{cov}\ (\alpha_i,\alpha_i') = E\ (\alpha_i\alpha_i') - E\ (\alpha_i)\ E(\alpha_i')$$

$$= \frac{l_i l_i'}{\Sigma l_i(\Sigma l_i+1)} - \frac{l_i}{\Sigma l_i}\ \frac{l_i'}{\Sigma l_i}$$

$$= -\frac{l_i\ l_i'}{\Sigma l_i(\Sigma l_i+1)}$$

If $l_i = l$ for $i = 1,2,\ldots n)$ then

$$\text{var}\ (\alpha_i) = \frac{nl^2 - l^2}{(nl)^2(nl+1)}$$

$$= \frac{(\ n\ -1\ )}{n^2(nl+1)}$$

and $\text{cov}(\alpha_i \alpha_i') = -\dfrac{l^2}{(nl)^2(nl+1)}$

$$= \frac{-1}{n^2(nl+1)}$$

Therefore correlation coeficient ρ is given by

$$\rho = \frac{-1}{n-1}$$

Hence results obtained in composite sampling can also be obtained by using Dirichlet distribution.

3.4 Special cases:-

Let us consider two particular cases one is mean plus random error model and other is simple mixed model for composite sampling procedure.

3.4.1 : Mean plus random error model:

In this case any observation on increment can be written as

$y_i = \mu + \varepsilon_i \qquad i = 1,2,..n$

Where μ is population mean and ε_i is random error associated with y_i having mean zero and variance σ^2

Hence $E(y_i) = E(\mu + \varepsilon_i)$

$= \mu$

that is $E(\underset{\sim}{y}) = \underset{\sim}{\mu}$ or $\mu \underset{\sim}{1}$ [3.4.1]

and $\text{var}(\underset{\sim}{y}) = \text{var}(\underset{\sim}{\mu} + \underset{\sim}{\varepsilon})$

$= \sigma^2 I_n$

$V = \sigma^2 I_n$ [3.4.2]

By using lemma (3.3.1) this results and equation 3.3.1 we get

$E(Z) = E(\alpha' y)$

$= \underset{\sim}{\mu}_c' \underset{\sim}{\mu}$

$= 1/n \, \underset{\sim}{1}' \mu \underset{\sim}{1}$

$= \mu$ [3.4.3]

and

$$\text{var}(Z) = \text{var}(\underset{\sim}{\alpha}' \underset{\sim}{x})$$

$$= \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c + \underset{\sim}{\mu}' V_c \underset{\sim}{\mu} + \text{tr}(V V_c)$$

But $\underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c = \underset{\sim}{1}/n \; \underset{\sim}{1}' \; \sigma^2 I_n \; 1/n \; \underset{\sim}{1}$

$$= \sigma^2/n^2 \; \underset{\sim}{1}' I_n \underset{\sim}{1}$$

$$= \sigma^2/n$$

$$\underset{\sim}{\mu}' V_c \underset{\sim}{\mu} = \mu \; \underset{\sim}{1}' \; V_c \; \underset{\sim}{1} \; \mu$$

$$= \mu^2 \; \underset{\sim}{1}' \; V_c \; \underset{\sim}{1}$$

$$= 0$$

$$\text{tr}(V V_c) = \text{tr}\left[\sigma_c^2 \left(\frac{n}{n-1} I_n - \frac{1}{n} E_{nn} \right)(\sigma^2 I_n)\right.$$

$$= n \; \sigma^2 \; \sigma_c^2$$

Hence

$$\text{Var}(z) = \frac{\sigma^2}{n} + n \; \sigma^2 \sigma_c^2 \tag{3.4.4}$$

If we assume that $\alpha_1 \alpha_2 \ldots \alpha_n$ have Dirichlet distribution that

$$E(z) = \underset{\sim}{\mu}_c' \; \underset{\sim}{\mu}$$

$$= \frac{1}{n} \underset{\sim}{1}' \underset{\sim}{1}$$

$$= \mu$$

and

$$\text{Var}(z) = \mu_c' \; V\mu_c + \mu' V_c \mu + \text{tr}(VV_c)$$

$$= \frac{\sigma^2}{n} + \text{tr}(VV_c)$$

$$= \frac{\sigma^2}{n} + \text{tr}\left[\frac{1}{n^2(nl+1)} [nI_n - E_{nn}] . \sigma^2 I_n \right]$$

$$= \frac{\sigma^2}{n} + \frac{n(n-1)}{n^2(nl+1)} . \sigma^2$$

$$= \frac{\sigma^2}{n} + \frac{(n-1)}{n(nl+1)} . \sigma^2$$

Suppose 'R' composites are formed from 'n' increments and observations are taken on each composite sample. Let $z_1, z_2, \ldots z_R$ be observations.

$$\text{where } z_i = \underset{\sim}{\alpha}_i' \underset{\sim}{Y} \qquad i = 1,2,\ldots R \tag{3.4.5}$$

$$\text{and } \bar{Z} = \frac{1}{R} \sum_{i=1}^{R} Z_i$$

$$\bar{Z} = \frac{1}{R} \left[\sum_{i=1}^{R} \underset{\sim}{\alpha}_i' \underset{\sim}{Y} \right]$$

$$= (1/R \Sigma \underset{\sim}{\alpha}_i') \underset{\sim}{Y}$$

$$= \underset{\sim}{\nu}' \underset{\sim}{Y} \tag{3.4.6}$$

where ν' is vector of $\nu_j = 1/R \Sigma \alpha_i \quad j = 1,2,,,,R$

Note that as $\underset{\sim}{\alpha}' \underset{\sim}{1} = 1$

then $\underset{\sim}{\nu}' \underset{\sim}{1} = 1$

and $\underset{\sim}{\mu}' \underset{\sim}{1} = 1$ where μ_ν is mean of ν

also as $\underset{\sim}{1}' V_c \underset{\sim}{1} = 0$

then $\underset{\sim}{1}' V_\nu \underset{\sim}{1} = 0$

Hence $E(\bar{Z}) = E(\underset{\sim}{\nu}' \underset{\sim}{Y})$

$$= \underset{\sim}{\mu}_\nu' \underset{\sim}{\mu}$$

If we assume each ν_j have same expectation and equally correlated then

$$\underset{\sim}{\mu}_\nu = 1/n \underset{\sim}{1}$$

and

$$V_\nu = \text{var}(\underset{\sim}{\nu})$$

$$= \text{var}(1/R \Sigma \underset{\sim}{\alpha}_i$$

$$= 1/R^2 \text{ var}(\Sigma \underset{\sim}{\alpha}_i)$$

$$= 1/R \, V_c$$

$$= 1/R \, \sigma_c^2 \left(\frac{n}{n-1} I_n - \frac{1}{n-1} E_{nn} \right)$$

Hence $E(\bar{Z}) = \underset{\sim}{\mu}_\nu' \underset{\sim}{\mu}$

$$= 1/n \underset{\sim}{1}' \underset{\sim}{1} \mu$$

$$= \mu \tag{3.4.7}$$

and $\text{var}(\bar{Z}) = \text{var}(\underset{\sim}{\nu}' \underset{\sim}{x})$

$$= \underset{\sim}{\mu}_\nu' V \underset{\sim}{\mu}_\nu + \underset{\sim}{\mu}' V_\nu \underset{\sim}{\mu} + tr(V V_\nu)$$

$$= 1/n \underset{\sim}{1}' \sigma^2 I_n \underset{\sim}{1}\, 1/n + \mu 1' V_\nu \underset{\sim}{1} \mu + tr(V V_\nu)$$

$$= \frac{\sigma^2}{n} + 0 + tr\left\{\sigma^2 I_n \frac{\sigma_c^2}{R}\left[\frac{n}{n-1} I_n - \frac{n}{n-1} E_{nn}\right]\right\}$$

$$= \frac{\sigma^2}{n} + \frac{n\,\sigma^2\sigma_c^2}{R} \qquad [3.4.8]$$

Hence as number of composite sample increases the variance reduces.

Theorem 3.4.1:

Show that $\quad E\left[\frac{(Z_i-\bar{Z})^2}{R-1}\right] = n\,\sigma^2\sigma_c^2$

Proof:

By definition of variance

$$\text{var}(Z_i) = E(Z_i^2) - E(Z_i)^2$$

Hence

$$E(Z_i^2) = \text{var}(Z_i) + (E(Z_i))^2$$

$$= \frac{\sigma^2}{n} + n\,\sigma^2\sigma_c^2 + \mu^2$$

and $\quad \text{var}(\bar{Z}) = E(\bar{Z}^2) - E(\bar{Z})^2$

Hence $\quad E(\bar{Z}^2) = \text{var}(\bar{Z}_i) + E(\bar{Z})^2$

$$= \sigma^2/n + n\sigma^2\sigma^2c/R + \mu^2$$

Now consider

$$E\left(\sum_{i=1}^{R} \frac{(Z_i-\bar{Z})^2}{R-1}\right) = 1/(R-1)\, E\left[\Sigma Z_i^2 - R\,\bar{Z}^2\right]$$

$$= 1/(R-1)\left[\Sigma E(Z_i^2) - R\,E(\bar{Z}^2)\right]$$

$$= 1/(R-1)\left[R\left(\frac{\sigma^2}{n} + n\sigma^2\sigma_c^2 + \mu^2\right) - R\left(\frac{\sigma^2}{n} + \frac{n\sigma^2\sigma_c^2}{R} + \mu^2\right.\right.$$

$$= 1/(R-1)\left[n\,\sigma^2\sigma_c^2\,(R-1)\right]$$

$$= n\,\sigma^2\sigma^2c \qquad [3.4.9]$$

Thus we can use $\Sigma\ (Z_i-\bar{Z})^2 = S_z^2$ as underestimate of variance of $(\bar{Z})$

The estimate of σ_c^2 can be obtained from laboratory experiment as given by Brown and Fisher (1972). We may use s_z^2 to obtain estimate of σ^2.

3.4.2 Simple mixed model:-

In this method select J sites (places) and from j^{th} site, select n_j increments $j= 1,2,..J$. Mix these $\Sigma n_j = n$ increments to form R composite samples.

Let y_{jk} be observation on k^{th} increment which is selected from j^{th} site. Let us assume that y_{jk} are such that

$$y_{jk} = \mu + t_j + \varepsilon_{jk} \qquad j = 1,2,...J,\ k= 1,2,,,,n_j \qquad [3.4.10]$$

Where t_j denote effect of jth sampling site and ε_{jk} be random effect of kth increment selected from j^{th} site. Such that

$$E(t_j)= 0, \qquad E\ (\varepsilon_{jk}) = 0$$
$$\text{var}\ (t_j)= \sigma_t^2\ ,\ \text{var}\ (\varepsilon_{jk})= \sigma^2$$
$$\text{cov}\ (t_j, \varepsilon_{jk}) = 0 \quad \text{for } j = 1,2,...\ J$$
$$k = 1,2,...n_j$$

Let Z_i be observation on i^{th} composite sample then

$$Z_i = \sum_{jk} \alpha_{ijk}\ y_{jk} \qquad i = 1,2,..\ R$$

$$= \underset{\sim}{\alpha}_i'\ \underset{\sim}{y} \qquad [3.4.11]$$

Assume that $\underset{\sim}{\alpha}_i$'s are identical and independently distributed random variables (IIDrv's) with

$$E(\underset{\sim}{\alpha}_i) = \underset{\sim}{\mu}_c \quad \text{and var}\ (\underset{\sim}{\alpha}_i) = V_c$$
$$E(\underset{\sim}{y}) = \underset{\sim}{\mu} \quad \text{and var}\ (\underset{\sim}{y}\ ) = V$$

Now $\text{var}\ (y_{jk}) = \text{var}\ (\mu + t_j + \varepsilon_{jk})$

$$= \sigma_t^2 + \sigma^2 \qquad j= 1,2,...J,\ k= 1,2,..n_j \qquad [3.4.12(a)]$$

$$\begin{aligned}
\text{cov}\,(y_{jk}, y_{jk'}) &= E\,(y_{jk}\, y_{jk'}) - E\,(y_{jk})\, E(y_{jk'}) \\
&= E[\,(\mu+t_j+\varepsilon_{jk})\,(\mu+t_j+\varepsilon_{jk'})] - \mu^2 \\
&= \mu^2 + \sigma_t^2 - \mu^2 \\
&= \sigma_t^2 \quad \text{for } k \neq k' \qquad [3.4.12.(b)
\end{aligned}$$

$$\begin{aligned}
\text{cov}\,(y_{jk}, y_{j'k'}) &= E\,(y_{jk}\, y_{j'k'}) - E\,(y_{jk})\, E(y_{j'k'}) \\
&= E[\,(\mu+t_j+\varepsilon_{jk})\,(\mu+t_{j'}+\varepsilon_{j'k'})] - \mu^2 \\
&= \mu^2 - \mu^2 \\
&= 0 \quad \text{for } j \neq j_i,\ j \neq i' \text{ and } k \neq k' \qquad [3.4.12.(b)
\end{aligned}$$

Hence matrix V becomes

$$\begin{bmatrix}
\sigma^2+\sigma_t^2 & \sigma_t^2 \cdots\cdots & \sigma_t^2 & 0 \cdots & 0 & 0 \text{----} & 0 \text{----} & 0 \text{----} & 0 \\
\sigma_t^2 & \sigma^2+\sigma_t^2 \cdots\cdots & \sigma_t^2 & 0 \cdots & 0 & 0 \text{----} & 0 \text{----} & 0 \text{----} & 0 \\
\sigma_t^2 & \sigma_t^2 \cdots\cdots & \sigma^2+\sigma_t^2 & 0 \cdots & 0 & 0 \text{----} & 0 \text{----} & 0 \text{----} & 0 \\
0 \cdots\cdots & 0 & \cdots\cdots & \sigma^2+\sigma_t^2 & \sigma_t^2 & \sigma_t^2 & 0 \cdots & 0 \cdots & 0 \\
0 \cdots\cdots & 0 & \cdots\cdots & \sigma_t^2 & \sigma^2+\sigma_t^2 & \sigma_t^2 & \cdots & & \\
0 \cdots\cdots & & & \sigma_t^2 & \cdots\cdots & \sigma^2+ & \sigma_t^2 & \cdots & 0 \\
0 \cdots\cdots & 0 & \cdots\cdots & & & & & \sigma^2+\sigma_t^2 & \sigma_t^2 \cdots \sigma_t^2 \\
\cdots\cdots & 0 & \cdots\cdots & & & & & \sigma_t^2 & \sigma^2+\sigma_t^2 \cdot \sigma_t^2 \\
0 \cdots\cdots & & & & & & & \sigma_t^2 \cdots & \cdot\sigma^2+\sigma_t^2
\end{bmatrix}$$

$$= \sigma^2\, I_n + \sigma_t^2\, \text{diag}\,(v_j) \qquad [3.4.13]$$

where v_j is $n_j x n_j$ matrix of 1's $j = 1, 2, \ldots$]

Therefore

$$E(z_i) = E(\underset{\sim}{\alpha} i' \underset{\sim}{y})$$
$$= \underset{\sim}{\mu}_c' \underset{\sim}{\mu}$$
$$= 1/n \; 1' \; 1 \; \mu$$
$$= \mu \qquad [3.4.14]$$

and $\text{var}(Z_i) = \text{var}(\underset{\sim}{\alpha} i' \underset{\sim}{y})$

$$= \underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c + \underset{\sim}{\mu}' V_c \underset{\sim}{\mu} + \text{tr}(V_c V_c)$$

But $\underset{\sim}{\mu}_c' V \underset{\sim}{\mu}_c = 1/n \; 1' \; [\sigma^2 I_n + \sigma^2 \text{diag}(V_j)] \; 1/n$

$$= 1/n^2 \; [n\sigma^2 + \Sigma n_j^2 \sigma_t^2]$$

$$= \sigma^2/n + \Sigma \frac{n_j^2}{n^2} \sigma_t^2$$

$$\underset{\sim}{\mu}_c V \underset{\sim}{\mu}_c = 0$$

and $\text{tr}(V_c V) = \sigma_c^2 \; (n\sigma^2 + n\sigma_t^2 - \frac{\Sigma n_j^2 - n}{n-1} \sigma_t^2$

$$= \sigma_c^2 \; [n\sigma^2 + \frac{\sigma_t^2}{n-1} (n(n-1) - \Sigma n_j^2 + n)]$$

$$= n \sigma^2 \sigma_c^2 + \frac{\sigma_t^2 \sigma_c^2}{n-1} (n^2 - \Sigma n_j^2)$$

Hence

$$\text{var}(Z_i) = \frac{\sigma^2}{n} + \frac{\Sigma n_j^2}{n^2} \sigma_t^2 + n \sigma^2 \sigma_c^2 + \frac{n^2 - \Sigma n_j^2}{n-1} \sigma_c^2 \sigma_t^2 \qquad [3.4.15]$$

Now since $\underset{\sim}{\alpha}_i'$ s are IIDrv's then

$$\text{cov}(\underset{\sim}{\alpha}_i, \underset{\sim}{\alpha}_i') = 0$$

Hence

$$\text{cov}(Z_i, Z_i') = \text{cov}(\underset{\sim}{\alpha}_i' \underset{\sim}{y}, \alpha_i' y)$$
$$= \underset{\sim}{\mu}_{c\alpha}' V \underset{\sim}{\mu}_{c\alpha} + \underset{\sim}{\mu}' \; c_{\alpha_i \alpha_i} \underset{\sim}{\mu} + \text{tr}(V \; C_{\alpha_i \alpha_i}')$$
$$= \underset{\sim}{\mu}_{c\alpha}' V \underset{\sim}{\mu}_{c\alpha}$$
$$= \frac{\sigma^2}{n} + \Sigma \frac{n_j^2}{n^2} \sigma_t^2 \qquad [3.4.16]$$

The mean of R composite sample is

$$\bar{Z} = 1/R \sum_{i=1}^{R} Z_i$$

Hence $E(\bar{Z}) = 1/R \sum_{i=1}^{R} E(Z_i)$

$$= \mu$$

$$\text{var}(\bar{Z}) = \text{var}(1/R \Sigma Z_i)$$

$$= 1/R^2 \left[\Sigma \text{var}(Z_i) + \sum_i \sum_{i \neq i'} \text{cov}(Z_i, Z_{i'}) \right]$$

$$= 1/R^2 \left[R\left(\frac{\sigma^2}{n^2} + \frac{\Sigma n_j^2}{n^2} \sigma_t^2 + n\,\sigma^2\sigma_t^2 + \frac{n^2 - \Sigma n_j^2}{n-1} \sigma_t^2 \sigma_c^2 \right) + R(R-1)\left(\frac{\sigma^2}{n} + \frac{\Sigma n_j^2}{n^2} \sigma_t^2 \right) \right]$$

$$= \frac{\sigma^2}{n} + \frac{\Sigma n_j^2}{n^2} \sigma_t^2 + \frac{n\,\sigma^2\sigma_c^2}{R} + \frac{n^2 - \Sigma n_j^2}{R(n-1)} \sigma_t^2 \sigma_c^2 \qquad (3.4.17)$$

If from each site we select k increments that is $n_j = k$ so that

$$n = \sum_{j=1}^{J} n_j = JK$$

then above equation (3.4.16) becomes

$$\text{var}(\bar{Z}) = \frac{\sigma^2}{JK} + \frac{\sigma_t^2}{J} + \frac{JK\sigma^2\sigma_c^2}{R} + \frac{JK^2(J-1)}{R(JK-1)} \sigma_t^2 \sigma_c^2 \qquad (3.4.18)$$

The same result is obtained by Brown and Fisher (1972) for $N \to \infty$ and $r = 1$.

3.5 Asymptotic distribution of mean of the composite sample :

Let us assume that $y_1, y_2 \ldots y_n$ are IID rv's with

$$E(y_i) = \mu \text{ and } \text{var}(y_i) = \sigma_y^2 \qquad [3.5.1]$$

If $\nu_1, \nu_2, \ldots \nu_n$ are IID gamma variates with scale parameter 'β' and shape parametr 1

Then Probability distribution function of ν_i is

$$f(\nu_i) = \frac{1}{\Gamma 1\, \beta^1} e^{-\nu_i/\beta} (\nu_i)^{1-1} \qquad [3.5.2]$$

so that

$$\sum_{i=1}^{n} \nu_i \sim G\ (nl, \beta)$$

Let $\alpha_i = \dfrac{\nu_i}{\Sigma \nu_i}$

Then joint distrubution of $\alpha_1, \alpha_2, \ldots \alpha_n$ is

$$f(\alpha_1, \alpha_2, \ldots \alpha_n) = \prod_{i=1}^{n} f(\alpha_i)$$

$$= \prod_{i=1}^{n} f\left(\frac{\nu_i}{\Sigma \nu_i}\right)$$

$$= \frac{1}{f(\Sigma \nu_i)} \text{ II } f(\nu_i)$$

$$= \frac{1}{1/\Gamma nl\ 1/\beta nl\ \bar{e}^{\Sigma \nu_i/\beta}} (\Sigma \nu_i)^{nl-1} x (1/\Gamma l)^n (1/\beta^l)^n\ \bar{e}^{\frac{\Sigma \nu_i}{\beta}} \text{II} \nu_i^{l-1}$$

$$= \frac{\Gamma nl}{(\Gamma l)^n} \text{ II } (\nu_i)^{l-1} \qquad [3.5.4]$$

Which is Dirichlet distribution with parameters (1,...1)

Now consider the distribution of

$$\begin{aligned} W &= \sqrt{n}\ (Z-\mu) \\ &= \sqrt{n}\ (\Sigma \alpha_i y_i - \mu) \\ &= \sqrt{n}\ \left(\Sigma \frac{\nu_i}{\Sigma \nu_i}\ y_i - \mu\right) \\ &= \frac{\sqrt{n}}{\Sigma \nu_i} (\Sigma\ \nu_i y_i - \mu \Sigma \nu_i) \\ &= \frac{\sqrt{n}}{\Sigma \nu_i} (\Sigma \nu_i (y_i - \mu)) \qquad [3.5.6] \end{aligned}$$

Where ν_i's are iid gamma variates and independent of y_i's

But $E(\nu_i(y_i-\mu)) = E(\nu_i)\ E(y_i-\mu)$

$= 0$

and $\text{var}\ (\nu_i(y_i-\mu) = E\ (\nu_i^2(y_i-\mu)^2) - E(\nu_i(y_i-\mu))^2$

$= E\ (\nu_i^2(y_i-\mu)^2)$

$= E\ (\nu_i^2)\ E(y_i-\mu)^2$

$= (\sigma_\nu^2 + \mu_\nu^2)\sigma_y^2 = \sigma^2(\text{say})$ [3.5.7]

Hence $\sqrt{n}\ (\Sigma\nu_i(y_i-\mu) \xrightarrow{D} N\ (0,\sigma^2)$

Also by strong law of large numbers

$$\frac{\Sigma\ \nu_i}{n} \xrightarrow{P} \mu_r$$

Now $W = \frac{\sqrt{n}}{\Sigma\nu_i}\ (\Sigma\nu_i(y_i-\mu)$

$$= \frac{1/\sqrt{n}(\Sigma\nu_i(y_i-\mu)}{1/n\ \Sigma\nu_i}$$

Hence by mann-wald theorem

$$W = \sqrt{n}\ (Z-\mu) \xrightarrow{D} \frac{N(0,\sigma^2)}{\mu_r}$$

$$\xrightarrow{D} N\ (0,\ \sigma^2/\mu_r^2\ )\ .$$

Where μ_r and σ_ν^2 are mean and variance of gamma distribution.

A COMPUTER PROGRAM IN BASIC

```
10 INPUT "Give Probability : ";P
20 LET N = 1
30 LET RCF1 = 1 + (1/N) - (1-P)^N
40 N = N + 1
50 LET RCF2 = 1 + (1/N) - (1-P)^N
60 IF RCF1 < RCF2 THEN 80
70  GOTO 30
80 PRINT "N = ";N, "P = ";P, " RCF1 = ";RCF1," RCF2 = ";RCF2
90 END
```